GLASNIK MATEMATIČKI
Vol. 21 (41) (1986), 283 – 300.

ON SUMS INVOLVING RECIPROCALS OF THE LARGEST PRIME FACTOR OF AN INTEGER

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Abstract. Sum of reciprocals of $P(n)$, the largest prime factor of n, is precisely evaluated asymptotically. Asymptotic formulas for some related sums, involving the function $\Omega(n)$ and $\omega(n)$ (the number of distinct and the total number of prime factors of n) are also derived.

§1. Introduction and statement of results

Let as usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors of n and the total number of prime factors of n, respectively. Let $P(n)$ denote the largest prime factor of an integer $n \geqslant 2$, and let $P(1)=1$. Several results involving sums of reciprocals of $P(n)$ and some related additive functions were obtained recently in [4], Ch. 6, [6], [7], [9] and [10]. Thus it was shown in [9] that

$$\sum_{n \leqslant x} 1/P(n) = x \exp\{-(2 \log x \operatorname{loglog} x)^{1/2} + O((\log x \log\log\log x)^{1/2})\}. \tag{1.1}$$

The proof of this result depended on estimates for $\psi(x, y)$, the number of positive integers $n \leqslant x$ with $P(n) \leqslant y$. The connection is seen via the easy identity

$$\sum_{n \leqslant x} 1/P(n) = 1 + \sum_{p \leqslant x} p^{-1} \psi(xp^{-1}, p), \tag{1.2}$$

where p denotes a general prime throughout the paper. By using a better estimate for $\psi(x, y)$ (see [3]), the result (1.1) was slightly sharpened and more general sums were estimated in [10], namely

$$S_r(x) = \sum_{n \leqslant x} 1/P^r(n), \qquad T_r(x) = \sum_{n \leqslant x,\, P^2(n) | n} 1/P^r(n),$$

Mathematics subject classifications (1980): 10H15, 10H25.

Key words and phrases: The largest prime factor of an integer, the number of prime factors of an integer, asymptotic formulas for summatory functions, the number of integers not exceeding x all of whose prime factors do not exceed y.

* Research supported in part by a grant from the National Science Foundation.

where $r \geqslant 0$ is an arbitrary, fixed real number. It was proved in [10] that

$$S_r(x) = x \exp\{-(2r)^{1/2} (\log x \log_2 x)^{1/2} (1 + g_{r-1}(x) + O(\log_3^3 x / \log_2^3 x))\} \tag{1.3}$$

and

$$T_r(x) = x \exp\{-(2r+2)^{1/2} (\log x \log_2 x)^{1/2} (1 + g_r(x) + O(\log_3^3 x / \log_2^3 x))\}, \tag{1.4}$$

where $\log_k x = \log(\log_{k-1} x)$ is the k-fold iterated logarithm and

$$g_r(x) = \frac{\log_3 x + \log(1+r) - 2 - \log 2}{2 \log_2 x}\left(1 + \frac{2}{\log x}\right) - \frac{(\log_3 x + \log(1+r) - \log 2)^2}{8 \log_2^2 x}.$$

Recently H. Maier [11] and A. Hildebrand [8] obtained independently much better results concerning $\psi(x, y)$, which may be used in connection with our problems. It is now possible to obtain asymptotic formulas for the sums $S_r(x)$ and $T_r(x)$. We shall work out the details only for the sum in (1.1), namely $S_1(x)$. The other sums can be handled by the same method. We prove

THEOREM 1.

$$\sum_{n \leqslant x} 1/P(n) = x\delta(x)\left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right), \tag{1.5}$$

where

$$\delta(x) = \int_2^x \rho\left(\frac{\log x}{\log t}\right) t^{-2}\, dt, \tag{1.6}$$

and $\rho(u)$ is the continuous solution to the differential delay equation

$$u\rho'(u) = -\rho(u-1) \tag{1.7}$$

with the initial condition $\rho(u) = 1$ for $0 \leqslant u \leqslant 1$.

Here $\rho(u)$ is the so-called Dickman-de Bruijn function, for which the latter [1] obtained the estimate

$$\rho(u) = \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u}{\log u} - \frac{1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right)\right)\right\}. \quad (u \to \infty) \tag{1.8}$$

By comparing (1.3) and (1.5) (or by direct evaluation with the aid of (1.8)) we find that

$$\delta(x) = \exp\{-(2 \log x \log_2 x)^{1/2} (1 + g_0(x) + O(\log_3^3 x / \log_2^3 x))\}. \tag{1.9}$$

Although $\delta(x)$ is a fairly complicated function, (1.9) gives for most purposes a sufficiently sharp approximation. Moreover $\delta(x)$ is slowly oscillating, i.e. for any constant $C>0$ we have

$$\lim_{x\to\infty} \delta(Cx)/\delta(x)=1, \tag{1.10}$$

which is obtained in §3. As a corollary of (1.5) and (1.10) we have, for example,

$$\sum_{n\leqslant x} 1/P(n) \sim \sum_{x<n\leqslant 2x} 1/P(n),$$

which seems to be difficult to obtain without using finer information about $\psi(x, y)$. The notation $f(x)\sim g(x)$ means as usual that

$$\lim_{x\to\infty} f(x)/g(x)=1,$$

and other notation used throughout the paper is also standard. For example, $f(x)=O(g(x))$ and $f(x)\ll g(x)$ both mean that $|f(x)|\leqslant Cg(x)$ for some absolute $C>0$ and $x\geqslant x_0$, while $f(x)=o(g(x))$ means that

$$\lim_{x\to\infty} f(x)/g(x)=0.$$

It seems interesting to investigate how much the sum in (1.1) changes when $1/P(n)$ is replaced by $\omega(n)/P(n)$ or $\Omega(n)/P(n)$. This problem has already been investigated in [7], where it was shown that

$$\left(\frac{\log x}{\log\log x}\right)^{1/2} \sum_{n\leqslant x} 1/P(n) \ll \sum_{n\leqslant x} \Omega(n)/P(n) \ll (\log x \log\log x)^{1/2} \sum_{n\leqslant x} 1/P(n),$$

and the method would yield the same result if $\Omega(n)$ is replaced by $\omega(n)$. By using Theorem 1, we prove

THEOREM 2. *There is a positive constant c such that*

$$\sum_{n\leqslant x} (\Omega(n)-\omega(n))/P(n) \sim cx\,\delta(x), \tag{1.11}$$

$$\sum_{n\leqslant x} \Omega(n)/P(n) \sim \sum_{n\leqslant x} \omega(n)/P(n) \sim (2\log x/\log\log x)^{1/2}\, x\delta(x), \tag{1.12}$$

where $\delta(x)$ is defined by (1.6).

The results (1.5) and (1.12) imply that in a certain sense the main contribution to the sums in (1.12) comes from those $n\leqslant x$ with about $(2\log x/\log\log x)^{1/2}$ prime factors. This fact is interesting in view of the classical result of G. H. Hardy and S. Ramanujan (see [12]) that both the normal and average order of $\omega(n)$ and $\Omega(n)$ is $\log\log n$.

Theorem 2 shows that sums of $\omega(n)/P(n)$ and $\Omega(n)/P(n)$ behave similarly. On the other hand, quite a different situation arises when one estimates sums of $P(n)^{-\omega(n)}$ and $P(n)^{-\Omega(n)}$. The corresponding summatory functions turn out to be completely different, as shown by

THEOREM 3.

$$\sum_{n\leq x} P(n)^{-\omega(n)} = \exp\{(4+o(1))(\log x)^{1/2}/(\log\log x)\}, \qquad (1.13)$$

$$\sum_{n\leq x} P(n)^{-\Omega(n)} = \log\log x + D + O(1/\log x). \qquad (1.14)$$

Here $D>0$ denotes an absolute constant that is effectively computable.

It seems interesting to compare (1.13) with the estimate

$$\sum_{n\leq x} 1/\alpha(n) = \exp\{(2\sqrt{2}+o(1))(\log x/\log\log x)^{1/2}\}, \qquad (1.15)$$

where $\alpha(n)=\prod_{p|n} p$ is the largest square-free divisor of n. This result is due to N. G. de Bruijn [2] and was sharpened by W. Schwarz [13]. Since evidently

$$\alpha(n)\leq P(n)^{\omega(n)},$$

the sum in (1.13) is majorized by the sum in (1.15), but it turns out that even the logarithms of these sums are of a different order of magnitude. The sum in (1.13) is more difficult to estimate than the sum in (1.14), where the main contribution comes from primes.

For our last result, let $Q(n)$ denote the largest prime power which divides $n\geq 2$, and let $Q(1)=1$. One naturally expects sums of $1/P(n)$ and $1/Q(n)$ to behave similarly, and this is precisely what is established in

THEOREM 4. *There is a constant $C>0$ such that*

$$\sum_{n\leq x} 1/Q(n) = \{1+O(\exp(-C(\log x\log\log x)^{1/2}))\}\sum_{n\leq x} 1/P(n). \quad (1.16)$$

§2. Proof of Theorem 1

We begin by establishing the identity (1.2). We have

$$\sum_{n\leq x} 1/P(n) = 1+\sum_{p\leq x} 1/p \sum_{n\leq x,\, P(n)=p} 1 = 1+\sum_{p\leq x} 1/p \sum_{m\leq x/p,\, P(m)\leq p} 1 =$$
$$= 1+\sum_{p\leq x} p^{-1}\psi(x/p, p),$$

so that (1.2) holds. To facilitate notation, let from now on

$$L = L(x) = \exp\{(\log x\log\log x)^{1/2}\}.$$

Using (1.2) and following the proof of (1.1) given in [9] we see that the contribution from the primes p with $p<L^{1/2}$ or $p>L$ is at most

$$xL^{-3/2+o(1)} \qquad (x\to\infty).$$

Thus in view of (1.1) it suffices to consider only those values of p with $L^{1/2} \leqslant p \leqslant L$.

We now state the new result of A. Hildebrand [8] (H. Maier's theorem [11] is slightly weaker) mentioned in the introduction.

THEOREM (A. Hildebrand). *The estimate*

$$\psi(x, x^{1/u}) = x\rho(u)\left(1 + O_\varepsilon\left(\frac{u \log(u+1)}{\log x}\right)\right) \tag{2.1}$$

holds uniformly in the range

$$x \geqslant 3,\ 1 \leqslant u \leqslant \log x / (\log\log x)^{5/3+\varepsilon},$$

where ε *is any fixed positive number.*

The importance of this result lies in the wide range for u. By N. G. de Bruijn [1] (2.1) was known to hold for $1 \leqslant u \leqslant (\log x)^{3/8-\varepsilon}$, but this range is not sufficient for our purposes.

Applying (2.1) to the $\psi(x/p,\ p)$ for $L^{1/2} \leqslant p \leqslant L$ we obtain the uniform estimate

$$\psi(x/p,\ p) = \frac{x}{p}\rho\left(\frac{\log x}{\log p} - 1\right)\left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right).$$

Thus from the above comments we have

$$\begin{aligned}
\sum_{n \leqslant x} 1/P(n) &= \left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right) \sum_{L^{1/2} \leqslant p \leqslant L} xp^{-2}\rho\left(\frac{\log x}{\log p} - 1\right) \\
&= \left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right) \int_{L^{1/2}}^{L} xt^{-2}\rho\left(\frac{\log x}{\log t} - 1\right) d\pi(t) \qquad (2.2) \\
&= \left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right) \int_{L^{1/2}}^{L} \frac{x}{t^2 \log t}\rho\left(\frac{\log x}{\log t} - 1\right) dt.
\end{aligned}$$

Using (1.7) the last integral becomes

$$\begin{aligned}
&\int_{L^{1/2}}^{L} \frac{-x\log x}{t^2\log^2 t}\rho'\left(\frac{\log x}{\log t}\right) dt = \int_{L^{1/2}}^{L} \frac{x}{t} d\rho\left(\frac{\log x}{\log t}\right) \\
&= \frac{x}{L}\rho\left(\frac{\log x}{\log L}\right) - \frac{x}{L^{1/2}}\rho\left(\frac{\log x}{\log L^{1/2}}\right) + \int_{L^{1/2}}^{L} \frac{x}{t^2}\rho\left(\frac{\log x}{\log t}\right) dt.
\end{aligned} \tag{2.3}$$

Using (1.8) we see that the right-hand side of (2.3) is equal to

$$x\delta(x)+O(xL^{-3/2+\varepsilon})$$

for any $\varepsilon>0$, hence Theorem 1 follows then from (2.2) and (2.3).

§3. Proof of Theorem 2

A positive integer m is called square-full if $p^2|m$ for every prime $p|m$. Let $s(n)$ denote the largest square-full divisor of n. Then we have

$$\Omega(n)-\omega(n)=\Omega(s(n))-\omega(s(n)). \tag{3.1}$$

To estimate the sum in (1.11) we first show that those n with $s(n)>\log^3 x$ or with $P^2(n)|n$ contribute only $o(x\delta(x))$ to the sum. To do this first note that, for $n\leqslant x$,

$$\Omega(n)-\omega(n)\leqslant\frac{\log x}{\log 2}-1,$$

hence

$$\sum_{n\leqslant x,\,P^2(n)|n}(\Omega(n)-\omega(n))/P(n)\ll T_1(x)\log x=xL^{-2+o(1)}, \tag{3.2}$$

where we used (1.4) with $r=1$.

Next note that there are $\sim Cx^{1/2}$ square-full integers not exceeding x. Therefore using partial summation we obtain

$$\begin{aligned}\sum_{n\leqslant x,\,s(n)\geqslant L^3}(\Omega(n)-\omega(n))/P(n)&\ll\log x\sum_{n\leqslant x,\,s(n)\geqslant L^3}1\\ &=\log x\sum_{s\geqslant L^3}{}'\sum_{n\leqslant x,\,s(n)=s}1\leqslant x\log x\sum_{s\geqslant L^3}{}'1/s\\ &=xL^{-3/2+o(1)}=o(x\delta(x)),\end{aligned} \tag{3.3}$$

where $\sum'$ denotes a sum over square-full integers.

To estimate the sum in (1.11) for $s(n)<L^3$, we need the following

LEMMA. *Uniformly for $x\geqslant 3$ and $1\leqslant s\leqslant L^3$, we have*

$$\delta(x/s)\leqslant(1+o(1))\,s^{2(\log\log x/\log x)^{1/2}}\delta(x).$$

Proof. We use the following result, which is Lemma $1-(v)$ in Hildebrand [8]:

$$-\rho'(u)/\rho(u)\leqslant\log(u\log^2 u)\qquad(u\geqslant e^4).$$

Integrating this inequality over $[u-\lambda,u]$ we obtain, for $0\leqslant\lambda\leqslant u-e^4$,

$$\rho(u-\lambda)/\rho(u)\leqslant(u\log^2 u)^{\lambda}.$$

Thus for $1 \leqslant s \leqslant L^3$ and for large x we have

$$\delta(x/s) = \int_2^{x/s} t^{-2} \rho\left(\frac{\log x}{\log t} - \frac{\log s}{\log t}\right) dt$$

$$= (1+o(1)) \int_{L^{1/2}}^{L} t^{-2} \rho\left(\frac{\log x}{\log t}\right) \rho\left(\frac{\log x}{\log t} - \frac{\log s}{\log t}\right) / \rho\left(\frac{\log x}{\log t}\right) dt$$

$$\leqslant (1+o(1)) \int_{L^{1/2}}^{L} t^{-2} \rho\left(\frac{\log x}{\log t}\right) \left(\frac{\log x}{\log t} \log^2\left(\frac{\log x}{\log t}\right)\right)^{\log s/\log t} dt$$

$$\leqslant (1+o(1)) \left(\frac{\log x}{\log L^{1/2}} \log^2\left(\frac{\log x}{\log L^{1/2}}\right)\right)^{\log s/\log L^{1/2}} \delta(x)$$

$$\leqslant (1+o(1)) (\log x)^{\log s/\log L^{1/2}} \delta(x) = (1+o(1))\, s^{2(\log\log x/\log x)^{1/2}} \delta(x),$$

which establishes the lemma.

By Theorem 1 and the Lemma, we have

$$\begin{aligned} \sum_{n \leqslant x,\ \log^3 x < s(n) < L^3} (\Omega(n) - \omega(n)) &\ll \log x \sum_{n \leqslant x,\ \log^3 x < s(n) < L^3} 1/P(n) \\ &\leqslant \log x \sum_{\log^3 x < s < L^3}{}' \sum_{n \leqslant x,\ s \mid n} 1/P(n) \\ \leqslant \log x \sum_{\log^3 x < s < L^3}{}' \sum_{m \leqslant x/s} 1/P(m) &\ll \log x \sum_{\log^3 x < s < L^3}{}' (x/s)\,\delta(x/s) \\ \ll x\delta(x) \log x \sum_{s > \log^3 x}{}' s^{-1+2(\log\log x/\log x)^{1/2}} &\ll x\delta(x) \log^{-1/2} x. \end{aligned} \tag{3.4}$$

Therefore to show (1.11) it will be sufficient to restrict the sum to those $n \leqslant x$ for which $P^2(n) \nmid n$ and $s(n) \leqslant \log^3 x$. Also, as in Section 2, we may assume $L^{1/2} \leqslant P(n) \leqslant L$. If $\sum''$ denotes a sum over integers k with the restrictions $P^2(k) \nmid k$ and $P(k) \geqslant L^{1/2}$, then by (3.1) we have

$$\begin{aligned} \sum_{n \leqslant x,\ s(n) \leqslant \log^3 x}{}'' (\Omega(n) - \omega(n))/P(n) &= \sum_{s \leqslant \log^3 x}{}' \sum_{n \leqslant x,\ s(n) = s}{}'' (\Omega(n) - \omega(n))/P(n) \\ &= \sum_{s \leqslant \log^3 x}{}' (\Omega(s) - \omega(s)) \sum_{m \leqslant x/s,\ (m,s)=1}{}'' \mu^2(m)/P(m) \\ &= \sum_{s \leqslant \log^3 x}{}' (\Omega(s) - \omega(s)) \dot{\sum_{m \leqslant x/s}}{}'' 1/P(m) \sum_{d^2 \mid m\alpha(s)} \mu(d) \\ &= \sum_{s \leqslant \log^3 x}{}' (\Omega(s) - \omega(s)) \sum_d \mu(d) \sum_{\substack{k \leqslant x(d,\alpha(s))/sd^2 \\ P(k) > P(d)}}{}'' 1/P(k). \end{aligned} \tag{3.5}$$

Here $\alpha(s)=\prod_{p|s} p$ and the last equality follows from the fact that $d^2|m\alpha(s)$ if and only if m is of the form $kd^2/(d, \alpha(s))$ (and so $P(m)=P(k)$ since $P^2(m)\nmid m$, $P(m)\geqslant L^{1/2}$ and $s\leqslant \log^3 x$).

The last and innermost sum in (3.5) is majorized by x/d^2, so that the contribution to (3.5) by those $d>L^{3/2}/\log^3 x$ is $O(xL^{-3/2+o(1)})$ and is thus negligible. For $\log^3 x\leqslant d\leqslant L^{3/2}/\log^3 x$ and $s\leqslant \log^3 x$ we have

$$\log^6 x\leqslant sd^2/(d, \alpha(s))\leqslant L^3,$$

so that from the proof of (3.4), the last sum in (3.5) for such a value of d is

$$O(x\delta(x)\, d^{-2+2(\log\log x/\log x)^{1/2}}).$$

Thus the contribution to (3.5) from these values of d is $\ll x\delta(x)\log^{-1/2} x$, which is also negligible.

Hence we may restrict attention in (3.5) to those d with $d<\log^3 x$. For such d's we have

$$sd^2/(d, \alpha(s))<\log^9 x,$$

and so by Theorem 1 and (3.2) the last sum in (3.5) for these d's is uniformly

$$(1+o(1))\frac{x(d, \alpha(s))}{sd^2}\delta\left(\frac{x(d, \alpha(s))}{sd^2}\right). \tag{3.6}$$

Note that from the definition of $\delta(x)$ it is possible to show that $\delta(x)$ is decreasing for $x\geqslant x_0$, so that using the Lemma we have $\delta(x/t)=(1+o(1))\delta(x)$ uniformly for $1\leqslant t\leqslant \log^9 x$. In particular, this remark implies that (1.10) holds. Thus (3.6) is

$$(1+o(1))\frac{x(d, \alpha(s))}{sd^2}\delta(x).$$

Putting this estimate in (3.5) we have

$$\sum_{n\leqslant x,\, s(n)\leqslant \log^3 x}'' (\Omega(n)-\omega(n))/P(n)=$$

$$=(1+o(1))x\delta(x)\sum_{s\leqslant \log^3 x}' (\Omega(s)-\omega(s))\sum_{d<\log^3 x}\frac{(d, a(s))\mu(d)}{sd^2}$$

$$=(1+o(1))\frac{6}{\pi^2}x\delta(x)\sum_{s\leqslant \log^3 x}'\frac{\Omega(s)-\omega(s)}{s}\prod_{p|s}\frac{p}{p+1}$$

$$=(1+o(1))\frac{6}{\pi^2}x\delta(x)\sum_{s=1}^{\infty}{}'\frac{\Omega(s)-\omega(s)}{s}\prod_{p|s}\frac{p}{p+1},$$

since by multiplicativity

$$\sum_{d=1}^{\infty} \mu(d)(d, \alpha(s)) d^{-2} = \prod_{p} (1+\mu(p)(p, \alpha(s)) p^{-2})$$

$$= \prod_{p \mid s}\left(1-\frac{1}{p^2}\right) \prod_{p \mid s}\left(1-\frac{p}{p^2}\right) = \frac{1}{\zeta(2)} \prod_{p \mid s}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1} = \frac{6}{\pi^2} \prod_{p \mid s} \frac{p}{p+1}.$$

Thus from this calculation and (3.2), (3.3) and (3.4) we obtain (1.11) with the constant

$$c = \frac{6}{\pi^2} \sum_{s=1}^{\infty}{}' \frac{\Omega(s)-\omega(s)}{s} \prod_{p \mid s} \frac{p}{p+1} > 0.$$

A more careful analysis shows that $\sim$ in (1.11) can be replaced by $1+O((\log\log x)^{3/2} \log^{-1/2} x)$, since for $1 \leqslant t \leqslant \log^{12} x$ we obtain by following the proof of the Lemma

$$\delta(x/t) = (1+O((\log\log x)^{3/2} \log^{-1/2} x)) \delta(x).$$

We now turn our attention to the proof of (1.12). From the proof of (1.1) or from the proof of Theorem 1 it may be seen that only the values of $n \leqslant x$ for which

$$L^{\sqrt{2}/2-\varepsilon} < P(n) < L^{\sqrt{2}/2+\varepsilon} \tag{3.7}$$

for any fixed $\varepsilon > 0$ make a non-negligible contribution to the sums in (1.12).

We next note that if $n \leqslant x/L^2$, then

$$\sum_{n \leqslant x/L^2} \omega(n)/P(n) \leqslant \sum_{n \leqslant x/L^2} \Omega(n)/P(n) \ll xL^{-2} \log x = o(x\delta(x)),$$

so we may assume that

$$x/L^2 < n < x. \tag{3.8}$$

Thus combining (3.7), (3.8) and using the trivial inequality $n \leqslant (P(n))^{\Omega(n)}$, we obtain

$$\Omega(n) \geqslant (\sqrt{2}-2\varepsilon)(\log x/\log\log x)^{1/2},$$

and so for $x \geqslant x_0(\varepsilon)$ it follows that

$$\sum_{n \leqslant x} \Omega(n)/P(n) \geqslant (\sqrt{2}-3\varepsilon)(\log x/\log\log x)^{1/2} \sum_{n \leqslant x} 1/P(n). \tag{3.9}$$

To estimate the sum of $\omega(n)/P(n)$ from above we use the classical elementary inequality of G. H. Hardy and S. Ramanujan ([12], p. 265):

$$\sum_{n \leqslant x, \omega(n)=k} 1 \leqslant \frac{Ex(\log\log x+F)^k}{k! \log x}, \tag{3.10}$$

where E, $F>0$ are absolute constants. Using (3.10) we obtain that the number of $n \leqslant x$ with

$$\omega(n) > (\sqrt{2}+5\varepsilon)(\log x/\log\log x)^{1/2} \tag{3.11}$$

is at most $xL^{-\sqrt{2}/2-2\varepsilon}$. Thus the sum of $\omega(n)/P(n)$ for $n \leqslant x$ and such that (3.7) and (3.11) both hold is at most

$$x \log x L^{-\sqrt{2}-\varepsilon} = o(x\delta(x)).$$

On the other hand, the sum of $\omega(n)/P(n)$ for $n \leqslant x$ and (3.11) failing is clearly at most

$$(\sqrt{2}+5\varepsilon)(\log x/\log\log x)^{1/2} \sum_{n \leqslant x} 1/P(n). \tag{3.12}$$

Combining (3.9) and (3.12) with Theorem 1 and (1.11), we have (1.12), completing the proof of Theorem 2. We finally remark that we can prove (1.12) without using the Hildebrand-Maier result on $\psi(x, y)$. However this result seems to be essential for the proof of (1.11).

§4. Proof of Theorem 3

We denote by $\sum(x)$ the sum in (1.13) and proceed first to derive the lower bound of the correct order of magnitude. In what follows p will always denote primes and p_r will denote the r-th prime. Let A be a large positive integer, and consider integers $m \leqslant x$ such that $\omega(m)=k$ and $P(m) \leqslant p_{(A+1)k}$, where $k=k(x)$ is an integer which will be suitably determined later. If

$$m = p_{i_1}^{a_1} \cdots p_{i_k}^{a_k} \tag{4.1}$$

is the canonical decomposition of m, then there are $\binom{(A+1)k}{k}$ ways we can choose $p_{i_1} \ldots p_{i_k} = \alpha(m)$. Once $\alpha(m)$ is fixed, we can choose the exponents $a_1, \ldots, a_k$ in (4.1) by considering positive integer solutions of

$$a_1 \log p_{i_1} + \ldots + a_k \log p_{i_k} \leqslant \log x.$$

Note that the number of positive, integer solutions of the above inequality is certainly not less than the corresponding number of solutions of

$$a_1 \log p_{(A+1)k} + \ldots + a_k \log p_{(A+1)k} \leqslant \log x,$$

which is $\binom{v}{k}$, where $v = [\log x/\log p_{(A+1)k}]$. Therefore

$$\sum(x) \geqslant \sum_{m \leqslant x} (P(m))^{-k} \geqslant \binom{v}{k}\binom{Ak+k}{k} p_{(A+1)k}^{-k}. \tag{4.2}$$

To evaluate binomial coefficients which appear in the last inequality we shall use Stirling's formula in the form

$$n! = \exp(n\log n - n + \log\sqrt{2\pi n} + o(1)). \qquad (n\to\infty) \tag{4.3}$$

When $k, v\to\infty$ and $k = o(v)$, (4.3) gives

$$\log\binom{Ak+k}{k} = k(A+1)\log k(A+1) - k(A+1) - kA\log(kA) + kA - {} \\ - k\log k + k + O(\log k) \\ = k(A+1)\log(A+1) - kA\log A + O(\log\log x),$$

$$\log\binom{v}{k} = v\log v - k\log k - (v-k)\log(v-k) + O(\log v) \\ = k\log v - k\log k + k + o(k) + O(\log\log x).$$

From the prime number theorem we have

$$p_r = r(\log r + O(\log\log r)), \qquad \log p_r = \log r + \log\log r + O(\log\log r/\log r),$$
$$\log\log p_r = \log\log r + O(\log\log r/\log r).$$

Hence from (4.2) we obtain

$$\begin{aligned}\sum(x) \geq \exp(O(\log_2 x))\exp\{&k\log_2 x - k\log k + k + o(k) - k\log_2 p_{(A+1)k} + \\ &+ k((A+1)\log(A+1) - A\log A) - k\log p_{(A+1)k}\} = \\ =\exp\{&k\log_2 x - 2k\log k - 2k\log_2 k + 2k + o(k) + \\ &+ O(k/A) + O(k\log A/\log k) + O(\log_2 x)\},\end{aligned} \tag{4.4}$$

where the o and O-notation is uniform in A.

Suppose now that $\varepsilon > 0$ is given. Then (4.4) implies, for $k \geq k_0(\varepsilon)$,

$$\sum(x) \geq \exp(O(\log_2 x))\exp\{k(\log\log x - {} \\ - 2\log k - 2\log\log k + 2k + R(k, A))\},$$

where for some $B > 0$

$$|R(k, A)| < \varepsilon/3 + B/A + B\log A/\log k.$$

At this point we choose $A = [3B\varepsilon^{-1} + 1]$. Then for $k \geq k_1(\varepsilon)$ we have $B\log A/\log k < \varepsilon/3$, hence for $k \geq \max(k_0, k_1)$

$$\sum(x) \geq \exp(O(\log_2 x))\exp(f(k)), \tag{4.5}$$

where

$$f(k) = k\log_2 x - 2k\log k - 2k\log_2 k + (2-\varepsilon)k, \tag{4.6}$$

so that

$$f'(k) = \log_2 x - 2\log k - 2\log_2 k - 2/\log k - \varepsilon,$$

and $f''(k)<0$ for k sufficiently large. This means that $f(k)$ attains its maximum for $k=k_2$, where $k_2=k_2(x)$ is the solution of $f'(k)=0$, which yields

$$\log x = k_2^2 \log^2 k_2 \exp(\varepsilon + 2/\log k_2),\ \log x = \left(\frac{1}{4}+o(1)\right) k_2^2 (\log\log x)^2,$$

hence

$$k_2 = k_2(x) = (2+o(1)) \log^{1/2} x (\log\log x)^{-1}, \qquad (x \to \infty). \qquad (4.7)$$

Taking $k=k_3=[k_2(x)]$ we see that $k=o(v)$ holds (this is needed in the evaluation of $\binom{v}{k}$), and therefore from (4.5) and (4.6) we obtain

$$\begin{aligned} \sum(x) &\geqslant \exp(O(\log_2 x)) \exp((2+o(1))k_3) \geqslant \\ &\geqslant \exp((4-\varepsilon_1)\log^{1/2} x (\log\log x)^{-1}), \end{aligned}$$

for $x \geqslant x_1(\varepsilon_1)$, $\varepsilon_1 = \varepsilon_1(\varepsilon)$ and $\lim\limits_{\varepsilon \to 0} \varepsilon_1 = 0$. Therefore we have proved the lower bound for $\sum(x)$.

In proving the upper bound for $\sum(x)$ we shall make use of

$$\begin{gathered} \psi(x, p_t) = \exp(t \log_2 x - t \log t - t \log_2 t + t + o(t)). \\ (1 \ll t \leqslant \log^{1-\varepsilon} x) \end{gathered} \qquad (4.8)$$

Actually we need only the upper bound implied by (4.8), but the lower bound follows from a simple combinatorial argument (e.g. see [1] or [5]) which gives

$$\psi(x, p_t) \geqslant \binom{t+[\log x/\log t]}{t}, \qquad (4.9)$$

and then evaluating the binomial coefficient by (4.3) we arrive at the lower bound implied by (4.8). The upper bound could be also obtained from known results on $\psi(x,y)$, but it seems more appropriate to proceed directly. Note that $\psi(x,p_t)$ represents the number of lattice points $(a_1, \ldots, a_t) \in (N \cup \{0\})^t$ such that

$$a_1 \log p_1 + \ldots + a_t \log p_t \leqslant \log x.$$

Each such lattice point lies in a "lower left corner" of a unit hypercube. If $(w_1, \ldots, w_t) \in (Re^+)^t$ is in one of these hypercubes, then

$$\sum_{i \leqslant t} w_i \log p_i \leqslant \log x + \sum_{i \leqslant t} \log p_i \leqslant \log x + (1+\varepsilon) p_t,$$

by the prime number theorem. Thus $\psi(x,p_t)$ does not exceed the t-dimensional volume of

$$\{(w_1, \ldots, w_t) \in (Re^+)^t : \sum_{i \leqslant t} w_i \log p_i \leqslant \log x + (1+\varepsilon) p_t\},$$

and consequently

$$\psi(x, p_t) \leqslant \prod_{i \leqslant t} \frac{\log x + (1+\varepsilon) p_t}{t! \log p_i} =$$
$$= \exp(-t \log t + t + o(t) + t \log(\log x + (1+\varepsilon) p_t) - \sum_{i \leqslant t} \log\log p_i)$$
$$= \exp(t \log\log x - t \log t - t \log\log t + t + o(t) + O(tp_t/\log x)).$$

By hypothesis $p_t \leqslant \log^{1-\varepsilon} x$, hence $O(tp_t/\log x) = o(t)$ and (4.8) follows.

Having (4.8) at our disposal we may obtain the upper bound for $\sum(x)$ as follows. Let $\omega(n) = t$, $P(n) = p$ for n counted by $\sum(x)$. Then obviously $p \geqslant p_t$, and moreover for a fixed p there are $\binom{\pi(p)-1}{t-1}$ choices for the remaining $t-1$ prime factors of n. Once the t prime factors of n are known, there are at most $\psi(x, p_t)$ numbers with those prime factors counted by $\sum(x)$, giving

$$\sum(x) \leqslant \sum_{t \leqslant 2 \log x/\log_2 x} \psi(x, p_t) \sum_{p_t \leqslant p \leqslant x} p^{-t} \binom{\pi(p)-1}{t-1}, \tag{4.10}$$

since $t = \omega(n) \leqslant 2 \log n/\log_2 n \leqslant 2 \log x/\log_2 x$. To estimate the inner sum in (4.10) we use the prime number theorem to obtain

$$\begin{aligned} \sum_{p_t \leqslant p \leqslant x} p^{-t} \binom{\pi(p)-1}{t-1} &\leqslant \sum_{p_t \leqslant p \leqslant x} p^{-t} ((1+\varepsilon) p/\log p)^{t-1}/(t-1)! \\ &\leqslant \frac{(1+\varepsilon)^{t-1}}{(t-1)!} \sum_{p_t \leqslant n \leqslant x} n^{-1} \log^{1-t} n \ll (1+\varepsilon)^t (t! \log^{t-2} t)^{-1}. \end{aligned} \tag{4.11}$$

To be able to use (4.8) we restrict t in (4.10) to the range $t_0 \leqslant t \leqslant \log^{1-\varepsilon} x$. The contribution of t's for which $t < t_0$ is seen to be negligible by using the trivial bound

$$\psi(x, y) < \left(\frac{\log x}{\log 2} + 1\right)^{\pi(y)},$$

while for the range $t > \log^{1-\varepsilon} x$ we may use the estimates of N. G. de Bruijn [1] or the elementary estimate

$$\psi(x, y) < \binom{\pi(y) + u}{u}^{1+\varepsilon}, \qquad u = [\log x/\log y]$$

of [5] and the trivial inequality $\binom{n}{k} \leqslant n^k/k!$. For the range $t_0 \leqslant t \leqslant \log^{1-\varepsilon} x$ in (4.10) we use (4.8) to obtain a contribution which is

$$\begin{aligned} &\ll \sum_{t_0 \leqslant t \leqslant \log^{1-\varepsilon} x} \psi(x, p_t) \sum_{p_t \leqslant p \leqslant x} p^{-t} \binom{\pi(p)-1}{t-1} \\ &\ll \log x \max_{t_0 \leqslant t \leqslant \log^{1-\varepsilon} x} \exp(t \log_2 x - t \log t - 2t \log_2 t + t + \\ &\quad + o(t) - \log t!) \leqslant \log x \max_{t_0 \leqslant t \leqslant \log^{1-\varepsilon} x} \exp(g(t)), \end{aligned} \tag{4.12}$$

where

$$g(t) = t \log\log x - 2t \log t - 2t \log\log t + (2+\varepsilon)t. \tag{4.13}$$

The function $g(t)$ differs from $f(t)$ (as defined by (4.6)) only that it has ε in place of $-\varepsilon$, and its maximal value is determined analogously by solving the equation $g'(t)=0$. This gives the value

$$t = t_1(x) = (2+o(1)) \log^{1/2} x (\log\log x)^{-1}, \qquad (x \to \infty)$$

thus completing the proof of (1.13), since with the above value (4.12) gives

$$\sum(x) \leqslant \exp((4+\varepsilon) \log^{1/2} x (\log\log x)^{-1}).$$

The proof of (1.14) is considerably simpler that the proof of (1.13). It is sufficient to prove

$$\sum_{n>x}{}' 1/(P(n))^{\Omega(n)} \ll 1/\log x,$$

where $\sum'$ denotes summation over composite n, since

$$\sum_{p \leqslant x} 1/p^{\Omega(p)} = \sum_{p \leqslant x} 1/p = \log\log x + B + O(1/\log x). \ (B = 0.26419 \ldots) \tag{4.14}$$

Write

$$S = \sum_{n>x}{}' 1/(P(n))^{\Omega(n)} = S_1 + S_2,$$

say, where in S_1 we have $P(n) \leqslant y$ and in S_2, $P(n) > y$, and $y = y(x)$ will be suitably determined in a moment. Using the trivial

$$(P(n))^{\Omega(n)} \geqslant n$$

and partial summation, we obtain

$$S_1 \leqslant \int_x^\infty t^{-1} d\psi(t, y) \ll x^{-1} \psi(x, y) + \int_x^\infty \psi(t, y) t^{-2} dt. \tag{4.15}$$

From [1] one has, for $y \leqslant x$ and some absolute $C > 0$,

$$\psi(x, y) < x \exp(-C \log x/\log y), \tag{4.16}$$

so that (4.15) gives

$$S_1 \ll \exp(-C\log x/\log y) = 1/\log x,$$

if we choose

$$y = y(x) = \exp(C\log x/\log\log x).$$

To estimate S_2 note that the number of n with $\Omega(n) = k$ and $P(n) = p$ is at most $\pi^{k-1}(p)$. If $\Omega(n) = 2$ and $n > x$, then $P(n) > x^{1/2}$. From these elementary observations, we have

$$\begin{aligned} S_2 &\leqslant \sum_{n>x,\,\Omega(n)=2} (P(n))^{-\Omega(n)} + \sum_{n>x,\,\Omega(n)>2,\,P(n)>y} (P(n))^{-\Omega(n)} \\ &\leqslant \sum_{p>x^{1/2}} p^{-2}\pi(p) + \sum_{k=3}^{\infty}\sum_{p>y} p^{-k}\pi^{k-1}(p) \\ &= \sum_{p>x^{1/2}} p^{-2}\pi(p) + \sum_{p>y} \pi^2(p)/(p^3 - p^2\pi(p)) \\ &\ll \sum_{p>x^{1/2}} 1/(p\log p) + \sum_{p>y} 1/(p\log^2 p) \\ &\ll 1/\log x + 1/\log^2 y \ll 1/\log x, \end{aligned}$$

using elementary estimates on the distribution of primes. Therefore

$$S = S_1 + S_2 \ll 1/\log x,$$

which proves (1.14). We finally remark that the error term $O(1/\log x)$ is best possible since

$$\sum_{n>x}{}' (P(n))^{-\Omega(n)} > \sum_{p>x} p^{-2}\pi(p) \ll 1/\log x.$$

With more work we could replace the term $O(1/\log x)$ in (1.14) with $(C + o(1))/\log x$ for some positive constant C, but we do not undertake this here.

§5. Proof of Theorem 4

We write the summatory function of $1/Q(n)$ as

$$\sum_{n\leqslant x} 1/Q(n) = \sum_{n\leqslant x,\,Q(n)=P(n)} 1/Q(n) + \sum_{n\leqslant x,\,Q(n)=P^k(n),\,k>1} 1/Q(n) + \sum_{n\leqslant x,\,Q(n)\neq P^k(n),\,k\geqslant 1} 1/Q(n) = S_1 + S_2 + S_3,$$

say. By (1.4) with $r = 1$ we have first

$$S_2 \ll x\exp(-(2+o(1))(\log x\log_2 x)^{1/2}) \ll L^{-1/2}\sum_{n\leqslant x} 1/P(n). \quad (5.1)$$

Next by writing

$$\sum_{n\leqslant x,\, Q(n)=P(n)} 1/Q(n) = \sum_{n\leqslant x} 1/P(n) - \sum_{n\leqslant x,\, Q(n)>P(n)} 1/P(n),$$

we see that (1.16) follows from (5.1) and

$$\sum_{n\leqslant x,\, Q(n)\neq P^k(n),\, k\geqslant 1} 1/P(n) \ll L^{-C} \sum_{n\leqslant x} 1/P(n), \tag{5.2}$$

where $C>0$ is some absolute constant. As in the proof of (1.12) we may consider only those n for which (3.7) holds. Since $Q(n)$ is the largest prime power dividing n, we may assume that $q^a = Q(n) > p = P(n)$ for some $a \geqslant 2$. Observe that in estimating the left-hand side of (5.2) we may also assume that $q^a \leqslant L^4$, since

$$\sum_{q^a > L^4,\, a\geqslant 2} q^{-a} \ll L^{-2}.$$

If $0 < C_1 < C_2$ are any two fixed constants, then from (1.8) and (2.1) or from earlier estimates on $\psi(x, y)$ (see [1], [3]) we have, for $L^{C_1} \leqslant y \leqslant L^{C_2}$ and $u = \log x/\log y$,

$$\psi(x, y) = x \exp(-(1+o(1))\, u \log u), \qquad (x\to\infty). \tag{5.3}$$

In view of (1.2) and the remarks above it follows that we are left with $O(\log x)$ sums of the form

$$\sum\nolimits_a = \sum_p{}' \sum_{p<q^a\leqslant L^4} p^{-1} \psi(xp^{-1}q^{-a}, p),$$

where $a \geqslant 2$ is a fixed integer and $\sum_p{}'$ denotes summation over primes p which satisfy (3.7). Using (5.3) to estimate $\psi(xp^{-1}q^{-a}, p)$, we obtain

$$\sum\nolimits_a \ll \sum_p{}' x \exp\{-(2^{-1/2} - 2\varepsilon)(\log x \log_2 x)^{1/2}\}\, p^{-2} \sum_{q^a>p} q^{-a}$$
$$\ll x L^{-(2^{-1/2}-2\varepsilon)} \sum_p{}' p^{-5/2} \ll x L^{-(5\cdot 2^{-3/2} - 5\varepsilon)}.$$

Since $5\cdot 2^{-3/2} = 2^{1/2} + \frac{1}{4} 2^{1/2}$, we obtain in view of (1.1)

$$\sum_{a\geqslant 2} \sum\nolimits_a \ll L^{-C(\varepsilon)} \sum_{n\leqslant x} 1/P(n)$$

for some $C(\varepsilon) > 0$, if ε in (3.7) is sufficiently small. This proves (5.2) and completes the proof of (1.16). Presumably by methods similar to those used to prove Theorem 1 we could improve (1.16) and obtain an asymptotic formula for the sum

$$\sum_{n\leqslant x} (1/P(n) - 1/Q(n)).$$

Finally we remark that by methods similar to those used in the proof of Theorem 4 we may obtain

$$\sum_{2\leqslant n\leqslant x} 1/f(n)=\{1+\exp(-C(\log x\log\log x)^{1/2})\} \sum_{2\leqslant n\leqslant x} 1/P(n). \quad (5.4)$$

Here $C>0$ is an absolute constant and $f(n)$ denotes either $\beta(n)=\sum_{p\mid n} p$ or $B(n)=\sum_{p^{\alpha}\| n} \alpha p$ (see [6], Ch. 6 and [7] for some results concerning these functions). Thus (5.4) and Theorem 1 provide an asymptotic formula for sums of reciprocals of $\beta(n)$ and $B(n)$.

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(Received September 14. 1984)

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