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BOUNDS ON THRESHOLD DIMENSION AND DISJOINT THRESHOLD COVERINGS*

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Abstract. The threshold dimension (threshold covering number) of a graph G is the least number of threshold graphs needed to edgecover the graph G. If tc (n) is the greatest threshold dimension of any graph of n vertices, we show that for some constant A,

$$n - A\sqrt{n}\log n < \text{tc}(n) < n - \sqrt{n} + 1.$$

We establish the same bounds for edge-disjoint coverings of graphs by threshold graphs (threshold partitions). We give an example to show there exist planar graphs on n vertices with a smallest covering of An threshold graphs and a smallest partition of Bn threshold graphs, with $B = 1.5A$. Thus the difference between these two covering numbers can grow linearly in the number of vertices.

Key words. threshold graph, threshold dimension, threshold partition, graph partition

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1. Preliminaries. By a graph $G = (V, E)$ we mean a finite set V of vertices and a collection E of edges: distinct unordered pairs of distinct vertices. A subgraph of a graph G is a subset V' of V together with a subset E' of E that consists only of edges between vertices of V'. An *induced subgraph* of a graph is a subset of the vertices together with all edges of the original graph that connect those vertices. For further notation see [6].

If x is a vertex of a graph G, the *star* of x is the subgraph consisting of x, the edges containing x, and the other vertices contained in those edges. A *stable set* of vertices (also called an independent set) is a set of vertices which induces no edges. A *dominating set* of vertices is one such that every vertex in the graph is connected to at least one of them by an edge. If a single vertex is a dominating set, it is called a *dominating vertex*. To build a *cone* on G means to add a new vertex to V and connect it to all other vertices by edges.

Threshold graphs were introduced in [2], [3], [8]. A graph is a *threshold graph* if it meets one of the following equivalent conditions:

a) It does not have as an induced subgraph a square (C_4), two disconnected edges ($2K_2$) or a path of three consecutive edges (P_4).

b) The vertices can be labelled with integers $l(v)$, and there is an integer constant t (the threshold) such that a set $\{v_1, v_2, \cdots, v_k\}$ of vertices is stable if and only if $l(v_1) + \cdots + l(v_k) < t$.

c) The vertices can be labelled with integers $l(v)$, and there is an integer constant t (these numbers may be different than those in (b)) such that any two vertices x and y are connected by an edge if and only if $l(x) + l(y) \geqq t$.

d) Every induced subgraph of G, including G itself, has at most one nontrivial component (there may be isolated vertices) and this component has a dominating vertex.

Since every edge of G is, taken by itself, a threshold graph, every graph G may be covered by threshold graphs. The smallest number of threshold subgraphs (not necessarily induced subgraphs) of G that cover G is called the *threshold dimension* of G; we will also call it the *threshold covering* number of G and denote it by tc (G). From an applied perspective, tc (G) is the smallest number of semaphores needed to synchronize a system

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of parallel processes definable by the graph G using PV-chunk synchronizing primitives [8]; alternatively, it is the smallest number of 0-1 simultaneous linear inequalities which can replace such a system of linear inequalities represented by G; see [3], [7], or [6, Chap. 10]. For other prior results on tc (G), see [3].

Two subgraphs of G are called *edge-disjoint* (or simply disjoint) if they have no edges in common. Since the covering of a graph G by its edges is a covering by disjoint threshold graphs, it follows that for every graph there is defined a unique integer tp (G), the *disjoint threshold dimension* or *threshold partition* number of G, the smallest number of edge-disjoint threshold graphs that will cover G.

Since every threshold partition is a threshold covering, tp $(G) \geqq$ tc (G). One goal of this paper is to begin exploring the questions, when is tp (G) = tc (G)? How different can they be? For example, for some corresponding results for clique coverings and clique partitions, see [1].

It should be noted that while it is easy to determine if G is a threshold graph (that is, if tc $(G) = 1$), determining tc (G) is in general NP-complete [3]; in fact, it is NP-complete to test if tc $(G) = 3$ [10] or even if tc $(G) = 2$ [4].

LEMMA 1. *If G is a triangle-free graph*, tc (G) = tp (G).

Proof. As observed in [2], if G contains no triangle, every threshold graph contained in G is a star. Suppose G is covered by k stars $S_1, S_2, \cdots, S_k$. Define $S'_1 = S_1$, $S'_2 = S_2 - S_1$, and in general $S'_j = S_j - (S_1 \cup \cdots \cup S_{j-1})$ for $j = 2$ to k. Clearly the various S'_j are disjoint stars and cover G, so tp $(G) \leqq$ tc (G) as required.

2. The size of a required threshold covering. In [3], Chvátal and Hammer raise the issue: how big need tc (G) be? They prove [3, Thm. 3] that if $\alpha(G)$ is the size of the largest stable set in a graph G with n vertices, then tc $(G) \leqq n - \alpha(G)$ with equality holding if G is triangle-free (and in some other cases). They also observe [3, Cor. 3A] that for every positive ε, there is a graph G on n vertices with tc $(G) > (1 - \varepsilon)n$. In fact, the proof of their Corollary 3A shows more than this. We restate it as follows:

THEOREM 1. *There is a constant A such that for large enough n there is a graph G with n vertices and*

$$\text{tp}\,(G) = \text{tc}\,(G) > n - A\sqrt{n}\log(n).$$

Proof. In [5], Erdös shows that for a sufficiently large fixed constant A, there is an integer N such that for $n > N$ there is a graph G on n vertices with no triangle and with no stable set of $A\sqrt{n}\log(n)$ vertices. Thus tp (G) = tc (G), and

$$\alpha(G) < A\sqrt{n}\log(n) \quad \text{and} \quad \text{tc}\,(G) > n - A\sqrt{n}\log(n)$$

as desired.

This shows that there are graphs with relatively large values of tc (G). We now turn to improving the upper bound on tp (G).

THEOREM 2. *Let G be an arbitrary graph on n vertices. Then*

$$\text{tp}\,(G) < n - \sqrt{n} + 1.$$

Proof. Suppose there is a stable set A in G of size $\sqrt{n}$ or larger. Then Theorem 3 of [3] points out that the stars on $V - A$ provide a covering of G by no more than $n - \sqrt{n}$ threshold graphs; Lemma 1 above shows how to make this a threshold partition.

Now by contrast suppose that no stable set in G has as many as $\sqrt{n}$ elements. Pick a vertex z in G; let $x_1, \cdots, x_k$ be a maximal stable set in the star of z; hence $k < \sqrt{n}$. For each x_i, in turn, we construct a graph T_i consisting of all edges starting at x_i together with any triangles including the edge (z, x_i); omit from this any edges included in a previous T_j to keep the T_i's disjoint. (To see that T_i is threshold, use definition (c). Label

x_i with 4; z with 3; any vertex which neighbors z and x_i but no previous x_j, $j < i$, with 2; other points adjoining x_i with 1. Let $t = 5$.)

We have now constructed k edge-disjoint threshold graphs which cover the union of the stars of the $k + 1$ vertices $z, x_1, \cdots, x_k$. Delete the covered edges from G. This eliminates at least $k + 1$ vertices. Since it deletes an edge only when deleting at least one vertex on it, the reduced graph G' cannot have a bigger independent set than G had.

Reduce G' by choosing a new z. At each stage, we eliminate $k + 1$ vertices by covering them with k threshold graphs;

$$k < \sqrt{n} \quad \text{so} \quad \frac{k}{k+1} < \frac{\sqrt{n}}{\sqrt{n}+1}$$

and the total number of graphs needed to cover all n vertices is not greater than

$$\frac{n\sqrt{n}}{\sqrt{n}+1} < n - \sqrt{n} + 1$$

which completes the proof of Theorem 2.

We now let tc (n) denote the largest tc (G) for any G with n vertices; tp (G) is defined similarly. The above results show that

$$n - A\sqrt{n}\log(n) < \mathrm{tc}\,(n) < n - \sqrt{n} + 1$$

and

$$n - A\sqrt{n}\log(n) < \mathrm{tp}\,(n) < n - \sqrt{n} + 1.$$

It remains of interest to tighten these bounds, and to know whether the limits for tc (n) and tp (n) are actually the same. A private communication from János Pach [9] improves the upper bound in each case to $n - \sqrt{n \log n}$ for triangle-free graphs only.

3. The difference between tc (*G*) and tp (*G*). Since the bounds we have established for tc (G) and tp (G) are identical, it is reasonable to ask whether tc (G) and tp (G) are ever very different. Our object in this section is to show that tp (G)–tc (G) can grow proportionally to the number of vertices n in G, even if G is a planar connected graph or a very highly-connected graph of low diameter.

We will make heavy use of a threshold graph H constructed as follows: consider six vertices $x_1, \cdots, x_6$ and connect x_i and x_j if $i + j \leqq 7$. Note that the deletion of the single edge x_2x_3 would make it cease to be threshold since then $x_5x_2x_4x_3$ would be an induced path.

Example 1. Let G_{10} be the graph made by taking two copies of H and identifying the two copies of x_2, x_3, and the edge between them. This graph is shown in Fig. 1; it is planar. Clearly tc $(G_{10}) = 2$, since it is covered by two copies of H. The reader may verify that tp $(G_{10}) = 3$; two graphs in the partition are a copy of H and a path $x_4x_3x_1$. The proof that there is no partition into two threshold graphs hinges on the fact that x_2x_3 would have to be in the same graph as one "wing" x_1x_6; the side of G_{10} lacking x_2x_3 cannot then be covered by one threshold graph.

The reader may also wish to verify that G_{10} is a critical example; deleting an x_1x_6 from G_{10} results in tc = tp = 2, deleting any other edge yields tc = tp = 3.

The graph G_{10} may be used to build various examples in which the difference between tc (G) and tp (G) grows linearly in the number of vertices or edges of G. For example, if G' is the disjoint union of r copies of G_{10}, tp $(G') = 3r$ and tc $(G') = 2r$. This example may be made planar and connected by joining successive copies G_{10} together at the "wingtips" (identify an x_6 of one G_{10} with an x_6 from another). To build more highly

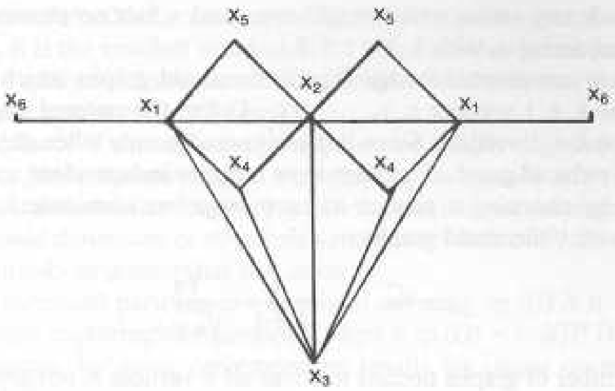

FIG. 1. *The graph* G_{10}.

connected (but nonplanar) examples, we use the following lemma motivated by a discussion with V. Chvátal:

LEMMA 2. *Let G' denote the cone on the (arbitrary) graph G. Then*

$$\mathrm{tc}\,(G') = \mathrm{tc}\,(G) \quad \text{and} \quad \mathrm{tp}\,(G') = \mathrm{tp}\,(G).$$

Proof. Any threshold covering of G' induces a (no larger) threshold covering of G since an induced subgraph of a threshold graph is a threshold graph. Given a (disjoint) threshold cover of G, we obtain a (disjoint) threshold cover of G' by picking any threshold graph D in the cover of G and enlarging it to include the new vertex of G' and its star in G'. That the enlarged D remains a threshold graph is easily seen by definition (d) of threshold graphs; the new vertex of G' is a dominating vertex in the enlarged version of D.

Using this lemma, we can create an arbitrarily highly connected graph with $\mathrm{tc} = 2r$, $\mathrm{tp} = 3r$, by taking G^r and erecting a cone on it as many times as desired (that is, add 5 new points all connected to all original points and each other, to make it 5-connected).

It is now clear that there is a constant c_1 such that a graph G on n vertices can have $\mathrm{tp}\,(G) - \mathrm{tc}\,(G) \geqq c_1 n$. How big can c_1 be? Example G_{10} shows it can be at least $\frac{1}{10}$. What upper bound can be put on $\mathrm{tp}\,(G) - \mathrm{tc}\,(G)$? We know it cannot exceed $n - \sqrt{n} - 1$, but we believe this can be improved. Finally, can $\mathrm{tp}\,(G)/\mathrm{tc}\,(G)$ ever exceed $\frac{3}{2}$? If so, how big can it be?

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