The Vertex Independence Sequence of a Graph Is Not Constrained

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Abstract

We consider a sequence of parameters $a_1, a_2, \cdots, a_m$ associated with a graph G. For example, m can be the maximum number of independent vertices in G and each a_i is then the number of independent sets of order i. Sorting this list into nondecreasing order determines a permutation π on the indices so that $a_{\pi(1)} \le a_{\pi(2)} \le \cdots \le a_{\pi(m)}$. We call a sequence constrained if certain permutations π cannot be realized by any graph. It is well known that the edge independence sequence is constrained to be unimodal. The vertex independence sequence was conjectured to be likewise, but we show that, quite the contrary, it is totally unconstrained. That is, every permutation is realized by some graph.

1. Constrained Sequences.

We wish to study sequences of parameters associated with a specific graph. For example, in the vertex independence sequence $a_1, a_2, \cdots, a_m$, each a_i denotes the number of independent sets of order i of vertices in G and m is the maximum order of any independent set. Similarly, in the edge independence sequence, $b_1, b_2, \cdots, b_m$, each b_i counts the number of ways to select i independent edges. Again m denotes the maximum order, but it is probably a different numerical value than m in the vertex independence sequence. The approach we shall develop to study the vertex independence sequence might well be applied to other graphical sequences, for example, to $c_3, c_4, \cdots, c_m$ where c_i counts the number of i-cycles in G.

The edge independence sequence was shown [1] to be unimodal, that is $b_1 < b_2 < \cdots < b_r \ge b_{r+1} > b_{r+2} > \cdots > b_m$.

H. Wilf asked whether the vertex independence sequence was likewise unimodal. He seemed to be somewhat sceptical of this conjectured unimodality. We shall not only show that the unimodal conjecture is false, but that, unlike the edge sequence, the numbers in the vertex independence sequence are totally unconstrained in the following sense:

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Sort the sequence into nondecreasing order. This defines a permutation π on the indices such that

$$a_{\pi(1)} \leq a_{\pi(2)} \leq a_{\pi(3)} \leq \cdots \leq a_{\pi(m)}.$$

We call the family of sequences being investigated constrained if certain permutations π are never realized by any graph. The family is unconstrained if, for each m, every permutation on m indices can be realized by some graph.

We know the edge sequence must be unimodal. Which permutations π correspond to unimodal sequences? To characterize them, let S_1 be the set $\{1, 2, \cdots, m\}$ and let $S_{i+1} = S_i - \{\pi(i)\}$. Each S_{i+1} is the set of indices remaining after the i smallest terms have been removed. It is not hard to see that the sequence is unimodal if and only if $\pi(i) = \min S_i$ or $\max S_i$

for each i. After all, where can the smallest term in a unimodal sequence be located? It must be either first or last, for otherwise, if j gives the smallest term, we have $a_{j-1} > a_j < a_{j+1}$ and the sequence cannot be unimodal. Having selected one end term to be smallest, we observe that the second smallest must be at either end of the remaining terms. In this way we find that exactly 2^{m-1} of the $m!$ permutations can be associated with unimodal sequences. As already noted, the edge independence sequence is known to be unimodal. Thus, π is constrained to be among the 2^{m-1} permutations that give unimodal sequences. We do not, however, know if all 2^{m-1} inimodal π's can actually be realized. Quite possibly the edge independence sequence is even more constrained than the unimodal property requires. We leave this question for future research. We may now state the main theorem of this article.

Theorem. The vertex independence sequence for graphs is totally unconstrained. That is for each m and for each permutation π on $\{1, 2, \cdots, m\}$, there exists a graph G with vertex independence number equal to m and $a_{\pi(1)} < a_{\pi(2)} < a_{\pi(3)} < \cdots < a_{\pi(m)}$.

Proof. For each permutation π we shall realize π by a graph of the form

$$G = 1K_{n_1} + 2K_{n_2} + 3K_{n_3} + \cdots + mK_{n_m}.$$

That is, G is the join of m subgraphs, and the jth subgraph is j copies of a complete graph K_{n_j}. We must assign the parameters $n_1, n_2, \cdots, n_m$ carefully to assure that G realizes π. How can we select k independent vertices in G? If we try to use vertices from more than one of the m subgraphs, the join operation prevents them from being independent. Thus, all k must come from a single jK_{n_j}. Moreover, if $j < k$, there do not exist k independent vertices. But for $j \geq k$ we select k of the j components, and then choose on vertex from each component to find $\binom{j}{k}n_j^k$ independent k-sets in jK_{n_j}. Summing over $j \geq k$ we have

$$a_k = \sum_{j=k}^{m} \binom{j}{k} n_j^k. \tag{1}$$

We can now explain our strategy to realize π. Let T be a large parameter (to be specified soon). Choose

$$n_k = ((\pi(k) - 1)T) \quad . \tag{2}$$

Of course, n_k must be rounded to the nearest integer, but we have neglected this in our notation because the formulas are already quite involved. When T is large, this round off effect is negligible. We must make one other adjustment. If $\pi(m) = 1$, equation (2) gives $n_m = 0$ and we would have $a_m = 0$. Since we promised to construct a graph with vertex independence number equal to m, in this one case redefine

$$n_m = 1 \quad \text{whenever} \quad \pi(m) = 1. \tag{3}$$

Each n_k has been defined so that the first term in (1) for a_k gives $(\pi(k) - 1)T$. That is, each a_k has a leading term in (1) giving a distinct integral multiple of T. Moreover, these terms fall in the proper order to realize π, provided that the remaining terms do not scramble the order. This will be true if we can show that for each $k < m$,

$$0 \leq \sum_{j=k+1}^{m} \binom{j}{k} n_j^k < T. \tag{4}$$

In other words, the first term $(j = k)$ throws a_k onto the number line at $(\pi(k) - 1)T$ and the remaining terms $(j > k)$ do not total enough to reach the next integral multiple of T. In this way we guarantee π has been realized by the graph G defined by (2) and (3).

If we set T very large, (4) is easy to verify, but the order of G may be extravagently large. As we reduce T, more work is required to verify (4). Although we know it is not the smallest T that works, we have found that $T = m^{2m}$ is roughly the smallest T that leaves the task of verifying (4) manageable. For this value of T, notice that if $\pi(k) > 1$ equation (2) gives $n_k \geq m^2$.

We shall verify (4) first for $k = m$ and then work backwards down the list. For $k = m$, the sum is empty and hence equals 0. For $k = m - 1$, we have a single term.

$$\binom{m}{m-1} n_m^{m-1} = m\, n_m^m / n_m = \frac{m(\pi(m) - 1)T}{n_m} .$$

Now either $\pi(m) = 1$ and the term is 0, or $2 \leq \pi(m) \leq m$ and so $n_m \geq m^2$ and we have
$$\binom{m}{m-1} n_m^{m-1} \leq \frac{m(m-1)}{m^2} T < T.$$

Do not despair. We are not going to continue to verify one a_k at a time. For all $k \leq m - 2$ we may estimate quite crudely:

$$\sum_{j=k+1}^{m} \binom{j}{k} n_j^k = \sum_{j=k+1}^{m} \binom{j}{k} (\pi(j) - 1)T/n_j^{j-k} .$$

Replace $\pi(j) - 1 \leq m$ and $n_j \geq m^2$, so

$$\sum_{j=k+1}^{m} \binom{j}{k} n_j^k \leq \sum_{j=k+1}^{m} \binom{j}{k}\ T/m^{2j-2k-1} .$$

Now the binomial coefficient is at most $\binom{j}{k} = \binom{j}{j-k} < (k + 1)m^{j-k-1}$ since there are $j - k$ factors in the numerator, the smallest equals $k + 1$, and each of the rest is at most m. We have generously suppressed $(j - k)!$ in the denominator. This gives

$$\sum_{j=k+1}^{m} \binom{j}{k} n_j^k \leq T \sum_{j=k+1}^{m} (k + 1)m^{j-k-1}/m^{2j-2k-1}.$$

Finally, letting the upper summation limit of m be replaced by infinity allows us to make the inequality strict, and we evaluate the geometric series to get

$$< T(k+1)/(m-1).$$

Since this case required $k \leq m-2$, we have obtained T as the required bound. We have demonstrated that the graph constructed by (2) and (3) does in fact realize the permutation π.

Examples. Table 1 illustrates our construction for each permutation of length 3, and Table 3 shows the construction for length 4. Noticing that a_1 is just the number of vertices in G, we see that our construction uses as many as 1515 for $m = 3$ and 197456 for $m = 4$. In general, to obtain the sequence with $a_1 > a_2 > a_3 > \cdots > a_m$ we use order

$$a_1 = \sum_{j=1}^{m} j\, n_j = \sum_{j=1}^{m-1} j[(m-j)T]^{1/j} + m$$

where the last term of m results from condition (3). For $T = m^{2m}$ this is on the order of $a_1 \approx m^{2m+1}$.

This is equivalent to saying $m \sim O(\frac{\log a_1}{2 \log \log a_1})$. But this is certainly a much larger value of a_1 than necessary. By examining the permutations one by one, we can produce the parameters in Table 2 requiring no more than 65 vertices. In fact, the use of jK_{n_j} in the construction is a convenience, not a necessity. If we allow the union of j complete graphs of varying sizes we can get even smaller examples. Thus $K_{24} + 2K_3 \cup K_4$ realizes $\pi = 213$ with 34 vertices and $K_{27} + 2K_3 \cup K_4$ realizes 231 with 37 vertices. Similar careful choices for $m = 4$ are shown in Table 4 where every permutation is achieved using at most 302 vertices. However it is quite possible that even smaller graphs might succeed if we liberalize the construction to allow graphs other than joins of unions of complete graphs. Determining the smallest order that is large enough to realize every permutation of order m is likely to remain exceedingly difficult.

It appears that permutations starting with $m - 1$ followed by m require the most vertices. Each independent m-set contains m independent $m - 1$ subsets. Thus a_{m-1} tends to exceed a_m unless a_m is quite large and there is lots of overlap. For mK_m we have $a_{m-1} = a_m = m^m$. We suspect $a_{m-1} < a_m$ requires $a_{m-1} > m^m$. If so, then any permutation of the form $\pi = m - 1 \ m \cdots 1$ must have at least $m^m + m$. This suggests lower bounds of 30 for $m = 3$ and 260 for $m = 4$. Perhaps our best examples are not so far from the truth.

2. Related Problems

It is natural to wonder if the sequence $e_1, e_2, \cdots, e_m$ giving the number of maximal independent sets of vertices of each order behaves similar to the ordinary independence sequence. An independent i-set that is always counted in a_i is also counted in e_i only if it happens to be maximal, that is, no vertex can be added to it to form an independent $(i + 1)$-set. In some contexts, maximal independent sets are harder to analyze because an additional property is involved. However, in the present context it happens that the maximal independence sequence is easier to control. Not only is the maximal independence sequence totally unconstrained, but we can even select $e_1, e_2, \cdots, e_m$ to be any nonnegative sequence and construct a graph having this specified maximal independence sequence. Specifically, we let $G = H_1 + H_2 + \cdots + H_m$ where each subgraph $H_i = (i - 1)K_1 \cup K_{e_i}$. As constructed, each H_i contains precisely e_i independent i-sets, each of which is maximal, and no other H_j contains a maximal independent i-set. This last feature of the graph allows each H_i to be selected to produce e_i without affecting any other terms in the sequence.

Many open problems remain in this area. The first two we list have already been mentioned above:

Problem 1. Determine the smallest order large enough to realize every permutation of order m as the sorted indices of the vertex independence sequence of some graph.

Problem 2. Characterize the permutations realized by the edge independence sequence. In particular, can all 2^{m-1} unimodal permutations be realized?

It is possible that the nature of the vertex independence sequence is totally different for trees or forests.

Problem 3. For trees (or perhaps forests), is the vertex independence sequence unimodal?

At one point we suspected that unimodality of G and H would imply $G \cup H$ is unimodal. If so, the unimodal conjecture for trees would imply the one for forests. This is tempting because it is easy to verify that setting $a_0 = 1$ by convention gives

$$a_k(G \cup H) = \sum_{i=0}^{k} a_i(G)a_{k-i}(H) .$$

But such a convolution of unimodal sequences need not be unimodal. For example, $G = K_{95} + 3K_7$ has the unimodal sequence $a_0 = 1$, $a_1 = 116$, $a_2 = 147$, $a_3 = 343$, whereas $G \cup G$ has the nonunimodal sequence $a_0 = 1$, $a_1 = 232$, $a_2 = 13750$, $a_3 = 34790$, $a_4 = 101185$, $a_5 = 100842$, $a_6 = 117649$. Thus, it would seem that the tree and forest conjectures need to be attacked separately.

In closing, we suggest that permutation constraint offers a new perspective for investigating other sequences associated with graphs.

References

1. Allen J. Schwenk, On unimodal sequences of graphical invariants, J. Combinatorial Theory B. 30(1981), 247-250.

π	n_1	n_2	n_3	a_1	a_2	a_3
123	0	27	11	87	1092	1331
132	0	38	9	103	1687	729
213	729	0	11	762	363	1331
231	1458	0	9	1485	243	729
312	729	38	1	808	1447	1
321	1458	27	1	1515	732	1

Table 1. For $m = 3$ and $T = 3^6 = 729$,
the maximum order $= 1515 = 1 \cdot 2T + 2\sqrt{T} + 3 \cdot 1$

π	n_1	n_2	n_3	a_1	a_2	a_3
1 2 3	0	0	4	12	48	64
1 3 2	1	0	2	7	12	8
2 1 3	37	0	4	49	48	64
2 3 1	53	0	4	65	48	64
3 1 2	0	3	1	9	12	1
3 2 1	1	0	1	4	3	1

Table 2. For $m = 3$, more carefully chosen parameters give maximum order = 65.

π	n_1	n_2	n_3	n_4	a_1	a_2	a_3	a_4
1234	0	256	50	21	746	75682	162044	194481
1243	0	256	58	19	762	77794	222548	130321
1324	0	362	40	21	928	138490	101044	194481
1342	0	443	40	19	1082	203215	91436	130321
1423	0	362	58	16	962	142672	211496	65536
1432	0	443	50	16	1100	205285	141384	65536
2134	65536	0	50	21	65770	10146	162044	194481
2143	65536	0	58	19	65786	12258	222548	130321
2314	131072	0	40	21	131276	7446	101044	194481
2341	196608	0	40	19	196804	6966	91436	130321
2413	131072	0	58	16	131310	11628	211496	65536
2431	196608	0	50	16	196822	9036	141384	65536
3124	65536	362	0	21	66344	133690	37044	194481
3142	65536	443	0	19	66498	198415	27436	130321
3214	131072	256	0	21	131668	68182	37044	194481
3241	196608	256	0	19	197196	67702	27436	130321
3412	131072	443	0	16	132022	197785	16384	65536
3421	196608	362	0	16	197396	132580	16384	65536
4123	65536	362	58	1	66438	141142	195116	1
4132	65536	443	50	1	66576	203755	125004	1
4213	131072	256	58	1	131762	75634	195116	1
4231	196608	256	50	1	197274	73042	125004	1
4312	131072	443	40	1	132082	201055	64004	1
4321	196608	362	40	1	197456	135850	64004	1

Table 3. For $m = 4$ and $T = 4^8 = 65536$, maximum order = 197456

π	Graph	a_1	a_2	a_3	a_4
1234	$3K_4 \cup K_5$	17	108	304	320
1243	$4K_3$	12	54	108	81
1324	$2K_{14} + K_3 \cup K_4 \cup 2K_5$	45	303	295	300
1342	$2K_{22} + 4K_5$	64	634	500	625
1423	$4K_2$	8	24	32	16
1432	$2K_3 + 4K_2$	14	34	32	16
2134	$K_{91} + K_3 \cup K_4 \cup 2K_5$	108	107	295	300
2143	$K_{43} + 4K_3$	55	54	108	81
2314	$K_{279} + K_3 \cup K_4 \cup 2K_5$	296	107	295	300
2341	$K_{284} + K_3 \cup K_4 \cup 2K_5$	301	107	295	300
2413	$K_{70} + 4K_3$	82	54	108	81
2431	$K_{97} + 4K_3$	109	54	108	81
3124	$K_{250} + K_{10} \cup K_{19} + K_3 \cup K_4 \cup 2K_5$	296	297	295	300
3142	$K_{251} + 2K_{14} + K_3 \cup K_4 \cup 2K_5$	296	303	295	300
3214	$K_{250} + K_9 \cup K_{21} + K_3 \cup K_4 \cup 2K_5$	297	296	295	300
3241	$K_{254} + K_9 \cup K_{21} + K_3 \cup K_4 \cup 2K_5$	301	296	295	300
3412	$K_{256} + 2K_{14} + K_3 \cup K_4 \cup 2K_5$	301	303	295	300
3421	$K_{186} + K_2 \cup K_{97} + K_3 \cup K_4 \cup 2K_5$	302	301	295	300
4123	$3K_4 + 4K_1$	16	54	68	1
4132	$K_2 \cup 3K_1$	5	9	7	2
4213	$K_{17} + 4K_2$	25	24	32	16
4231	$K_{25} + 4K_2$	33	24	32	16
4312	$K_1 + 4K_1$	5	6	4	1
4321	$K_3 + 4K_1$	7	6	4	1

Table 4. More careful choices for $m = 4$ give maximum order $= 302$.