

BULLETIN D'INFORMATIQUE APPROFONDIE ET APPLICATIONS

COMPUTATION - INFORMATION

N° 86 - JUIN 2010

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Chamonix. Janvier 1982

par Jean - Pierre Petit, avec l'amabilité de Georges Chappaz

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DIRECTEUR

Jean - Michel Knippel

FONDATEUR

Edmond Bianco

SERVEUR DE PUBLICATION

Christian Blanvillain

SECRETARIAT

Kalassoumi Adjilani

Université de Provence
Equipe Hermès. Case 33
3 place Victor Hugo
F - 13331 Marseille Cedex 3
Téléphone : +33 (0) 413 550 252
Télécopie : +33 (0) 491 509 110

DEPOSITAIRE

Université de Provence
Bibliothèque Universitaire
3 place Victor Hugo
F - 13331 Marseille Cedex 3
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Téléphone : +33 (0) 413 550 626
Télécopie : +33 (0) 413 550 650

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EDITORIAL

Lettre 31. Fraude et déontologie

Michelle Bergadaà

Chers collègues, chers étudiants, chers amis,

Depuis bientôt sept ans, j'accompagne et je conseille les personnes qui choisissent, en toute liberté, de dénoncer des cas de plagiat.

Reconnaissons le courage de ces pairs qui risquent menaces et brimades de la part de certains pour leur non-respect des règles du jeu du «non-dit» académique. Milgram n'est pas loin pour nous rappeler que l'obéissance est l'ennemi du juste et qu'il faut toujours un authentique courage pour être en mesure, parfois, de s'opposer à l'«ordre».

Que deviennent les dénonciateurs de fraude ? En fait, tous sortent renforcés de l'épreuve de confrontation au système. Prenons ici deux exemples.

Ainsi, nous vous informions en 2008 que le Tribunal de Grande Instance de Versailles avait donné raison à Hélène Maurel-Indart contre Bernard Edelman, lequel n'avait pas apprécié le parallèle public entre son livre et la thèse de Laurent Pfister (sur l'histoire du droit d'auteur). Edelman ayant fait appel, cette affaire peut durer longtemps. Mais Hélène Maurel-Indart poursuit avec une ferveur renforcée son travail d'investigation sur le plagiat littéraire. Non seulement elle s'impose comme référence dans les médias (lire par exemple :

« <http://responsable.unige.ch/index.php?main=b-9-1> Les règles du savoir-plagier »), mais son <http://www.leplagiat.net/> site Web « Le plagiat » est de plus en plus visité.

Autre exemple, celui de Jean-Noël Darde qui en avait assez de l'inertie du système sur le plan des thèses de doctorat plagiées. Grâce à :

<http://archeologie-du-copier-coller.blogspot.com/> son blog où il publie en toute transparence son travail d'archéologue du plagiat, il a réussi à faire voter, le 25 mars 2010, l'annulation d'une thèse soutenue en 2008. L'annulation d'une thèse de doctorat est, à notre connaissance, une première en France et nous saluons ici le courage de Paris VIII d'avoir ouvert la voie.

Mais comment faire comprendre aux dirigeants des institutions académiques qui n'auraient pas lu notre

<http://responsable.unige.ch/documents/cultureSocietes2009.pdf> précédente supplique, qu'il leur faut maintenant s'impliquer véritablement ? Ainsi, peut-être :

Je dis devant vous, qu'un « chercheur », Directeur général adjoint d'une Grande École française, éditeur d'une revue scientifique, a traduit mot à mot un chapitre de livre publié chez un éditeur français (chapitre tiré de la thèse de doctorat de son auteur), puis qu'il l'a publié chez McGraw-Hill sous son nom.

Je dis devant vous, que si McGraw-Hill n'avait pas eu la bonne idée de mettre sur le Web ce chapitre pour promouvoir le livre américain, nous n'aurions pu le détecter de manière si éclatante.

Je dis devant vous, que l'auteur plagié ne cherche pas la vengeance ; il voudrait faire son travail de chercheur sans perdre de temps en procédures et sans avoir à se référer à ce texte indélicat dans tous ses futurs écrits.

Je dis devant vous, que l'éditeur français, propriétaire devant la loi de cet écrit original, se serait bien passé d'une telle confrontation avec McGraw-Hill.

Je dis devant vous, que l'établissement d'enseignement supérieur où exerce ce collègue indélicat ne peut pas sanctionner des étudiants plagieurs tant qu'il n'a pas réglé ce cas.

Je dis devant vous, que tous les représentants de la Conférence des Grandes Écoles et tous les Présidents d'universités sont dès lors concernés. Ils doivent maintenant nous rassurer quant à leur volonté de protéger la connaissance, de nous protéger et de protéger nos étudiants.

Merci à vous qui choisissez de lutter contre ce plagiat endémique en vous souvenant d'Aristote pour situer votre courage au juste milieu entre couardise et témérité.

Merci d'avoir compris que le courage, ce n'est pas d'affronter nos pairs, mais c'est de s'élever au-dessus de nous-mêmes pour comprendre, dans l'altérité, la «raison» de cette mutation de nos connaissances et de nos comportements.

Vos témoignages et vos contributions régulières à notre travail collaboratif via le site « <http://responsable.unige.ch/> Responsable » ont permis de développer cette prise de conscience collective. Pour faire circuler cette lettre ou inscrire des collègues, il vous suffit d'utiliser <http://responsable.unige.ch/index.php?main=b-4-1> le lien d'inscription au site.

Bien cordialement

Michelle Bergadaà
Professeure de communication et de marketing
Directrice de l'OVSM
Faculté des SES, Université de Genève

« Square trisection » de Abu'l - Wafa' à Christian Blanvillain

Jean - Michel Knippel

Ce numéro du bulletin, très helvète, commence par un éditorial dénonçant les cas de plagiat dans notre monde universitaire. Je remercie Madame Michelle Bergadaà¹, Professeur à l'Université de Genève, de nous l'avoir offert à la diffusion. Le travail de Christian Blanvillain² exposé dans les pages suivantes est un contre-exemple de plagiat, création scientifique au sein de l'équipe du Professeur János Pach à l'Ecole Polytechnique Fédérale de Lausanne (EPFL). Pour terminer, un dessin de Jean - Pierre Petit³ illustre les journées sur l'éducation scientifique de Chamonix ... à quelques vallées de Lausanne.

" N'est-il pas étonnant que l'on publie, en ce début de XXI^e siècle, un livre sur les problèmes posés par la géométrie du triangle et que de grandes revues de mathématiques y consacrent encore des articles ? Le triangle ressuscité illustre la singulière capacité de renouvellement des mathématiques et la possibilité de trouver, à propos de la plus simple des structures géométriques, des problèmes inconnus ... et difficiles. [...]

Un ouvrage récent " How Does One Cut a Triangle? "⁴ (Comment découper un triangle ?) écrit par Alexander Soifer, de l'Université du Colorado, concentre toute son attention sur le découpage d'un triangle. Ce livre détaille les découvertes récentes sur le sujet : on y trouve des énoncés surprenants de simplicité associés à des solutions étonnantes d'astuce, voire de génie mathématique. Certains problèmes n'étaient d'ailleurs pas résolus en 1990 dans la première édition du livre et le sont dans l'édition de 2009. D'autres sont toujours en suspens." [...]

Extrait d'un texte du Professeur Jean-Paul Delahaye⁵.

¹<http://responsable.unige.ch/index.php?main=b-21-1>

²<http://infoscience.epfl.ch/record/161493?ln=fr>

³<http://www.jp-petit.org/>

⁴<http://books.google.ch/books?id=bFifWgpVKisC>

⁵http://www.pourlascience.fr/ewb_pages/f/fiche-article-non-la-geometrie-du-triangle-n-est-pas-morte-23908.php

Passons sous silence la découpe des pizzas en parts égales⁶, autre sujet qui trouve sa place dans les "Lecture Notes in Computer Science" et venons-en à la géométrie du carré.

La question posée dans l'article qui suit est de savoir comment couper un carré en trois carrés congruents. Autrement dit, comment assembler trois carrés identiques dans un nouveau carré en utilisant un nombre minimal de dissections.

Je passe la plume à Christian Blanvillain qui est le mieux placé pour préciser le contexte de l'étude.

" Une première solution à ce problème a été proposée par Abu'l-Wafa' (940-998) en utilisant une dissection à 9 pièces. Cette dissection a été utilisée par Abu'l-Wafa' comme démonstration géométrique du théorème de Pythagore. Henry Perigal, Edouard Lucas et bien d'autres ont proposé de nouvelles solutions à ce problème, en essayant de diminuer le nombre de pièces de la dissection. La seule dissection en 6 pièces connue était celle de Henry Perigal, trouvée vers 1835. C'est une solution unique et non symétrique. Dans le laboratoire du Professeur János Pach, une nouvelle solution non connexe en 6 pièces a été trouvée qui est en fait une famille infinie de solutions symétriques. "

Christian Blanvillain a su nouer des contacts avec les meilleurs universitaires :

- Greg Frederickson⁷ de l'Université de Purdue à Indianapolis, la référence en problèmes de dissections ;
- János Pach⁸ de l'EPFL, l'un des meilleurs spécialistes mondiaux dans les domaines⁹ de la géométrie combinatoire et de l'algorithmique ;
- Alain Wegmann¹⁰ de l'EPFL, penseur et technologue de la programmation stratégique.

Les Professeurs Greg Frederickson et Alain Wegmann sont les relecteurs de cette "récréation" et j'en suis heureux.

⁶http://www.pourlascience.fr/ewb_pages/f/fiche-article-le-pizzaiolo-mathematicien-24927.php

⁷<http://www.cs.purdue.edu/people/faculty/gnf/>

⁸<http://dcg.epfl.ch/page-18448.html>

⁹<http://actualites.epfl.ch/presseinfo-com?id=537>

¹⁰<http://lams.epfl.ch/people/wegmann>

Nota :

Dans la langue allemande qui m'est chère, "travaux" se traduit par *Arbeiten*, ce qui désigne à la fois le processus (l'infinifitif du verbe *Arbeiten*, "travailler") et l'aboutissement (le pluriel du substantif *Arbeit*, "les travaux"). Nous pouvons passer à la lecture du processus et de l'aboutissement de ces travaux originaux qui auront une suite, je n'en doute pas.



Square Trisection

Dissection of a Square in Three Congruent Partitions

Christian Blanvillain, János Pach

Abstract—A square trisection is a problem of assembling three identical squares from a larger square, using a minimal number of pieces. This paper presents an historical overview of the square trisection problem starting with its origins in the third century. We detail the reasoning behind some of the main known solutions. Finally, we give a new solution and three ruler-and-compass constructions. We conclude with a conjecture of optimality of the proposed solution.

I. INTRODUCTION

Geometric problems are some of the earliest mathematical challenges undertaken by humanity. The most striking problems are often the simplest. The one we will discuss is correlated with the geometric demonstration of the Pythagorean theorem. Paper, pencil, ruler, compass and scissors are sufficient tools to explore the ways of transforming a square into three identical smaller squares. This problem is simple enough to be understood by a child, and mathematicians have studied it for over a thousand years.

As an introduction to the trisection problem, let us first consider a simpler one - how to divide a square to construct four identical smaller squares. The obvious answer is shown in Figure 4. We simply draw lines through the middle of opposing sides. The next problem asks how to divide a square to construct two identical smaller squares. The first solution that comes to mind is to draw the cross formed by the two diagonals (Figure 5). We assemble two identical squares by pairing the pieces. A second solution is to create a smaller square whose vertexes are the midpoints of each side of the big square (Figure 6). The second smaller square is assembled from the four triangles. The first solution is preferable because it uses four pieces instead of five.

The square trisection problem can be summarized in the following way:

Provide a solution for dividing a large square into a minimum number of small polygons that can be reassembled to make three identical smaller squares whose surface area is equal to one third of the large square's surface area.

II. HISTORY

The Gnomon of the Zhou by Liu Hui (around 263 AD) can be considered as the first general geometric proof by dissection of Pythagoras' theorem [1]. Thābit Ibn Qurra' (826 – 901) [2] and Bhāskara Āchārya (1114 – 1185) [5], also provides very famous geometric proofs by dissection. If we could ask to any of them how to solve this problem, they would probably reply "Use my dissection to divide the square into two squares of $1/3$ and $2/3$ of its surface, then simply cut the square of surface $2/3$ in two!". These proofs of the Pythagoras' theorem by dissection give solutions to the square trisection problem, but the solutions aren't minimal.

From the 8th century AD up to the 15th century¹, Muslims dominated the world through their science. The craftsmen who built the mosques used mosaics adorned with geometric patterns of great aesthetic value and sometimes found themselves faced with the complex problem of dissections. One such problem was figuring out how to assemble three identical squares to form a new one, using a minimal number of pieces.

Abū'l-Wafā' Al-Būzajānī (940 – 998), born in Iran, was the most skillful and knowledgeable professional geometry expert of his time. In his treatise "Kitāb fīmā ya ḥtaju ilayhi al-ṣānī' min a'māl al-handasa" (*On the Geometric Constructions Necessary for the Artisan*, chapter *On Dividing and Assembling Squares* [3]), he wrote:

"I was present at a meeting in which a number of geometers and artisans participated. They were asked about the construction of a square from three squares. Geometers easily constructed a line such that the square of it is equal to the three squares but none of the artisans was satisfied. They wanted to divide those squares into pieces from which one square can be assembled. [...] Some of the artisans locate one of these squares in the middle and divide the next one on its diagonal and divide the third square into one isosceles right triangle and two congruent trapezoids and assemble together."

Figure 1 details this reasoning. If we assume that the small central square has unit length, then the diagonal of the big square must be $1 + \sqrt{2}$ which is less than $\sqrt{6}$, the diagonal of a square of area 3. Thus this construction is wrong (see error in bold lines in Figure 7). Abū'l-Wafā' gave the first correct solution to this problem (Figure 2). One of the copies of his treatise can be seen at the "Bibliothèque Nationale Française de Paris".

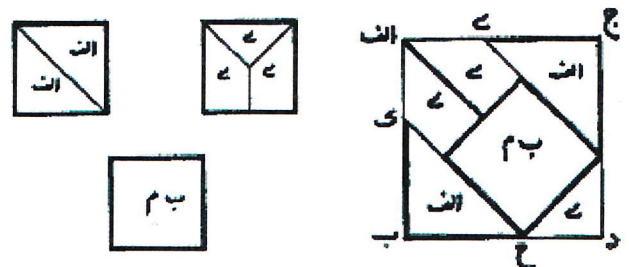


Fig. 1: Before Abū'l-Wafā's solution (credit Reza Sarhangi)

christian.blanvillain@epfl.ch, janos.pach@epfl.ch
Swiss Federal Institute of Technology in Lausanne

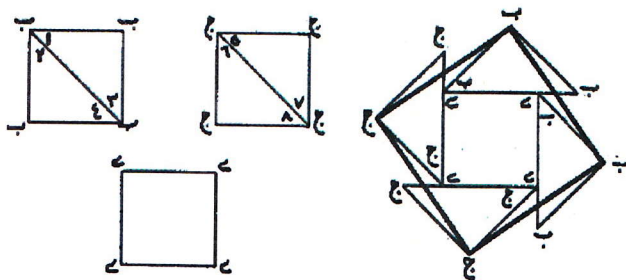


Fig. 2: Abū'l-Wafā' solution (credit Reza Sarhangi)

Abū'l-Wafā' has generalized this dissection for proving the Pythagorean theorem¹ (see Alpay Özdural in [3]). A representation of his generalization can be seen in many mosaics on the largest mosque in Iran, the Jameh Mosque of Isfahan (Figure 3) [4].

Around 1300, a 9-piece solution and a 8-piece solution were found, presumably² by Abū Bakr al-Khalīl al-Tājir al-Rasādī. In reference [6] we can see the details of these solutions; they are also reproduced in Figures 8 and 9.

It was not until the 18th - 19th centuries [7] that mathematicians such as J-É. Montucla, P. Kelland, P. Busschop, De Coatpont (Figure 10), and E. Lucas (Figure 11) revisited this issue. Henry Perigal found the first square trisection solution which uses only 6-pieces (Figure 23), around³ the 1840's, but published his technique only in 1875, and the solution itself in 1891. His solution is similar to one of the two proposed by Abū Bakr al-Khalīl. The Perigal version is asymmetric but, by shifting the diagonal of the cut, has two fewer pieces. It is noteworthy that Philippe Kelland published a very similar technique for dissecting a gnomon (L-shaped part of a square with one corner missing) in 1855 [8].

Henry Perigal rediscovered in 1835 the same dissection as Abū'l-Wafā' for proving Pythagoras' theorem. He considered this proof to be his best work (see [12] *On Geometric Dissections and Transformations* p.103) and later made an engraving of this dissection for his tomb⁴.

During the 20th century, H.E. Dudeney [13] and Sam Loyd's [14] both republished Perigal's 6-piece solution. More recently, Greg N. Frederickson found a 7-piece hinge-able and symmetric solution (Figure 18), and Nobuyuki Yoshigahara found a 9-piece dissection using exactly three time three identical pieces (which is indeed a 7-piece solution) (Figure 12). In recent times, new solution attempts were published in a few mathematics journals [16].

1. *Encyclopedia of the History of Arabic Science* Roshdi Rashed (1996), Vol. 2 Chap. 14, by B.A.Rosenfeld and A.P.Youschkevitch.

2. According to A. Özdural, Abū Bakr al-Khalīl was the assumed author of those dissections presented in the anonymous manuscript "Fitadākhul al-ashkāl al-mutashaābiha aw al-mutawaāfiq" [6]

3. According to the appendix after Rogers (1897) publication [12]

4. Online pictures <http://plus.maths.org/issue16/features/perigal/>

Below is a summary of the most significant solutions for this problem. For each, we provide a figure with demonstrating the solution. The only exception are J-É. Montucla and P. Busschop, who both provide a wonderful procedure for transform any rectangle in a square, but their final results are not very interesting (their solutions looks similar to Edouard Lucas' dissection, with an extra piece).

- 10th Abū'l-Wafā', 9-piece trisection [3]
- 14th Abū Bakr al-Khalīl, 9 & 8-piece trisection [6]
- 1778 Jean-Étienne Montucla, 8-piece trisection [7]
- 1873 Paul Busschop, 8-piece trisection [9]
- 1877 M. de Coatpont, 7-piece trisection [10]
- 1883 Edouard Lucas, 7-piece trisection [11]
- 1891 Henry Perigal, 6-piece trisection [12]
- 2002 Greg N. Frederickson, 7-piece hinge-able trisection [18]
- 2004 Nobuyuki Yoshigahara, 3x3 = 7-piece trisection [17]

For more information on this problem, its history, and on dissections in general, we refer the reader to three excellent books by Greg N. Frederickson [18]. We also recommend the fabulous chronologically arranged list of recreational problems by David Singmaster [15].

III. REVISITING KNOWN SOLUTIONS

It can be frustrating to view the results of a dissection problem without any idea about of how it was found. For example, given Nobuyuki Yoshigahara's 9-piece dissection (Figure 12), it would be difficult to even consider such a dissection before seeing it. One can see that the three small squares are dissected in the same way and might guess that the angles were carefully calculated, but it would be very difficult re-discover this solution even after having seen it.

The purpose of this chapter is to explain our reasoning process used to re-discover three wonderful historical dissections created by Abū'l-Wafā', Greg N. Frederickson and Henry Perigal. Finally, we will present a new original 6-piece non-convex square trisection solution.



Fig. 3: Detail Jameh Mosque of Isfahan (credit Alain Juhel)

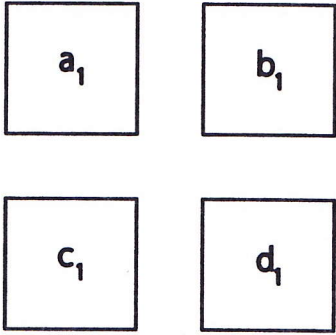
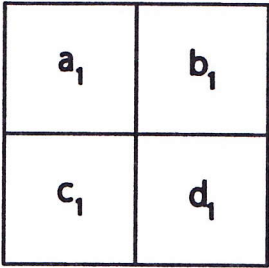


Fig. 4: Split in four

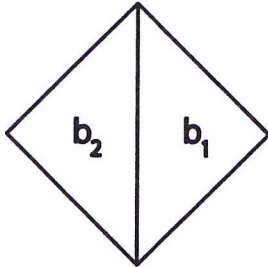
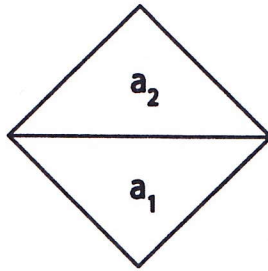
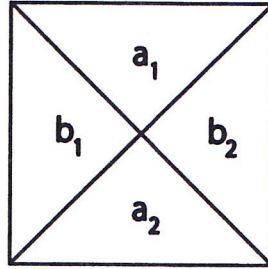


Fig. 5: Split in two
4-piece dissection

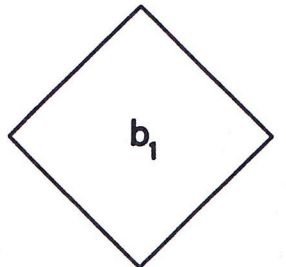
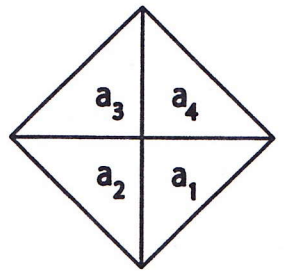
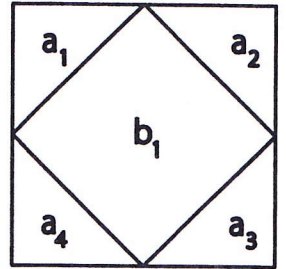


Fig. 6: Split in two
5-piece dissection

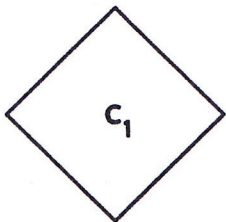
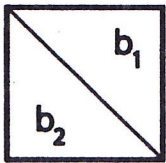
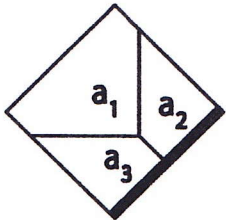
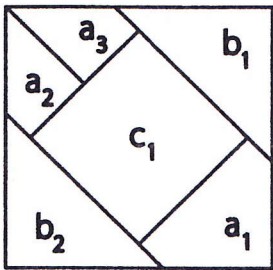


Fig. 7: Split in three: a *wrong* dissection

Bold lines show the error

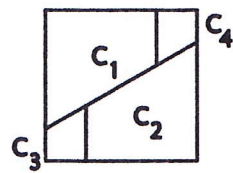
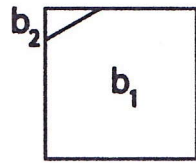
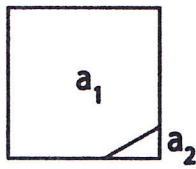
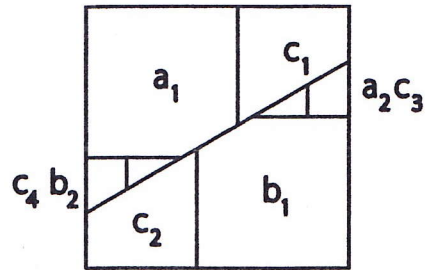


Fig. 8: Abū Bakr al-Khalīl (14th)

8-piece trisection

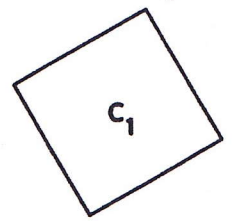
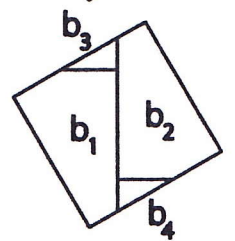
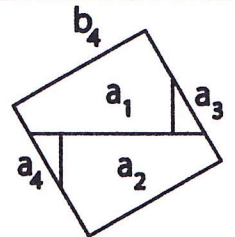
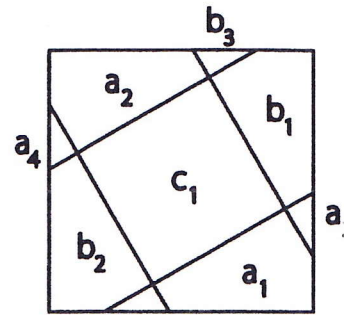


Fig. 9: Abū Bakr al-Khalīl (14th)

9-piece trisection

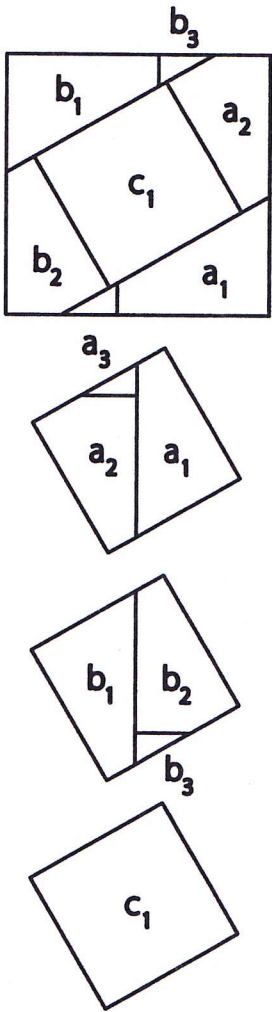


Fig. 10: Colonel De Coatpont (1877)
7-piece trisection

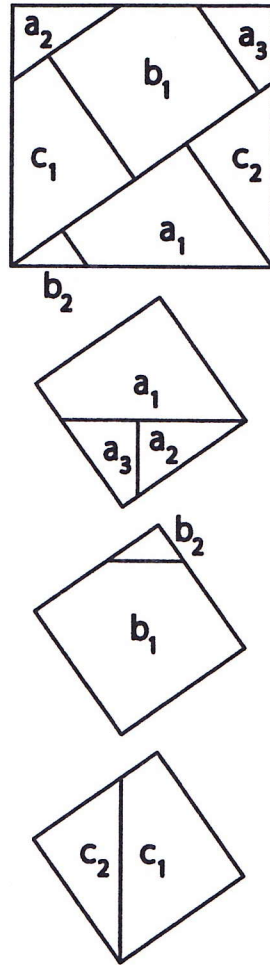


Fig. 11: Edouard Lucas (1883)
7-piece trisection

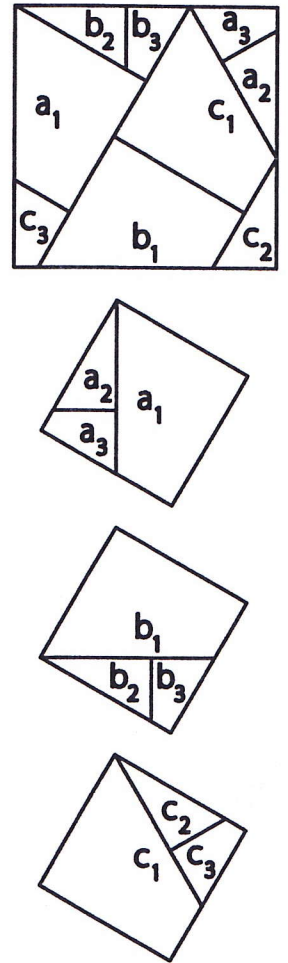


Fig. 12: Nobuyuki Yoshigahara (2003)
3x3 = 7-pieces trisection

Abū'l-Wafā

Our first idea was just try to place a small square at the center of the original square, and then to rotate it until we found a position with useful geometrical properties. We discovered that one specific angle enable us to create one new square of with $2/3$ of the original surface area by joining the remaining pieces. The angle that aligns an edge of the small square with the middle of one sides of the large square (Figure 14) enables the four vertexes of the large square to join together (Figure 15), and the vertexes in the original square become the center of the $2/3$ area square. We can then draw diagonals that cut this new square of area $2/3$ into two pieces. In Figure 16, those diagonals are drawn with dashed lines. Finally, Figure 17 shows the solution proposed by Abū'l-Wafā' one thousand years ago.

Note that Abū'l-Wafā' did not use this trick to solve the problem (see Figure 2). Tessellations were a well-known artistic technique during this time period. In Greg N. Frederickson's first book (see pp. 51-52 in [18]), the autor shows how Abū'l-Wafā' could have discovered dissections of $(a^2 + b^2)$ squares to a large square using tessellations. Thus, we could also use the technique of superimposing tessellations.

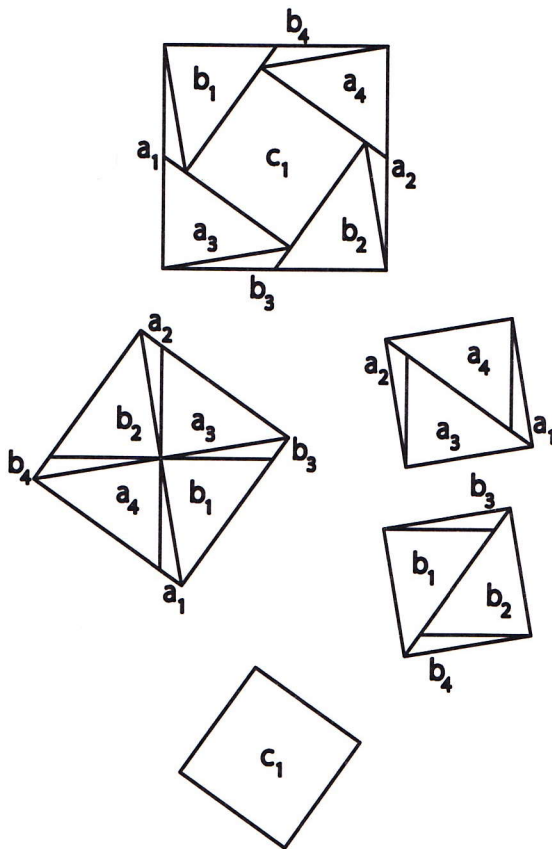


Fig. 13: Abū'l-Wafā' (10th)
9-piece trisection

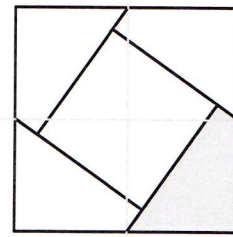


Fig. 14: Idea: This angle cut sides in the middle. AW 1/4

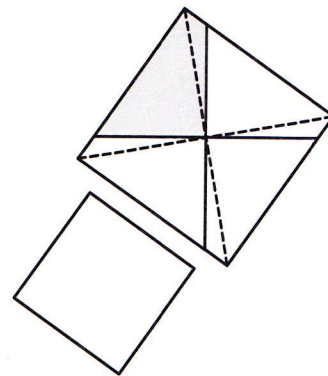


Fig. 15: Idea: Assemble corners to obtained a new $2/3$ square. AW 2/4

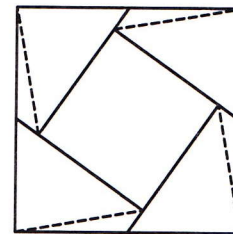


Fig. 16: Dashed lines are diagonals of the $2/3$ square's cut. AW 3/4

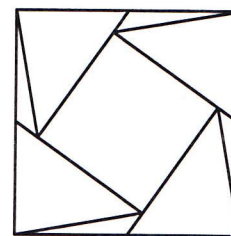


Fig. 17: Abū'l-Wafā' 's solution. AW 4/4

Greg N. Frederickson

The previous solution uses a square with the surface area being $2/3$ of the original square, raising the question of a plausibility of getting the two small squares of with $1/3$ of the original surface area directly. The central square allows us to have two edges of the right length that can be used to make the two smaller squares. Keeping the same angle as Abū'l-Wafā', if we extend the lines from two of the sides of the central square, we go through the midpoints of two sides of the large square (Figure 19). Then both symmetric grey shapes can be completed using the remaining triangles (Figure 20). We cut this triangle to obtain the desired length for the second side of the small square (Figure 21) and the last remaining piece complements the $1/3$ surface area square.

Greg N. Frederickson's dissection (Figure 22) is hinge-able, meaning that if we attach the pieces together with hinges, we can swing the pieces one way to form one figure, and swing them the other way to form a different figure. Note that Greg N. Frederickson did not use this trick to solve the problem. His goal was to discover a hinge-able dissection and he chose a specific method called T-strip that would guarantee a hinge-able dissection. More details are found in Greg N. Frederickson's second book [18].

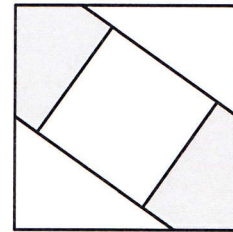


Fig. 19: Idea: Same angle used by Abū'l-Wafā'. GNF 1/4

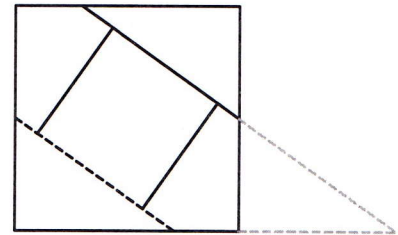


Fig. 20: Slide remaining triangle to complete square. GNF 2/4

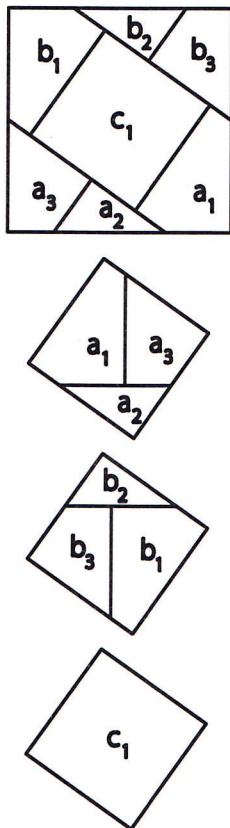


Fig. 18: Greg N. Frederickson (2002)
7-piece trisection

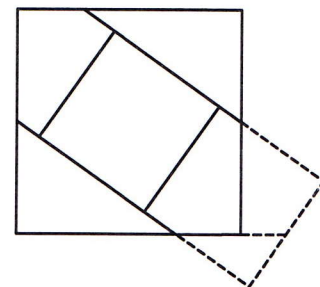


Fig. 21: Cut at right length and apply symmetry. GNF 3/4

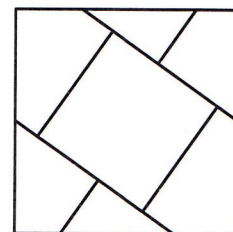


Fig. 22: Greg N. Frederickson's hingeable solution. GNF 4/4

Henry Perigal

Instead of starting with the larger square, now we consider the reverse problem: align three squares, each having $1/3$ of the original surface area, and try to assemble them. We fixed the final desired length on two edges: one top and one bottom. The edges are overlapping (Figure 24) so we cannot cut the two vertical lines that could be used as edges of the final square. Let's us draw only one vertical line with the right length to obtain a final square edge. Let us draw only one vertical line with the right length to obtain a final square edge. We remark that we could cut the block by an oblique line connecting a vertex to the extremity of the vertical one (Figure 25) and then drag the triangle obtained along the cut line until the two segments of the vertical constitute one final edge (Figure 26). To complete the square, it only remains to move the triangle along the same cut line (Figure 27). Figure 28 shows Henry Perigal's dissection.

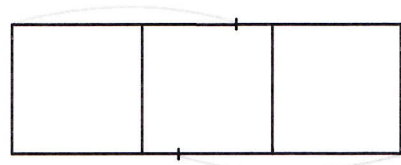


Fig. 24: Idea 1: Reverse process: start with three final squares. *HP 1/5*

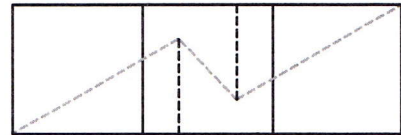


Fig. 25: Idea 2: What we would like to do, but can't. *HP 2/5*



Fig. 26: Idea 3: Do the cut for one side only. *HP 3/5*

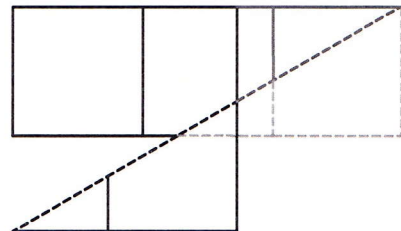


Fig. 27: Triangles will slide. *HP 4/5*

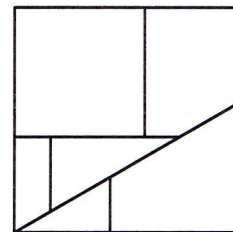


Fig. 28: Henry Perigal's solution. *HP 5/5*

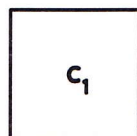
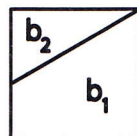
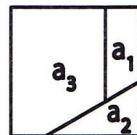
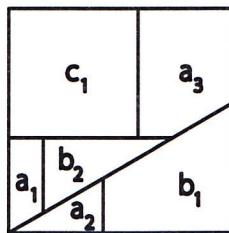


Fig. 23: Henry Perigal (1891)

6-piece trisection

IV. A NEW DISSECTION

This last Perigal's 6-pieces solution is not symmetrical, unlike all the previous solutions. We decided to attempt to find a symmetrical solution using 6 pieces. We begin by examining the wrong dissection in Figure 29, used by the artisans before Abū'l-Wafā' proposed his solution. The problem with that wrong dissection is that the area of the two half squares in the corners is too big. But if, as did the artisans in the 10th century, a square tile is cut using this solution, the error will be approximately twice 1.7% on each side of the middle band; it would be barely noticeable. We posed a problem of slightly reducing the length of the edges of the two half squares.

Our idea was to incline the middle band, without worrying about other pieces, until it reached red lines (we will discuss the final obtained angle later), so that it would be wide enough for the central square to be of the right size (Figure 30). As the whole area taken by one half square by the rotation of the central band, is given to the other half square, this transformation retains the property that interests us: the assembly of the two half square will form a square with exactly $1/3$ of the original area. Next, to make a square with the middle band, we just need to drag the complementary part to the center (Figure 31). That created the second small square. Its area is $1/3$ of the original square, and it does not overlap the center of the large square (Figure 32). By symmetry, we could now construct the third square and finally obtain the new 6-piece symmetrical solution in Figure 33.

The final rotating angle is $\pi/6$, implying that the construction of the dissection can use the trisection of the right angle. However, we found the correct angle was by augmenting the width of the middle band. This led us to find the same construction reasoning only on angles.

Assume the area of the big square is 3. To construct this dissection using only a ruler, the first step is to identify lengths 1 on its edges. Join the corners to these points as shown in Figure 35. The two last points can be obtained by intersecting the lighter segments. Figure 36 shows another construction using only a compass. On the figure, all the arcs have the radius equal to 1, and the two full lighter circles have the radius equal to $\sqrt{3}-1$.

V. FUTURE WORK

The new proposed dissection is not tight: we can slide the position of the both segments which are in the two symmetric parts of the middle band. Since this solution has one not-used degree of freedom, it generates an infinite number of different, but equivalent, dissections. This opens a possibility of finding a better solution. Perhaps there is a 5-piece dissection without any degree of freedom.

We conjecture however that a 5-piece solution can not exist. In the future, we would like to propose an algebraic proof, using the fact that we need to divide irrational lengths using integers. Another interesting approach would be to write an approximation algorithm to test all possible dissections. Unfortunately, the wealth of geometric problems has, so far, eluded the ingenuity of computer scientists, given its infinite number of possibilities. This research area is completely undeveloped, and the arrival of such algorithms would offer great opportunities for finding new unknown polygonal or polyhedral dissections.

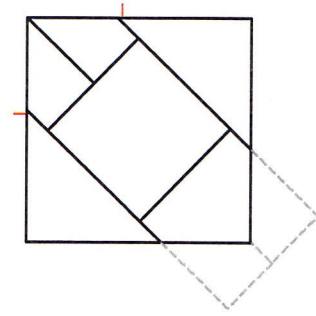


Fig. 29: Wrong dissection

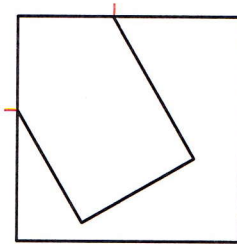


Fig. 30: Idea: Enlarge the central band. *CB 1/4*

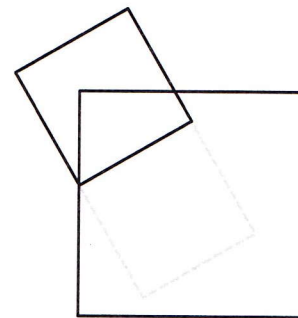


Fig. 31: Slide to complete square. *CB 2/4*

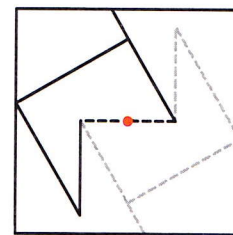


Fig. 32: Apply symmetry. *CB 3/4*

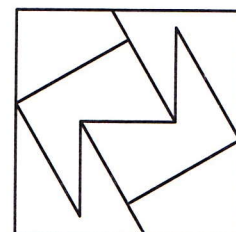


Fig. 33: A new solution. *CB 4/4*

VI. CONCLUSIONS

The incredible diversity of solutions presented in this paper shows the beauty and complexity of dissections. The question that arises whenever we find a new solution is: can we do better?

But what does “better” mean? We generally choose some metric to optimize, such as the number of pieces, and then break ties using another metric such as symmetry. But these preferences are not absolute requirements. We could instead try to minimize the total length of all cuts, or the total number of straight cuts using a pair of scissors on a folded paper. A metric could also be a property, such as requiring all pieces to be convex, so that an artisan can cut them easily, or requiring all pieces be hinge-able, to tessellate the plane. And so on.

For this particular problem, the number of pieces and the geometry aesthetics via symmetry appear to be important factors. The new solution presented in this paper has the dual advantage of being both optimal and symmetric. Moreover, this new 6-piece trisection is an infinite family of equivalent solutions. Professor Frederickson has very aptly remarked that in the particular position of Figure 37, all 6 pieces have exactly the same area, which is rather unique.

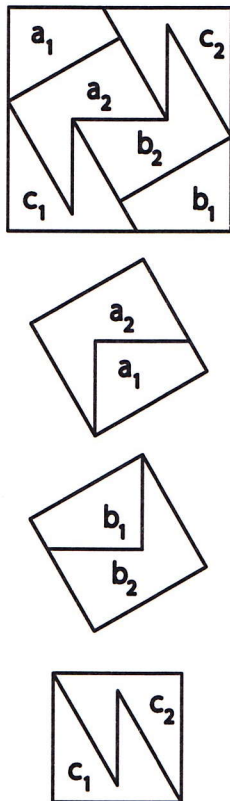


Fig. 34: Christian Blanvillain (2010)
6-piece trisection

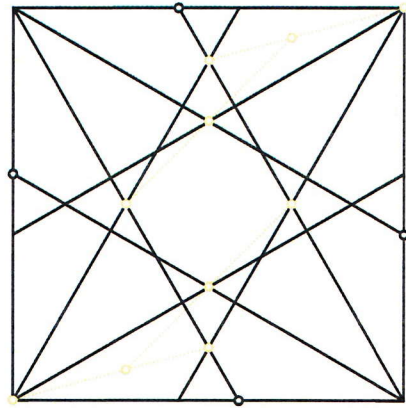


Fig. 35: Construction using only ruler

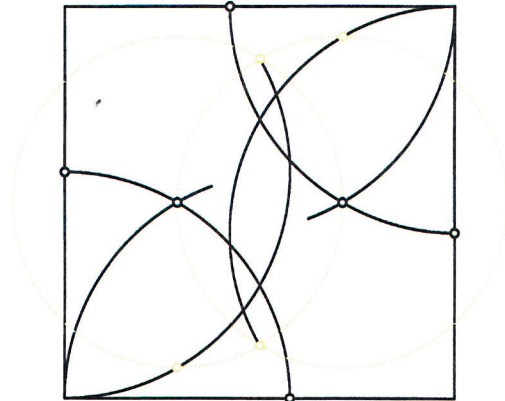


Fig. 36: Construction using only compass

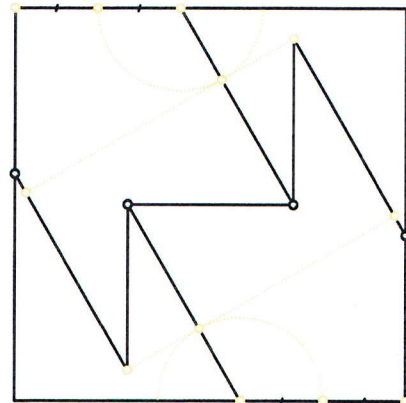


Fig. 37: Construction of six equal areas

VII. ACKNOWLEDGMENTS

I would like to particularly thank Professor János Pach to letting me work on this wonderful problem. I wish to express my warm and sincere gratitude to Professor Greg N. Frederickson and to Professor Alain Wegmann for their corrections and helpful comments. Finally, thank you to Natalia Grigorieva and Alban Gonin for having made all this possible.

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Chamonix. Janvier 1982

Jean - Pierre Petit, avec l'amabilité de Georges Chappaz



Chamonix Jan 82.

