Decomposition of Geometric Graphs into Star-Forests*

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Abstract. We solve a problem of Dujmović and Wood (2007) by showing that a complete convex geometric graph on n vertices cannot be decomposed into fewer than $n-1$ star-forests, each consisting of noncrossing edges. This bound is clearly tight. We also discuss similar questions for abstract graphs.

Keywords: Geometric graphs, Graph Decomposition, Graph Thickness, Star Forests

1 Introduction

To determine the smallest number of subgraphs of some special kind that a graph G can be partitioned into is a large and classical theme in graph theory. In particular, the parts may be required to be matchings (as in Vizing's theorem [12]), complete bipartite graphs (as in the Graham-Pollak theorem [7]), paths and cycles (as in Lovász' theorem [9]), forests (as in the Nash-Williams theorem [10]), etc.

Most likely, it was Erdős who first realized that one can ask many interesting new extremal questions for graphs drawn in the plane or in some other surface, if we replace the purely combinatorial conditions by geometric ones; see [11]. For instance, we may require that the edges participating in a matching or a path do not cross each other [4], [8]. In the 80s and 90s, the emergence of Graph Drawing as a separate discipline gave fresh impetus to this line of research.

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A *geometric graph* G is a graph whose vertex set is a set of points in the plane, no 3 of which are collinear, and whose edges are (possibly crossing) line segments connecting certain pairs of vertices. If the vertices of G are in *convex position*, that is, they form the vertex set of a convex polygon, then G is called a *convex geometric graph*. In the sequel, whenever we say that a graph or a geometric graph G *can be decomposed* into certain parts, we mean that its *edge set*, $E(G)$, can be partitioned into such parts. Each part can be regarded as a different *color class* in the corresponding coloring.

A *star* is a graph consisting of a vertex together with some edges incident to it. In particular, a single vertex is counted as a star. A graph whose every connected component is a star is called a *star-forest*. The (edge set of a) complete graph K_n with n vertices can be decomposed into $n-1$ stars. Akiyama and Kano [2] proved that fewer stars do not suffice. (This also follows from the Graham-Pollak theorem [7], mentioned above.) However, it was also shown in [2] that one can decompose K_n into much fewer *star-forests*: one needs only $\lceil n/2 \rceil + 1$ of them. Can one also decompose a *complete convex geometric graph* on n vertices into fewer than $n-1$ star-forests, if we insist that each star-forest is a *plane graph*, that is, its edges do not cross each other? This question was raised by Dujmović and Wood [6] (Section 10).

The aim of this note is to answer this question in the negative.

Theorem 1 *Let $n \geq 1$. The complete convex geometric graph with n vertices cannot be decomposed into fewer than $n-1$ plane star-forests.*

On the other hand, there are complete geometric graphs where fewer than $n-1$ plane star-forests suffice: consider $P = A_1 \cup A_2 \cup A_3 \cup A_4$ a point set consisting of four pairwise disjoint sets $A_1, \ldots, A_4$, each of size k, such that for every choice $P_1 \in A_1, \ldots, P_4 \in A_4$ we have that P_4 lies inside the convex hull of P_1, P_2 and P_3. Then, it can be seen that the complete geometric graph on P can be decomposed into $3k = 3n/4$ plane star-forests, which come in three families: the first family consists of stars emanating from points in A_1 connecting to all points in A_1 and A_2 together with stars emanating from points in A_3 connecting to all points in A_3 and A_4. Similarly, we draw stars emanating from points in A_2 connecting to all points in A_2 and A_3 and from points in A_4 connecting to all points in A_4 and A_1, and for the last family stars from points in A_1 connecting to all points in A_1 and A_3 and from points in A_2 connecting to all points in A_2 and A_4.

The most important unsolved question in this direction is, how much the bound in Theorem 1 can be improved if we drop the assumption that the vertices are in convex position. We conjecture that the above example is optimal.

Conjecture 2 *Let $n \geq 1$. There is no complete geometric graph with n vertices that be decomposed into fewer than $\lceil 3n/4 \rceil$ plane star-forests.*

Note that in the example above, all star-forests had exactly two components. A star-forest consisting of at most k connected components (stars) is said to be a *k-star-forest*.

It is also an interesting open problem to determine the minimum number of plane k-star-forests that a complete (convex) geometric graph of n vertices can be decomposed into. We do not even know the answer to the analogous question for abstract graphs.

Problem 3 *Let k and n be fixed positive integers. What is the minimum number of k-star-forests that a complete graph K_n of n vertices can be decomposed into?*

As was mentioned earlier, for $k = 1$, the minimum is $n - 1$. The following result settles the first nontrivial case.

Theorem 4 *The complete graph with $n > 3$ vertices can be decomposed into $\lceil 3n/4 \rceil$ 2-star-forests. This bound cannot be improved.*

In particular, this shows that any counterexample to Conjecture 2 would require the use of star-forests with more than 2 components.

Many other variants of decomposing complete geometric graphs have been studied in the literature, including decompositions into plane spanning trees. The conjecture that every complete geometric graph on $2m$ vertices can be decomposed into m plane spanning trees has been recently disproved in [1]. Several notions of thickness studied in [6] are concerned with decompositions of graphs into plane substructures. For many other interesting questions on abstract and geometric graph parameters, consult [3].

In Sections 2 and 3, we prove Theorems 1 and 4, respectively.

2 Covering with plane star-forests–Proof of Theorem 1

Recall that a *plane star-forest* is a star-forest which is a plane graph, i.e., its edges do not cross each other. In this section, in a slight abuse of notations, we will denote the complete convex geometric graph on n points as K_n. Instead of *decompositions* of K_n into plane star-forests, it will be more convenient to consider *coverings*, that is, to allow an edge to belong to more than one star-forest (to have more than one "color"). This does not change the problem, because by keeping just one color for each edge, we turn any covering of the edge set of K_n into a decomposition.

Definition 5 *A collection of plane star-forests, $F_1, F_2, \ldots, F_t$ forms a* covering *of K_n if every edge of K_n belongs to* at least one *F_i.*

For the proof, we need to introduce some simple terminology. The graphs consisting of just one vertex or a single edge are also regarded as stars. Every star S has a *center*. If S is a vertex, then it is its own center. If S is a single edge, we arbitrarily fix one of its endpoints and call it the center of S. The center of a star S is also said to be the *center of any edge* of S. Accordingly, if F is a (plane) star-forest, we always assume that each of its components is a star with a *fixed center*.

Proof (Proof of Theorem 1). For $n = 1, 2$, the statement is trivial. Assume for contradiction and let $n \geq 3$ be the smallest number for which the statement is not true. Let K_n be a complete convex geometric graph, and denote its vertices by $P_1, P_2, \ldots, P_n$, in clockwise order. The indices are taken modulo n, so that $P_{n+1} = P_1, P_{n+2} = P_2$, etc.

Suppose that K_n is covered by t plane star-forests, $F_1, F_2, \ldots, F_t$, for some $t < n - 1$. Our goal is to move some edges from one star-forest to another (i.e., to "recolor" them) in order to turn at least one F_i into a single star. We make sure that after each step of this process, we obtain a covering of K_n with plane star-forests. As soon as one of the F_is becomes a single star, we remove its center from K_n, and contradict with n being the smallest number for which we have a covering of K_n with fewer than $n - 1$ plane star-forests.

For every a, $1 \leq a \leq n$, and for every k, $1 < k < n$, we call the edge $P_a P_{a+k}$ a k-edge. Note that every k-edge is also a $(n - k)$-edge.

Definition 6 *A k-edge $P_a P_{a+k}$ is called* supported *if there exists F_i such that $P_a P_{a+k}$ belongs to F_i, and*

(i) either all edges $P_a P_{a+1}, P_a P_{a+2}, \ldots, P_a P_{a+k-1}$ belong to F_i,
(ii) or all edges $P_{a+1} P_{a+k}, P_{a+2} P_{a+k}, \ldots, P_{a+k-1} P_{a+k}$ belong to F_i.

Otherwise, we call it unsupported.

The goal is to recolor the edges step by step in order to make all the edges supported. For this purpose, the following observation is useful for the recoloring process.

Observation 7 *Suppose that the complete geometric graph K_n is covered by t plane star-forests, $F_1, F_2, \ldots F_t$. Let S be a connected component of F_i (that is, a star) where $1 \leq i \leq t$. Assume that no edge of S crosses an edge of F_j where $1 \leq j \leq t$, $j \neq i$. Remove the edges in S from F_i and add them to F_j. Then any edge that was supported before is still supported.*

Lemma 8 *Suppose that the complete geometric graph K_n can be covered by t plane star-forests, for some positive integer t.*

Then, for every k, $1 < k < n$, there exists a covering of K_n by t plane star-forests $F_1, F_2 \ldots, F_t$ such that every k'-edge with $1 < k' \leq k$ is supported.

Proof. We prove the lemma by induction on k.

Suppose that $k = 2$. By symmetry, it is sufficient to consider the 2-edge $P_1 P_3$ (that is, $a = 1$). We can assume without loss of generality that $P_1 P_3$ belongs to F_i, for some i, and its center is P_1 (which implies that $P_2 P_3$ is not in F_i). If $P_1 P_2$ belongs to F_i, condition (i) in Definition 6 is satisfied, and we are done. If $P_1 P_2$ does not belong to F_i, then add it to F_i. Obviously, it cannot cross any other edge in F_i. The only problem that may occur is that until now P_2 was a single vertex star in F_i, and now F_i has two stars that have a point in common. In this case, simply erase the single vertex star P_2 from F_i. Thus, the lemma is true for $k = 2$.

Suppose next that $k > 2$ and the statement has already been verified for $k-1$. We want to prove it for k.

By symmetry, it is enough to consider the k-edge P_1P_{k+1} and make it supported without making the already supported edges unsupported. Suppose without loss of generality that P_1P_{k+1} belongs to a star in F_i and the center of this star is P_1. The edges in F_i are marked *blue*.

Let $l < k+1$ be the *largest* index such that P_1P_l does *not* belong to F_i. Then the edges $P_1P_{l+1}, \ldots, P_1P_{k+1}$ are all blue. If there is no such index l, then we are done, because F_i satisfies condition (i) in Definition 6.

By the induction hypothesis the edge P_1P_l is supported, so there exists a star-forest F_j, $j \neq i$, which contains P_1P_l along with all the edges $P_1P_2, P_1P_3, \ldots, P_1P_{l-1}$ or along with all the edges P_2P_l, $P_3P_l, \ldots,$ $P_{l-1}P_l$. The edges of F_j are marked *red*. We distinguish two cases depending on these two possibilities.

Case 1: *The edges $P_1P_2, P_1P_3, \ldots, P_1P_l$ belong to F_j.*

We make two changes. See Figure 1.

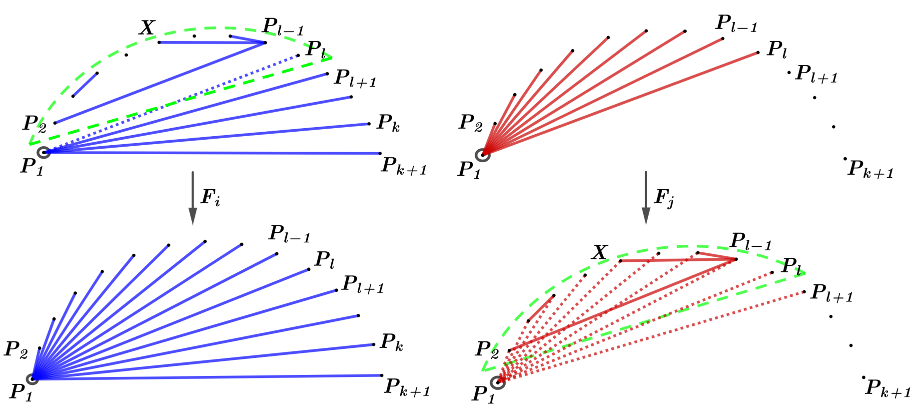

Fig. 1. In Case 1, recolor $P_1P_2, \ldots, P_1P_l$ from red to blue, and all blue stars spanned by $\{P_2, \ldots, P_l\}$ to red. A dotted line marks the absence of an edge.

Step 1: Remove the edges $P_1P_2, P_1P_3, \ldots, P_1P_l$ from F_j and add all of them to F_i (unless they were already in F_i).

Then P_1P_{k+1} will satisfy condition (i) of definition 6 in F_i (with $a = 1$). However, in the process, we may have created some crossings within F_i, and F_i may also cease to be a star-forest. Both of these problems can be avoided by performing

Step 2: Remove from F_i all (blue) edges connecting two elements of $\{P_2, P_3, \ldots, P_l\}$ and add them to F_j.

Note that by recoloring the blue edges within $\{P_2, P_3, \ldots, P_l\}$ to red, we do not violate the condition that F_j is a plane star-forest. Indeed, unless $l = 2$, originally, no element of $\{P_2, P_3, \ldots, P_l\}$ was connected by a red edge to any vertex other than P_1. Also by Observation 7, neither of the two steps results in any previously supported edge becoming unsupported.

Case 2: *The edges $P_2P_l, P_3P_l, \ldots, P_{l-1}P_l$ belong to F_j.*

First, we will modify F_i by including the edge P_1P_l. This will require some care, to make sure that the new covering does not violate the conditions. See Figure 2.

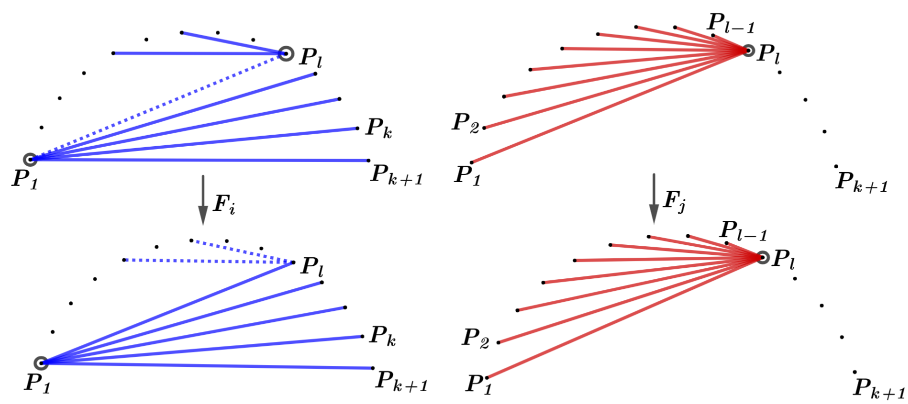

Fig. 2. In Case 2, P_1P_l will have two colors: red and blue. Remove the color blue from all previously blue edges incident to P_l.

Step 1: Add the edge P_1P_l to F_i, but also keep it in F_j. Remove from F_i all other edges incident to P_l.

Notice that after performing this step, we still have a covering of K_n by plane star-forests. It is a *covering*, because all edges deleted from F_i also belonged, and continue to belong, to F_j. Obviously, F_i remains a *star-forest*: its component containing P_1 remains a star, because we removed from F_i any other edge incident to P_l. Finally, F_i remains a *plane graph*, because its newly added edge, P_1P_l cannot cross any other blue edge. Indeed, such an edge should be incident to P_{l+1}, contradicting our assumption that P_1P_{k+1} originally belonged to a star in F_i, whose center is P_1. Also note that edges incident to P_l in F_i form a connected component which is already in F_j. So removing them is equivalent to recoloring them as red, which, by Observation 7, does not make any already supported edge unsupported.

Now we go back to the beginning of the proof, and again find the largest index l' such that $P_1P_{l'}$ does not belong to F_i. Obviously, we have $l' < l$. As before, we distinguish two cases. In Case 1, we conclude that P_1P_{k+1} satisfies condition (i) of Definition 6 in F_i (with $a = 1$), and we are done with the induction step. In Case 2, we can include the edge $P_1P_{l'}$ in F_i. Continuing like this, in fewer than k steps, we arrive at a situation where either P_1P_{k+1} satisfies condition (i) of definition 6 in F_i, or one by one, we manage to include all of the edges $P_1P_{k+1}, P_1P_k, \ldots, P_1P_3, P_1P_2$ in F_i, which again means that P_1P_{k+1} satisfies condition (i) of definition 6 in F_i. This completes the proof of Lemma 8. □

Applying the lemma with $k = n-1$ and $a = 1$, we can construct a covering of K_n by fewer than $n-1$ plane star-forests such that one of them, again denoted by F_i, has the property that either $P_1P_2, P_1P_3, \ldots, P_1P_n$ belong to F_i, or $P_1P_n, P_2P_n, \ldots, P_{n-1}P_n$ belong to F_i. That is, F_i is a single star of degree $n-1$, centered at P_1 or P_n. Deleting P_1 or P_n, resp., from K_n, we obtain a covering of K_{n-1} with fewer than $n-2$ plane star-forests, which contradicts our assumption that Theorem 1 is true for decompositions and, hence, for coverings of the complete convex geometric graph K_{n-1}. This completes the proof of Theorem 1. □

3 2-Star-Forests–Proof of Theorem 4

Proof. Let V be an n-element set, and let $V = V_1 \cup V_2 \cup V_3 \cup V_4$ be a partition of V into 4 subsets as equal as possible. Suppose without loss of generality that

$$\lfloor n/4 \rfloor \le |V_1| \le |V_2| \le |V_3| \le |V_4| \le \lceil n/4 \rceil.$$

Let $f : V_2 \to V_1$ be a surjection (onto mapping). For every $u \in V_2$, consider the two-star-forest F_u consisting of all edges connecting u to a every vertex in $V_2 \cup V_3$, and connecting $f(u)$ to every vertex in $V_1 \cup V_4$. These two-star-forests completely cover all edges within V_2 and V_1, and all edges in $V_1 \times V_4$ and in $V_2 \times V_3$. In a similar manner, we can construct $|V_4|$ two-star-forests that cover all edges within V_4 and V_3, and all edges in $V_4 \times V_2$ and $V_3 \times V_1$. Finally, with $|V_3|$ two-star-forests (with one center in V_3 and one in V_1), we can cover all edges in $V_3 \times V_4$ and $V_1 \times V_2$. Thus, we covered K_n with $|V_2| + |V_3| + |V_4| = \lceil 3n/4 \rceil$ two-star-forests, as required.

Next, we show that K_n cannot be covered by fewer than $\lceil 3n/4 \rceil$ two-star-forests, for any $n \ge 4$. The case $n = 4$ is easy. The proof is by contradiction. Let n be the smallest value greater than 4 for which there exists a covering of K_n by $t \le \lceil 3n/4 \rceil - 1$ two-star-forests. Denote the two-star-forests participating in such a covering by $F_1, \ldots, F_t$. If any F_i has only one center, then deleting it from K_n, together with all edges incident to it, we reduce the number of vertices by 1 and the number of two-star-forests by 1. This would contradict the minimal choice of n. Thus, we can and will assume that every F_i, $1 \le i \le t$, has two centers.

Now consider a graph G with the same set of vertices as K_n, and for every 2-star-forest F_i, draw an edge in G between the two centers of stars in F_i. The resulting graph G has at most $\lceil 3n/4 \rceil - 1$ edges and, therefore, at least $n - \lceil 3n/4 \rceil + 1$ connected components. Note that $3(n - \lceil 3n/4 \rceil + 1) > \lceil 3n/4 \rceil - 1$, so there exists a connected component C in G with fewer than 3 edges.

If C is a single vertex u, then by construction it cannot be the center of any two-star-forest. Thus, we would need at least $n-1$ two-star-forests just to cover the edges incident to u in K_n. If C consists of only one edge u_1u_2, then neither of these vertices can be the center of any other two-star-forest. Thus, the edge u_1u_2 was not covered by any two-star-forest F_j, which is a contradiction. Finally, if C consists of two edges, u_1u_2 and u_1u_3, say, then it is not difficult to see that at least one of the edges between u_1, u_2, u_3 in K_n is not covered by any two-star-forest F_j. In each of the above cases, we obtained a contradiction. This completes the proof of Theorem 4. □

In view of Theorem 4, we state the following conjecture.

Conjecture 9 *For any $n \geq k \geq 2$, the number of k-star-forests needed to cover the complete graph K_n is at least $\lceil \frac{(k+1)n}{2k} \rceil$.*

For $k = 2$, the conjecture is true, by Theorem 4. We construct an example inspired by the construction in [2], showing that Conjecture 9, if true, is best possible. For simplicity, we describe it only for the case where n is divisible by 2. Assuming $n = 2t$, and labeling the vertices by $\{v_1, v_2, \cdots, v_n\}$, we create t 2-star-forests $F_1, F_2, \cdots F_t$ by picking vertices v_i and v_{i+t} as centers of F_i, $1 \leq i \leq t$ and connecting v_i to all vertices v_j, $i < j < i + t$, and connecting v_{i+t} to all vertices v_{j+t}, $i < j < i + t$ (the indices are taken modulo n). The introduced 2-star-forests cover all edges of K_n, except the set of edges v_iv_{i+t} which can be simply decomposed into $\lceil \frac{n}{2k} \rceil$ k-star-forests. Altogether, K_n can be covered by $\frac{n}{2} + \lceil \frac{n}{2k} \rceil$ k-star-forests.

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