Note on k-planar crossing numbers*

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Dedicated to our colleague Ferran Hurtado (1951–2014)

Abstract

The *crossing number* $\text{CR}(G)$ of a graph $G = (V, E)$ is the smallest number of edge crossings over all drawings of G in the plane. For any $k \geq 1$, the *k-planar crossing number* of G, $\text{CR}_k(G)$, is defined as the minimum of $\text{CR}(G_0)+\text{CR}(G_1)+\ldots+\text{CR}(G_{k-1})$ over all graphs $G_0, G_1, \ldots, G_{k-1}$ with $\cup_{i=0}^{k-1} G_i = G$. It is shown that for every $k \geq 1$, we have $\text{CR}_k(G) \leq \left(\frac{2}{k^2} - \frac{1}{k^3}\right) \text{CR}(G)$. This bound does not remain true if we replace the constant $\frac{2}{k^2} - \frac{1}{k^3}$ by any number smaller than $\frac{1}{k^2}$. Some of the results extend to the rectilinear variants of the k-planar crossing number.

1 Introduction

Selfridge (see [9]) noticed that by Euler's polyhedral formula K_{11}, the complete graph on 11 vertices, cannot be written as the union of two planar graphs. Later Battle, Harary, and Kodama [1] and independently Tutte [21] proved that the same is true for K_9, but not for K_8. This led Tutte [22] to introduce a new parameter, the *thickness* of a graph G, which is the minimum number of planar graphs that G can be decomposed into. The notion turned out to be relevant for VLSI chip design, where it corresponds to the number of layers required for realizing a network so that there is no crossing within a layer. Consult Mutzel, Odenthal, and Scharbrodt [12] for a survey. If the thickness of G is at most 2, G is called *biplanar*. Mansfield proved that it is an NP-complete problem to decide whether a graph is biplanar; see [2, 11].

A *drawing* of a graph $G = (V, E)$ is a planar representation of G such that every vertex $v \in V$ corresponds to a point of the plane and every edge $uv \in E$ is represented by a simple continuous curve between the points corresponding to u and v, which does not pass through any point representing a vertex of G. We always assume for simplicity that (1) no two curves share infinitely many points, (2) no two curves are tangent to each other, and (3) no three curves pass through the same point. The *crossing number* of G is defined as the minimum number of edge

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crossings in a drawing of G, and is denoted by $\text{CR}(G)$. For a survey, see [17, 20]. Clearly, G is planar if and only if $\text{CR}(G) = 0$.

The *biplanar crossing number*, $\text{CR}_2(G)$, of G was defined by Owens [13] as the minimum sum of the crossing numbers of two graphs, G_0 and G_1, whose union is G. For the VLSI applications, we imagine that G_0 and G_1 are drawn (realized) in different planes. If G is biplanar, its biplanar crossing number is 0. The biplanar crossing number of random graphs was studied by Spencer [19]. Czabarka, Sýkora, Székely, and Vrťo [5] proved that for every graph G, we have

$$\text{CR}_2(G) \le \frac{3}{8}\text{CR}(G).$$

They also showed [4] that this inequality does not remain true if the constant $\frac{3}{8} = 0.375$ is replaced by anything less than $\frac{8}{119} \approx 0.067$.

Shahrokhi et al. [18] extended the notion of biplanar crossing number as follows. For any positive integer $k \ge 1$, define the *k-planar crossing number* of G as the minimum of $\text{CR}(G_0) + \text{CR}(G_1) + \ldots + \text{CR}(G_{k-1})$, where the minimum is taken over all graphs $G_0, G_1, \ldots, G_{k-1}$ whose union is G, that is, $\cup_{i=0}^{k-1} E(G_i) = E(G)$. This number is denoted by $\text{CR}_k(G)$. Obviously, $\text{CR}_1(G) = \text{CR}(G)$ and we have $\text{CR}_i(G) \ge \text{CR}_{i+1}(G)$ for all $i \in \mathbb{N}$ and every graph G.

In the present note, we investigate the relationship between the k-planar crossing number and the (ordinary) crossing number of a graph. For every $k \ge 1$, let

$$\alpha_k = \sup \frac{\text{CR}_k(G)}{\text{CR}(G)},$$

where the supremum is taken over all *nonplanar* graphs G. The above mentioned results yield $0.067 < \alpha_2 \le \frac{3}{8} = 0.375$. The next theorem implies that $\alpha_k = \Theta(k^{-2})$.

Theorem. *For every positive integer k, we have*

$$\frac{1}{k^2} \le \alpha_k \le \frac{2}{k^2} - \frac{1}{k^3}.$$

2 Proof of Theorem

Upper bound. First we prove the upper bound. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, and fix an optimal drawing of G in the plane with precisely $\text{CR}(G)$ crossings. We describe a randomized procedure to partition (the edge set of) G into k subgraphs $G_0, \ldots, G_{k-1}$ such that the expected value of the sum of their crossing numbers is at most $(\frac{2}{k^2} - \frac{1}{k^3})\text{CR}(G)$. We think of each G_i as a graph drawn independently so that edges of different subgraphs do not cross.

The idea of the proof is the following. We start by randomly partitioning the vertex set of G into k roughly equal classes. We associate with each class a vertex of a complete graph K_k. We consider a factorization of K_k into maximal matchings and then use these matchings to divide $E(G)$ into k classes, $G_0, \ldots, G_{k-1}$. It will follow from the definition that every G_i can be drawn independently in such a way that no two edges that correspond to distinct edges of the underlying matching of K_k will cross.

Let the vertex set of G be $V = V(G) = \{1, 2, \ldots, n\}$. Assign independent random variables ξ_v to the vertices $v \in V$ such that each ξ_v takes each of the values $0, 1, \ldots, k-1$ with probability $1/k$.

For every $i (0 \leq i < k)$, let $V_i = \{v \in V \mid \xi_v = i\}$, and define a subgraph G_i as follows. Let $V(G_i) = V$ and let the edge set $E(G_i)$ of G_i consist of all edges $uv \in E(G)$ for which

$$\xi_u + \xi_v \equiv i \bmod k.$$

Obviously, we have $\cup_{i=0}^{k-1} E(G_i) = E(G)$.

We define the *type* of an edge uv to be the unordered pair (ξ_u, ξ_v). For each i $(0 \leq i < k)$, first we draw G_i in the ith plane as it was drawn in the original drawing of G. Notice that for every index g, there is precisely one index $h = h(g)$ such that G_i has an edge connecting a vertex in V_g to a vertex in V_h. Thus, every connected component of G_i consists of edges of the same type. In the ith plane, we can translate the connected components of G_i sufficiently far from each other so that no two edges of different types intersect, and during the procedure no new crossings are introduced.

Calculate the expected value of the total number of crossings in the resulting drawing of G_i over all i $(0 \leq i < k)$. Every crossing arises from a crossing between two edges in the original drawing of G. Consider two edges $uv, u'v' \in E(G)$ that cross each other in the original drawing. A crossing between these edges will be present in the final drawing of one of the G_is if and only uv and $u'v'$ are of the same type. For every index g, this happens with probability $\Pr[\text{type}(uv) = (g, g)] = \frac{1}{k^2}$. For distinct indices g and h $(g \neq h)$, we have $\Pr[\text{type}(uv) = (g, h)] = \frac{2}{k^2}$.

Summing over all possible pairs of types, we obtain

$$\Pr[\text{type}(uv) = \text{type}(u'v')] = \binom{k}{2} \cdot \frac{2}{k^2} \cdot \frac{2}{k^2} + k \cdot \frac{1}{k^2} \cdot \frac{1}{k^2} = \frac{2}{k^2} - \frac{1}{k^3}.$$

Consequently, the expected value of the total number of crossings in the resulting drawings of all G_is is $(\frac{2}{k^2} - \frac{1}{k^3})\textsc{cr}(G)$. Hence, there exists a partition of (the edges of) G into $G_0, \ldots, G_{k-1}$ where

$$\textsc{cr}(G_0) + \ldots + \textsc{cr}(G_{k-1}) \leq \left(\frac{2}{k^2} - \frac{1}{k^3}\right) \textsc{cr}(G).$$

This completes the proof of the upper bound in the Theorem.

Lower bound. Next we establish the lower bound. For two functions $f(n)$ and $g(n)$, we write $f(n) \ll g(n)$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,

$$\kappa(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \textsc{cr}(G).$$

It was shown in [16] that there exists a positive constant K such that if $n \ll e \ll n^2$, the limit

$$\lim_{n\to\infty} \kappa(n, e) \frac{n^2}{e^3}$$

exists and is equal to K. The constant $K > 0$ is called the *midrange crossing constant*. The best known bounds for K are $0.032 \leq K \leq 0.09$; see [14, 15]. This result can be rephrased as follows.

Lemma. *For every ε $(0 < \varepsilon < 1)$, there exists a constant $N = N_\varepsilon$ satisfying the following condition. For every positive integers n and e with $\min(n, \frac{e}{n}, \frac{n^2}{e}) \geq N$, we have $\kappa(n, e) > K - \varepsilon$, and there is a graph G with n vertices and e edges such that $\textsc{cr}(G) < (K + \varepsilon)\frac{e^3}{n^2}$.*

Let $\varepsilon > 0$ be fixed, let

$$\min\left(n, \frac{e}{n}, \frac{n^2}{e}\right) > \frac{k}{\varepsilon} N_\varepsilon,$$

and let G be a graph with n vertices and e edges such that $\mathrm{CR}(G) < (K+\varepsilon)\frac{e^3}{n^2}$. Decompose G into k graphs $G = G_0 \cup G_1, \cdots \cup G_{k-1}$ such that $\mathrm{CR}(G_0) + \mathrm{CR}(G_1) + \cdots + \mathrm{CR}(G_{k-1}) = \mathrm{CR}_k(G)$. For simplicity, write e_i for $|E(G_i)|$.

We may assume, without loss of generality, that there is an integer t $(0 < t \leq k)$ such that $e_i \geq \frac{\varepsilon}{k} e$ for $i = 0, 1, \ldots, t-1$, and $e_i < \frac{\varepsilon}{k} e$ for $i = t, t+1, \ldots, k-1$.

For every $i < t$, we have $\min(n, \frac{e_i}{n}, \frac{n^2}{e_i}) > N_\varepsilon$, so we can apply the Lemma to conclude that $\mathrm{CR}(G_i) \geq (K-\varepsilon)\frac{e_i^3}{n^2}$. Using that $\sum_{i=t}^{k-1} e_i \leq \varepsilon e$, we have $\sum_{i=0}^{t-1} e_i \geq (1-\varepsilon)e$.

Hence, Jensen's inequality yields

$$\begin{aligned} \mathrm{CR}_k(G) &\geq \sum_{i=0}^{t-1} \mathrm{CR}(G_i) \geq \sum_{i=0}^{t-1} (K-\varepsilon)\frac{e_i^3}{n^2} \\ &\geq t(K-\varepsilon) \cdot \frac{((1-\varepsilon)e/t)^3}{n^2} > \frac{(1-3\varepsilon)(K-\varepsilon)}{k^2} \cdot \frac{e^3}{n^2}. \end{aligned}$$

Using that $\mathrm{CR}(G) < (K+\varepsilon)\frac{e^3}{n^2}$, the last inequality implies

$$\frac{\mathrm{CR}_k(G)}{\mathrm{CR}(G)} \geq (1-3\varepsilon)\frac{K-\varepsilon}{K+\varepsilon} \cdot \frac{1}{k^2}.$$

As $\varepsilon \to 0$, the lower bound in the Theorem follows.

3 Rectilinear Variants

Rectilinear k-planar crossing numbers. The *rectilinear crossing number*, $\mathrm{RCR}(G)$, of a graph G is the minimum number of crossings over all *straight-line* drawings of G, in which the edges are represented by line segments. Obviously, we have $\mathrm{CR}(G) \leq \mathrm{RCR}(G)$ for every graph G. For every $t \geq 4$, Bienstock and Dean [3] constructed families of graphs whose crossing number is at most t and whose rectilinear crossing number is unbounded.

Similarly to $\mathrm{CR}_k(G)$, we define the *rectilinear k-planar crossing number* of a graph G, denoted $\mathrm{RCR}_k(G)$, as the minimum of $\mathrm{RCR}(G_0) + \mathrm{RCR}(G_1) + \ldots + \mathrm{RCR}(G_{k-1})$, where the minimum is taken over all graphs $G_0, G_1, \ldots, G_{k-1}$ whose union is G. It is clear that $\mathrm{CR}_k(G) \leq \mathrm{RCR}_k(G)$ for every $k \in \mathbb{N}$. However, we do not know of any graph G where $\mathrm{CR}_k(G) < \mathrm{RCR}_k(G)$ and $k \geq 2$.

The analogue of α_k for every $k \in \mathbb{N}$ is

$$\beta_k = \sup \frac{\mathrm{RCR}_k(G)}{\mathrm{RCR}(G)},$$

where the supremum is taken over all *nonplanar* graphs G. The proof of our main theorem carries over verbatim to this variant, and yields

$$\frac{1}{k^2} \leq \beta_k \leq \frac{2}{k^2} - \frac{1}{k^3}.$$

Specifically, the upper bound starts from a fixed straight-line drawing of G with exactly $\mathrm{RCR}(G)$ crossings. Our randomized procedure decomposes G into k graphs $G_0, \ldots, G_{k-1}$, each of which consists of k vertex-disjoint subgraphs induced by the k edge types. These k^2 subgraphs can be translated independently to avoid any crossings between edges of different subgraphs, but maintain a straight-line drawing for each. The lower bound relies on the existence of a midrange crossing constant $\overline{K} > 0$ for the *rectilinear* crossing number, which is established by the argument in [16] even though the constants K and $\overline{K}$ are not necessarily the same.

Geometric k-planar crossing numbers. The *geometric thickness*, introduced by Kainen [10], is the smallest positive integer k such that G admits a k-edge-coloring *and* a straight-line drawing in which edges of the same color do not cross. The color classes define a decomposition of G into k planar graph $G_0, \ldots, G_{k-1}$ each of which admits a crossing-free straight-line drawing in such a way that corresponding vertices are represented by the same point in the plane. A straight-line drawing of a graph G is called *biplane* if G admits a 2-edge-coloring such that no two edges of the same color cross in this drawing; see [8]. Eppstein [7] constructed graphs with thickness 3 and geometric thickness at least t for every $t > 0$. Determining the geometric thickness of a graph is also an NP-hard problem [6].

The geometric thickness motivates the following variant of the k-planar crossing number. The *geometric k-planar crossing number* of a graph G, denoted $\textsc{gcr}_k(G)$, is the minimum number of crossings between edges of the same color over all k-edge-colorings of G and all straight-line drawings of G. It is clear that $\textsc{cr}_k(G) \leq \textsc{rcr}_k(G) \leq \textsc{gcr}_k(G)$ for every graph G and every $k \in \mathbb{N}$.

The analogue of α_k for every $k \in \mathbb{N}$ is

$$\gamma_k = \sup \frac{\textsc{gcr}_k(G)}{\textsc{rcr}(G)},$$

where the supremum is taken over all *nonplanar* graphs G. The lower bound of our main theorem carries over verbatim to this variant, since it relies on density results, namely the (rectilinear) midrange crossing number. But the upper bound argument does not extend to this variant. Our randomized procedure partitions the edge set $E(G)$ into k color classes $E(G_0), \ldots, E(G_{k-1})$, and crossings between edges of different colors do not count. But each color class consists of edges of up to k different types, and the crossings between edges of the same color and different types cannot be eliminated. A weaker upper bound easily follows from a uniform random k-coloring of the edges, and yields

$$\frac{1}{k^2} \leq \gamma_k \leq \frac{1}{k}.$$

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