Exchange properties of finite set-systems

Peter Frankl[*] *János Pach*[†] *Dömötör Pálvölgyi*[‡]

Abstract

In a recent breakthrough, Adiprasito, Avvakumov, and Karasev constructed a triangulation of the n-dimensional real projective space with a subexponential number of vertices. They reduced the problem to finding a small downward closed set-system $\mathcal{F}$ covering an n-element ground set which satisfies the following condition: For any two disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$, or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$. Denoting by $f(n)$ the smallest cardinality of such a family $\mathcal{F}$, they proved that $f(n) < 2^{O(\sqrt{n}\log n)}$, and they asked for a nontrivial lower bound. It turns out that the construction of Adiprasito *et al.* is not far from optimal; we show that $2^{(1.42+o(1))\sqrt{n}} \le f(n) \le 2^{(1+o(1))\sqrt{2n\log n}}$.

We also study a variant of the above problem, where the condition is strengthened by also requiring that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| > |B|$, there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$. In this case, we prove that the size of the smallest $\mathcal{F}$ satisfying this stronger condition lies between $2^{\Omega(\sqrt{n}\log n)}$ and $2^{O(n\log\log n/\log n)}$.

1 Introduction

It is an old problem to find a triangulation of the n-dimensional real projective space with as few vertices as possible. Recently, Adiprasito, Avvakumov, and Karasev [1] broke the exponential barrier by finding a construction of size $2^{O(\sqrt{n}\log n)}$. For the proof, they considered the following problem in extremal set theory.

What is the *minimum* cardinality of a system $\mathcal{F}$ of subsets of $[n] = \{1, 2, \dots, n\}$, which satisfies three conditions:

1. $\mathcal{F}$ is *atomic*, that is, $\emptyset \in \mathcal{F}$ and $\{a\} \in \mathcal{F}$ for every $a \in [n]$;

[*]Rényi Institute, P.O.Box 127 Budapest, 1364 Hungary and MIPT, Moscow, partially supported by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant 075-15-2019-1926. Email: `peter.frankl@gmail.com`.

[†]Rényi Institute, P.O.Box 127 Budapest, 1364 Hungary and MIPT, Moscow, partially supported by ERC Advanced Grant GeoScape, NKFI grants KKP-133864 and K-131529, and by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant 075-15-2019-1926. Email: `pach@cims.nyu.edu`

[‡]MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics, Eötvös Loránd University, Budapest, Hungary. Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under the grant LP2017-19/2017. Email: `domotorp@gmail.com`.

2. $\mathcal{F}$ is *downward closed*, that is, if $A \in \mathcal{F}$, then $A' \in \mathcal{F}$ for every $A' \subset A$;

3. for any two disjoint members $A, B \in \mathcal{F} \setminus \{\emptyset\}$, there exist $a \in A$ and $b \in B$ such that
either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$,
or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$.

Letting $f(n)$ denote the minimum size of a set-system $\mathcal{F}$ with the above three properties, Adiprasito *et al.* proved

$$f(n) \leq 2^{(1/2+o(1))\sqrt{n}\log n}, \tag{1}$$

where *log* always denotes the base 2 logarithm. They used the following construction. Let $s, t > 0$ be integers, $n = st$. Fix a partition $[n] = X_1 \cup \ldots \cup X_t$ of the ground set into t parts of equal size, $|X_1| = \ldots = |X_t| = s$. Let

$$\mathcal{F} = \cup_{i=1}^{t} \mathcal{F}_i, \quad \text{where} \quad \mathcal{F}_i = \{F \subseteq [n] : |F \cap X_j| \leq 1 \text{ for every } j \neq i\}, \tag{2}$$

for $1 \leq i \leq t$. (In the definition of $\mathcal{F}_i$, there is no restriction on the size of $F \cap X_i$.) It is easy to verify that $\mathcal{F}$ meets the requirements. We have

$$|\mathcal{F}| = (t2^s - (s+1)(t-1))(s+1)^{t-1} < 2^{s+t\log(s+1)+\log t}.$$

Substituting $s = (1/\sqrt{2} + o(1))\sqrt{n \log n}$ and $t = (\sqrt{2} + o(1))\sqrt{n/\log n}$, we obtain that

$$f(n) \leq 2^{(1/\sqrt{2}+o(1))\sqrt{n\log n}+(\sqrt{2}+o(1))\sqrt{n/\log n}\cdot\log\sqrt{n}} = 2^{(1+o(1))\sqrt{2n\log n}}. \tag{3}$$

This is slightly better than (1). (The authors of [1] remarked that their bound can be improved by a "subpolynomial factor.") Any further improvement on the upper bound would result in a smaller triangulation of the projective space.

Our first theorem implies that (3) is not far from optimal.

The *rank* of a set-system $\mathcal{F}$, denoted by $\mathrm{rk}(\mathcal{F})$, is the size of the largest set $F \in \mathcal{F}$; see, *e.g.*, [2].

We denote by $\lfloor x \rceil$ the integer closest to x.

Theorem 1. *Let $\mathcal{F}$ be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$,* either *there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$,* or *there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Then we have*

(i) $|\mathcal{F}| \geq e^{(2e^{-1/\sqrt{2}}+o(1))\sqrt{n}} \geq 2^{(1.42+o(1))\sqrt{n}}$;

(ii) $\mathrm{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rceil$, *and this bound is best possible.*

If we also assume that $\mathcal{F}$ is downward closed, then the inequality $\mathrm{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rceil$ immediately implies that $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rceil}$. This is only slightly weaker than the lower bound $f(n) \geq 2^{(1.42+o(1))\sqrt{n}}$, which follows from part (i).

Remark. We remark that the assumptions of Theorem 1 are weaker than those made by Adiprasito *et al.*, in two different ways: we do not require that $\mathcal{F}$ is downward closed (which is their condition 2), and the exchange condition between two disjoint sets is also less restrictive than condition 3.

Nevertheless, we know no significantly smaller set-systems satisfying these weaker conditions than the ones described in (2), for which $|\mathcal{F}| = 2^{(1+o(1))\sqrt{2n\log n}}$. We can, however, further weaken the conditions under which Theorem 1 holds; instead of the exchange property, it is sufficient to assume the following:

For any two disjoint members $A, B \in \mathcal{F}$ with $|A| = |B|$, there is a set $C \in \mathcal{F}$ such that $C \subset A \cup B$ and $|C| = |A| + 1$.

This answers Question 3.7 of Adiprasito *et al.* [1]: from Claim 3.1 of [1], one cannot obtain a significantly better construction, using another family. To see that condition (3) in Claim 3.1 implies our condition above, apply it with the unit vector X identified with $A \cup B$.

While part (ii) of Theorem 1 is tight, we suspect that part (i) and the lower bound $f(n) \geq 2^{\Omega(\sqrt{n})}$ can be improved. As a first step, we slightly strengthen the assumptions of Theorem 1, in order to obtain a better lower bound on $|\mathcal{F}|$.

Theorem 2. *Let $\mathcal{F}$ be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$,* either *there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$,* or *there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Moreover, suppose that if $|A| < |B|$, then the second option is true.*

Then we have $|\mathcal{F}| \geq 2^{(1/2+o(1))\sqrt{n}\log n}$.

This lower bound exceeds the upper bound in (3). Therefore, construction (2) cannot satisfy the stronger assumptions in Theorem 2. For example, set

$$A = \{a_1\} \cup \{a_2, a_2'\} \cup \{\emptyset\} \cup \ldots \cup \{\emptyset\} \in \mathcal{F}_2 \subset \mathcal{F},$$

$$B = (X_1 \setminus \{a_1\}) \cup \emptyset \cup \{\emptyset\} \cup \ldots \cup \{\emptyset\} \in \mathcal{F}_1 \subset \mathcal{F},$$

where $a_1 \in X_1$ and $a_2, a_2' \in X_2$. If $s > 4$, then $|A| < |B|$, but there is no element of B that can be added to A such that the resulting set also belongs to $\mathcal{F}$. If $s \leq 4$, then the conditions of Theorem 2 are satisfied, but the construction is uninteresting, as $|\mathcal{F}| = 2^{\Theta(n)}$ and $\mathrm{rk}(\mathcal{F}) = \Theta(n)$.

Our next result provides a nontrivial construction.

Theorem 3. *There exists an atomic downward closed set-system $\mathcal{F} \subset 2^{[n]}$ with the property that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| \leq |B|$, there is $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$, and*

(i) $|\mathcal{F}| \leq 2^{(2+o(1))n\log\log n/\log n}$,

(ii) $\mathrm{rk}(\mathcal{F}) \leq (2 + o(1))n/\log n$.

The proofs of Theorems 1, 2, and 3 are presented in Sections 2, 3, and 4, respectively.

2 Proof of Theorem 1

We start with a statement which immediately implies the inequality in part (ii).

Lemma 2.1. *Let $k \geq 1$ be an integer, $n > \binom{k}{2}$, and let $\mathcal{F}$ be an atomic family of subsets of $[n]$ satisfying the exchange property in Theorem 1, or the condition in the Remark.*

Then there is a set $F \in \mathcal{F}$ such that $|F| = k$. This bound cannot be improved: there are families satisfying the conditions, for which $\mathrm{rk}(\mathcal{F}) = k$.

Proof. By induction on k. For $k = 1$, the claim is trivial. Suppose that $k > 1$ and that the lemma has already been proved for $k - 1$.

Let $\mathcal{F} \subset 2^{[n]}$ be a family satisfying the conditions, where $n > \binom{k+1}{2}$. By the induction hypothesis, there is a member $A \in \mathcal{F}$ such that $|A| = k$. Consider the family $\mathcal{F}' = \{F \in \mathcal{F} : F \cap A = \emptyset\}$. Obviously, $\mathcal{F}'$ satisfies the conditions on the ground set $[n]\backslash A$, and we have $|[n]\backslash A| > \binom{k+1}{2} - k = \binom{k}{2}$. Hence, we can apply the induction hypothesis to $\mathcal{F}'$ to find a set $B \in \mathcal{F}'$ of size k which is disjoint from A. Using the exchange property in Theorem 1, or the condition in the Remark, for the sets A and B, we can conclude that $\mathcal{F}$ has a member with $k + 1$ elements.

Now we show the tightness of Lemma 2.1. Let $X_1, \ldots, X_{k-1}$ be pairwise disjoint sets with $|X_i| = i$, for every i. Then $V = X_1 \cup \ldots \cup X_{k-1}$ is a set of $\binom{k}{2}$ elements. For $i = 1, \ldots, k-1$, define

$$\mathcal{F}_i = \{F \subseteq V : |F \cap X_j| = 0 \text{ for every } j < i \text{ and } |F \cap X_j| \leq 1 \text{ for every } j > i\}. \tag{4}$$

In the definition of $\mathcal{F}_i$, there is no restriction on the size of $F \cap X_i$. Let $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{k-1}$. Obviously, every member of $\mathcal{F}_i$ has at most $|X_i| + k - 1 - i = k - 1$ elements, which yields that $\text{rk}(\mathcal{F}) = \max_{i=1}^{k} \text{rk}(\mathcal{F}_i) = k - 1$. Furthermore, $\mathcal{F}$ is atomic and any two disjoint members of $\mathcal{F}$ satisfy the exchange condition in Theorem 1. Hence, the lemma is tight. □

We remark that the maximal sets in the above $\mathcal{F}$ form the same hypergraph as the one defined in Example 3 of [4] for $v = 1$.

To prove the inequality $\text{rk}(\mathcal{F}) \geq \lfloor\sqrt{2n}\rceil$ in part (ii) of Theorem 1, we have to find the largest k for which we can apply Lemma 2.1. It is easy to verify by direct computation that

$$\max\{k : \binom{k}{2} < n\} = \lfloor\sqrt{2n}\rceil.$$

If $n = \binom{k}{2}$ for some $k \geq 1$, then the tightness of part (ii) of Theorem 1 follows from the tightness of Lemma 2.1. Suppose next that $\binom{k}{2} < n < \binom{k+1}{2}$. Let $X_1, \ldots, X_k$ be pairwise disjoint sets with $|X_i| = i$ for every $i < k$ and let $|X_k| = n - \binom{k}{2}$. Set $V = X_1 \cup \ldots \cup X_k$. For $i = 1, \ldots, k$, define $\mathcal{F}_i$ as in (4), and let $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$. Then $\mathcal{F}$ has the exchange property and $\text{rk}(\mathcal{F}) = k = \lfloor\sqrt{2n}\rceil$. This proves part (ii) of Theorem 1.

It remains to establish part (i). Let $\mathcal{F}$ be a family of subsets of $[n]$ satisfying the conditions. Let $\mathcal{F}'$ denote the *k-uniform hypergraph* (i.e., family of k-element sets) consisting of all sets F'.

The *independence number* $\alpha(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the maximum cardinality of a subset of its ground set which contains no element (hyperedge) of $\mathcal{H}$. It follows from Lemma 2.1 that any subset $S \subseteq [n]$ of size $|S| = \binom{k}{2} + 1$ contains at least one element of $\mathcal{F}$ whose size is k. Therefore, any such set contains at least one element of $\mathcal{F}'$, which means that $\alpha(\mathcal{F}') \leq \binom{k}{2}$.

We need a result of Katona, Nemetz, and Simonovits [3] which is a generalization of Turán's theorem to k-uniform hypergraphs.

Lemma 2.2. [3] *Let $\mathcal{H}$ be a k-uniform hypergraph on an n-element ground set. If the independence number of $\mathcal{H}$ is at most α, then we have*

$$|\mathcal{H}| \geq \binom{n}{k} \Big/ \binom{\alpha}{k}.$$

Applying Lemma 2.2 to the hypergraph $\mathcal{H} = \mathcal{F}'$ with $k = (\sqrt{2}e^{-1/\sqrt{2}} + o(1))\sqrt{n}$ and $\alpha = \binom{k}{2}$, we obtain

$$|\mathcal{F}| \geq |\mathcal{F}'| \geq e^{(2e^{-1/\sqrt{2}}+o(1))\sqrt{n}} \geq 2^{(1.42+o(1))\sqrt{n}},$$

completing the proof of part (i). This bound is slightly better than the inequality $|\mathcal{F}| \geq 2^{\lfloor\sqrt{2n}\rceil}$, which immediately follows from part (ii), under the stronger assumption that $\mathcal{F}$ is downward closed.

3 Proof of Theorem 2

Let $\mathcal{F}$ be an atomic set-system on an n-element ground set X, where n is large, and let s and t be two positive integers to be specified later. We describe a procedure to identify $\sum_{i=0}^{t} s^i$ distinct members of $\mathcal{F}$. To explain this procedure, we fix an s-ary tree T of depth t. At the end, each of the s^t root-to-leaf paths in T will correspond to a unique member of $\mathcal{F}$.

Each *non-leaf vertex* v will be associated with an s-element subset $X(v) \subset X$ such that along every root-to-leaf path $p = v_0v_1 \ldots v_t$, the sets $X(v_0), X(v_1), \ldots, X(v_{t-1})$, associated with the root and with the internal vertices of p, will be pairwise disjoint. See Figure 1 for an example.

Each *edge* $e = vu$ of T will be labelled with an element $x(e) \in X(v)$, in such a way that every edge from v to one of its s children gets a different label. Thus,

$$\{x(vu) : u \text{ is a child of } v\} = X(v).$$

Denoting the root by v_0, we choose $X(v_0)$ to be an arbitrary s-element subset of the ground set X, and set $F(v_0) = \emptyset \in \mathcal{F}$. For any non-root vertex v, let

$$F(v) = \{x(e) : e \text{ is an edge along the root-to-}v \text{ path}\}.$$

We will choose $X(v)$ such that $F(v) \in \mathcal{F}$ for every v. All of the sets $F(v)$ will be distinct, as any two different paths starting from the root diverge somewhere, unless one contains the other.

Suppose that we have already determined the set $X(u)$ for all ancestors of some non-leaf vertex v at level $\ell < t$ of T. At this point, we already know the set $F(v) \in \mathcal{F}$, where $|F(v)| = \ell$, and we want to determine $X(v)$. The next lemma guarantees that there is a good choice for $X(v)$.

Lemma 3.1. *There is an s-element subset $X(v) \subset X$ such that for every $x \in X(v)$, we have $F(v) \cup \{x\} \in \mathcal{F}$.*

Proof. Let $Z = \cup\{X(u) : u \text{ lies on the root-to-}v \text{ path}, u \neq v\}$. If v is at level $\ell < t$, we have $|Z| = \ell s \leq (t-1)s$. Let

$$Y = \{y \in X \setminus Z : F(v) \cup \{y\} \notin \mathcal{F}\}.$$

In other words, Y consists of all elements of $X \setminus Z$ that cannot be added to $F(v)$ to obtain a set in $\mathcal{F}$.

Consider the family $\mathcal{F}' = \{F \in \mathcal{F} : F \subseteq Y\}$. If $|Y| > \binom{t}{2}$, then Lemma 2.1 implies that there is a set $B \in \mathcal{F}'$ with $|B| = t > |F(v)|$. In this case, we can apply the exchange condition in Theorem

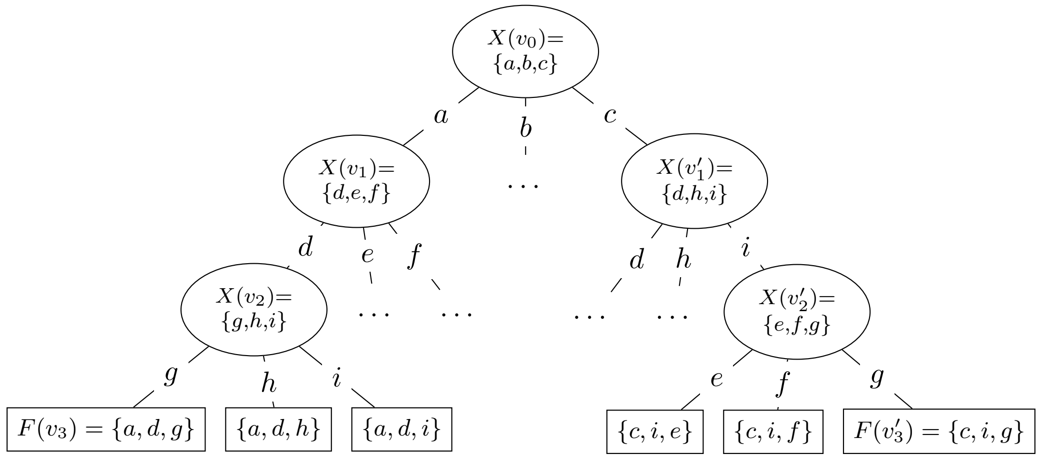

Figure 1: Construction of the auxiliary tree T for the proof of Theorem 2.

2 to the sets $F(v)$ and B, to conclude that there exists $b \in B$ for which $F(v) \cup \{b\} \in \mathcal{F}$. However, this contradicts the fact that $b \in Y$.

Thus, we can assume that $|Y| \leq \binom{t}{2}$. Now we have

$$|(X \setminus Z) \setminus Y| \geq n - (t-1)s - \binom{t}{2}.$$

If the right-hand side of this inequality is at least s, there is a proper choice for the set $X(v)$. For this, it is enough if $n \geq \frac{t^2}{2} + ts$, or, equivalently, $2 \geq (\frac{t}{\sqrt{n}})^2 + \frac{t}{\sqrt{n}}\frac{2s}{\sqrt{n}}$. To achieve this, let n be large, $s = \lfloor \sqrt{n}/\log^2 n \rfloor$, and $t = \lfloor (1 - 1/\log n)\sqrt{2n} \rfloor$. $\square$

By the above procedure, we can recursively assign a different set $F(v) \in \mathcal{F}$ to each vertex v of T. This gives

$$|\mathcal{F}| \geq \sum_{i=0}^{t} s^i \geq (n^{1/2}/\log^2 n)^{(1-o(1))\sqrt{2n}} = n^{\sqrt{(1/2+o(1))n}},$$

as desired.

4 Proof of Theorem 3

Assume for simplicity that n is a multiple of k, and fix a partition $[n] = X_1 \cup \ldots \cup X_{n/k}$ into n/k parts, each of size k. That is, let $|X_1| = \cdots = |X_{n/k}| = k$, where k is the largest number for which $2^{k-2} \leq n/k$; this gives $k = (1 + o(1)) \log n$. We will also assume $n, k \geq 3$.

For any $A \subset [n]$ and $0 \leq i \leq k$, let $p_A(i)$ and $s_A(i)$ denote the number of parts X_t which intersect A in *precisely* i elements and in *at least* i elements, respectively. Thus, we have $s_A(i) = \sum_{j=i}^{k} p_A(j)$

and $|A| = \sum_{i=1}^{k} i p_A(i) = \sum_{i=1}^{k} s_A(i)$. Define the *profile vector* of A, as

$$p_A = (p_A(k), p_A(k-1), \dots, p_A(0)),$$

and let

$$s_A = (s_A(k), s_A(k-1), \dots, s_A(0)).$$

That is, $p_A(0)$ is the number of parts that are disjoint from A, while $s_A(0)$ is always equal to n/k. We claim that the set-system

$$\mathcal{F} = \{A \subseteq [n] : s_A(k) \leq 1 \text{ and } s_A(i) \leq 2^{k-1-i} \text{ for every } 2 \leq i \leq k-1\}$$

meets the requirements of the theorem. Notice that for $i = 0$ and $i = 1$, there is no restriction on $s_A(i)$ other than the trivial bounds $0 \leq s_A(i) \leq n/k$. In particular, if A is an element of $\mathcal{F}$ with maximum cardinality, we have

$$s_A = (1, 1, 2^1, 2^2, 2^3, \dots, 2^{k-3}, n/k, n/k), \quad \text{and}$$

$$p_A = (1, 0, 1, 2^1, 2^2, \dots, 2^{k-4}, n/k - 2^{k-3}, 0).$$

Here, we used that $2^{k-3} \leq n/k$, by assumption. Thus, the size of such a largest set is

$$\text{rk}(\mathcal{F}) = |A| = \sum_{i=1}^{k} s_A(i) = n/k + \sum_{i=1}^{k-1} 2^{k-i-1} + 1 = n/k + 2^{k-2} \leq 2n/k = (2 + o(1))n/\log n.$$

This also gives the following simple bound on the size of the family.

$$|\mathcal{F}| \leq \sum_{i=0}^{\text{rk}(\mathcal{F})} \binom{n}{i} \leq \sum_{i=0}^{2n/k} \binom{n}{i} \leq \left(\frac{en}{2n/k}\right)^{2n/k} \leq \left(\frac{e \log n}{2}\right)^{(2+o(1))n/\log n} \leq 2^{(2+o(1))n \log\log n/\log n}.$$

To prove that $\mathcal{F}$ meets the requirements of the theorem, we consider any pair of disjoint sets $A, B \in \mathcal{F}$ such that there exists no $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. We need to show that in this case we have $|A| > |B|$. Suppose for contradiction that this is not the case, and fix a *counterexample* for which $|B| \geq |A|$ and $|B| - |A|$ is as large as possible. We refer to such a counterexample as a *maximal counterexample.* We say that $s_A(i)$ is *saturated* if $s_A(i) = 2^{k-1-i}$ for $i > 1$ and if $s_A(i) = n/k$ for $i = 1$.

Claim 4.1. *For any part $X \in \{X_1, \dots, X_{n/k}\}$, if $B \cap X \neq \emptyset$, then $A \cap X \neq \emptyset$. Hence, $s_B(k) = 0$. Moreover, if $|A \cap X| = i$, then $s_A(i+1)$ is saturated.*

Proof. Otherwise, we could add one element from $B \cap X$ to A, as $s_A(i+1)$ was not saturated. As A and B are disjoint, this also implies $s_B(k) = 0$. □

Claim 4.2. *There is a maximal counterexample $A, B \in \mathcal{F}$ such that, for every $X \in \{X_1, \dots, X_{n/k}\}$,*

(i) $A \cap X \neq \emptyset$;

(ii) *if* $|A \cap X| = 1$*, then* $B \cap X \neq \emptyset$.

Proof. Among all maximal counterexamples $A, B \in \mathcal{F}$, choose one for which $s_A(1)$ is as large as possible. Let $j \geq 1$ be the smallest positive integer for which there is a part X with $|A \cap X| = j$. Thus, we have $s_A(j) = s_A(j-1) = \ldots = s_A(1) \geq s_B(1)$, where the last inequality follows from Claim 4.1

Suppose first that $j > k/2$. Then we have

$$|A| = \sum_{i=1}^{n/k} |A \cap X_i| \; > \sum_{X_i \cap A \neq \emptyset} k/2 \; \geq \sum_{X_i \cap B \neq \emptyset} k/2 \; \geq \sum_{i=1}^{n/k} |B \cap X_i| = |B|,$$

where the inequality in the middle follows from Claim 4.1. Hence, in this case, the pair A, B did not constitute a counterexample.

From now on, we can assume $j \leq k/2$. If for any part X, we have $|A \cap X| = j$ and $B \cap X = \emptyset$, then $A' = A \setminus X$ and B would form a counterexample for which $|B| - |A'| > |B| - |A|$. Therefore, by the last statement of Claim 4.1, we can conclude that $s_A(j+1)$ is saturated.

Next, we show that $s_A(j)$ is also saturated. Indeed, otherwise we can pick a part X disjoint from A and B. (The existence of such a part follows from Claim 4.1 and our assumption that $n/k \geq 2^{k-3}$.) Add any $j \leq k/2$ elements of X to A and $j \leq k/2$ other elements of X to B, and denote the resulting sets by A' and B', so that $|B'| - |A'| = |B| - |A|$. In view of Claim 4.1, we have $s_{B'}(j) = s_B(j) + 1 \leq s_B(1) + 1 \leq s_A(1) + 1 = s_A(j) + 1 = s_{A'}(j)$. Thus, A' and B' belong to $\mathcal{F}$. To show that they constitute a counterexample, we need to prove that for every $b \in B'$, we have $A' \cup \{b\} \notin \mathcal{F}$. That is, if $b \in X'$ for some part X' and $|A' \cap X'| = i$, then we need to prove that $s_{A'}(i+1)$ is saturated. Since $i \geq j$, this can happen only if $i = j$. But $s_{A'}(j+1) = s_A(j+1)$ is saturated. Since $s_{A'}(1) > s_A(1)$, this contradicts the maximal choice of A.

Moreover, $j = 1$ must hold. Otherwise, pick a part X disjoint from A and B, and add any $j-1$ elements of X to A and $j-1$ other elements of X to B. Denote the resulting sets by A' and B', so that $|B'| - |A'| = |B| - |A|$. These new sets also belong to $\mathcal{F}$, as $s_A(j)$ was saturated. We get a contradiction again from $s_{A'}(1) > s_A(1)$. Since $s_A(1)$ is saturated, we have $s_A(1) = n/k$. This proves part (i).

To see (ii), it is enough to recall that if $|A \cap X| = 1$ and $B \cap X = \emptyset$, then $A' = A \setminus X$ and B would form a counterexample for which $|B| - |A'| > |B| - |A|$, contradicting the maximality of the pair A, B. □

Claim 4.3. *There is a maximal counterexample* $A, B \in \mathcal{F}$ *such that* $|B \cap X| > 1$ *implies* $|A \cap X| = 1$*, for every* $X \in \{X_1, \ldots, X_{n/k}\}$.

Proof. By definition, we have

$$s_B(2) \leq 2^{k-3} \leq n/k - 2^{k-3} \leq n/k - s_A(2) = p_A(1),$$

where the second inequality follows from our assumption $n/k \geq 2^{k-2}$.

Therefore, if $|B \cap X| > 1$ and $|A \cap X| > 1$ for some part X, then there exists another part X' for which $|B \cap X'| = 1$ and $|A \cap X'| = 1$. Choose any $|B \cap X| - 1$ elements of $B \cap X$, and remove them from B. Choose the same number of elements of $X' \setminus A$, and add them to B. By a repeated application of this procedure, we can achieve that the condition in the claim is satisfied. □

From now on, we consider a counterexample $A, B \in \mathcal{F}$ satisfying the condition in Claim 4.3. Let j be the smallest positive integer with the property that for every part X with $|A \cap X| = j$, we have $B \cap X = \emptyset$. (There always exists such a number, because $k-1$ meets the requirement. If there is no part X with $|A \cap X| = j$, then the property automatically holds.) By Claim 4.2 (ii), we know that $j > 1$.

For each part X intersecting A in more than j elements, remove $|A \cap X| - j$ elements of $A \cap X$ from A and remove all elements in $B \cap X$ from B. According to Claim 4.3, $|B| - |A|$ cannot decrease during this procedure, and we obtain another maximal counterexample for which Claims 4.2 and 4.3 remain valid.

Claim 4.4. *This counterexample satisfies the properties*

(i) $s_A(j+1) = 0$;
(ii) $p_B(0) \geq s_A(j)$;
(iii) $s_A(i) = 2^{k-1-i}$ *for every* $2 \leq i \leq j$.

Proof. Parts (i) and (ii) follow directly from the construction and Claim 4.2 (i), while (iii) follows from the last part of Claim 4.1. □

Now we can easily complete the proof of Theorem 3. We have

$$|A| = \sum_{i=1}^{k} s_A(i) = \sum_{i=2}^{j} 2^{k-1-i} + n/k.$$

On the other hand, $s_B(k) = 0$ holds, by Claim 4.1, and $s_B(1) \leq n/k - s_A(j) = n/k - 2^{k-1-j}$, by Claim 4.4 (iii). Thus,

$$|B| = \sum_{i=1}^{k} s_B(i) \leq n/k - 2^{k-1-j} + \sum_{i=2}^{k-1} 2^{k-1-i},$$

$$|A| - |B| \geq 2^{k-1-j} - \sum_{i=j+1}^{k-1} 2^{k-1-i} = 1.$$

This means that $|A| > |B|$, contradicting our assumption that $A, B \in \mathcal{F}$ is a counterexample.

5 Concluding remarks

If we strengthen the condition of our results by requiring that for any two non-empty disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \in \mathcal{F}$ *both* hold, then the problem becomes trivial. Any atomic set-system $\mathcal{F} \subset 2^{[n]}$ with this property must contain all subsets of $[n]$. Indeed, every set $F = \{x_1, \ldots, x_k\}$ can be built up, sequentially applying the condition to the sets $\{x_1, \ldots, x_i\}$ and $\{x_{i+1}\}$, for $i = 1, \ldots, k-1$.

In Theorems 1 and 2, we only assume that $\mathcal{F}$ is atomic. However, our best constructions have the stronger property that $\mathcal{F}$ is downward closed. Could we substantially strengthen these results under the stronger assumption? The proof of the bound $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rceil}$, which is only slightly weaker than Theorem 1 (i), becomes much easier if we assume that $\mathcal{F}$ is downward closed, and the proof of Theorem 2 can also be simplified if $\mathcal{F}$ is downward closed.

The property of the set-system described in Theorems 2 and 3 is reminiscent of the *independent set exchange property* of matroids; see [5]. A common generalization of these two properties would be to require that for any two members $A, B \in \mathcal{F}$, if either $|A| = |B|$ and $A \cap B = \emptyset$, or $|A| < |B|$ (but they are not necessarily disjoint), then there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. A downward closed set-system $\mathcal{F}$ has this property if and only if $\mathcal{F}$ is the family of independent sets in a matroid in which no subspace has two disjoint generators A and B, i.e., $A \cap B = \emptyset$ and $\mathrm{rk}(A) = \mathrm{rk}(B) = \mathrm{rk}(A \cup B)$ is forbidden. We do not know whether this question has been studied before.

Acknowledgment

We are grateful to Balázs Keszegh for his valuable remarks concerning Lemma 2.1.

References

[1] K. Adiprasito, S. Avvakumov, and R. Karasev: A subexponential size $\mathbb{RP}^n$, Combinatorica, online first, 2021.

[2] P. Duchet: Hypergraphs, in: Handbook of Combinatorics (R. Graham *et al.*, eds.), Elsevier Science, Amsterdam, 1995, pp. 381–432.

[3] G. Katona, T. Nemetz, and M. Simonovits: On a problem of Turán in the theory of graphs, Mat. Lapok 15 (1964), 228–238. (In Hungarian)

[4] L. Lovász: On minimax theorems of combinatorics. Mat. Lapok 26 (1975), no. 3–4, 209–264 (1978). (In Hungarian)

[5] J.G. Oxley: Matroid Theory. Oxford University Press, New York, 1992.