A RAMSEY-TYPE THEOREM FOR BIPARTITE GRAPHS

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Abstract

Let H be a fixed graph with k vertices. It is proved that every graph G with n vertices, which does not contain an induced subgraph isomorphic to H, has two disjoint sets of vertices, $V_1, V_2 \in V(G)$, such that $|V_1|, |V_2| \geq \lfloor (n/k)^{1/(k-1)} \rfloor$ and either all edges between V_1 and V_2 belong to G or none of them does. Some related geometric questions are also discussed.

1 Introduction

According to Ramsey's theorem [ES35], every graph G with n vertices has either a complete or an empty subgraph with at least $\frac{1}{2}\log_2 n$ vertices. Erdős and Hajnal [EH89] showed that a much stronger statement is true if we assume that G is H-*free*, i.e., it contains no *induced* subgraph isomorphic to a fixed graph H. In this case, one can guarantee the existence of a complete or an empty subgraph with $e^{c\sqrt{\log n}}$ vertices, where $c = c(H) > 0$ is a constant. They raised the possibility that this bound can be further improved to n^c. For some partial results is direction, see [G97], [APS99].

Although there is no strong evidence supporting the last conjecture, it is not hard to verify the following weaker statement.

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Theorem 1. *Let H be a fixed graph with k vertices.*

Any H-free graph with n vertices or its complement has a complete bipartite subgraph with $\lfloor (n/k)^{1/(k-1)} \rfloor$ vertices in its classes.

The proof of Theorem 1 is presented in Section 2. Essentially the same argument yields

Theorem 2. *Let H be a bipartite graph with vertex classes U_1 and U_2, $|U_1| = k \le |U_2| = l$, and let $n > l^{k+1}$.*

Then in any bipartite graph G with vertex classes V_1 and V_2, $|V_1| = |V_2| = n$, which contains no two subsets $U_1' \subseteq V_1, U_2' \subseteq V_2$ that induce an isomorphic copy of H, there exist $V_1' \subseteq V_1, V_2' \subseteq V_2$, $|V_1'| = |V_2'| = \lfloor (n/l)^{1/k} \rfloor$ such that either all edges between V_1' and V_2' belong to G or none of them does.

Given two tournaments, S and T, we say that T is *S-free* if S is not a subtournament of T.

Theorem 3. *Let S be a fixed tournament with k vertices.*

Any S-free tournament T with n vertices has two disjoint $\lfloor (n/k)^{1/(k-1)} \rfloor$-element subsets, $V_1, V_2 \subseteq V(T)$, such that every edge running between them is oriented towards its endpoint in V_2.

In Section 3, we discuss some related geometric problems.

2 Proof of Theorem 1

We prove a slightly stronger statement.

Theorem 2.1 *Let H be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_k\}$, and let G be a k-partite graph with t^{k-1}-element vertex classes, $V_1, V_2, \ldots, V_k$, for some $t, k \ge 2$. Suppose that no two different classes, V_i and V_j, contain two t-element subsets such that either all edges between them belong to G or none of them does.*

Then G has an induced subgraph isomorphic to H, whose vertex corresponding to v_i is in V_i, for every $i = 1, 2, \ldots, k$.

Note that it is sufficient to prove Theorem 2.1 in the special case when H is a *complete* graph. Otherwise, for every $i \ne j$ with $v_i v_j \notin E(H)$, replace in G the bipartite graph between V_i and V_j by its complement.

Thus, Theorem 2.1 follows by repeated application of the following

Lemma 2.2 *Let G be a k-partite graph with vertex classes, $V_1, V_2, \ldots, V_k$ of the same size, t^{k-1} $(t, k \geq 2)$. Suppose that no two different classes, V_i and V_j, contain two t-element subsets such that none of the edges between them belong to G.*

Then there is a vertex $v_1 \in V_1$ which has at least t^{k-2} neighbors in each $V_i,\ i > 1$.

Proof: Suppose, to obtain a contradiction, that for every $v \in V_1$ there exists $i(v),\ 1 < i(v) \leq k$, such that v has at most $t^{k-2} - 1$ neighbors in $V_{i(v)}$. Since $t^{k-1}/(k-1) \geq t$, we can find an index $i > 1$ and a t-element subset $V_1' \subseteq V_1$ such that $i(v) = i$ for all $v \in V_1'$.

Let V_i' denote the set of all vertices in V_i *not* connected to any element in V_1'. Clearly, we have

$$|V_i'| \geq |V_i| - t(t^{k-2} - 1) = t.$$

Thus, V_1' and V_i' induce an empty subgraph in G, contradicting our assumption. $\square$

Let $v_1 \in V_1$ satisfy the conditions in Lemma 2.2. For every $i > 1$, choose a t^{k-2}-element subset $V_i^* \subseteq V_i$, all of whose vertices are connected to v_1. Applying Lemma 2.2 to the $(k-1)$-partite subgraph of G induced by $V_2^* \cup V_3^* \cup \ldots \cup V_k^*$, we find a point $v_2 \in V_2^*$ with at least t^{k-3} neighbors in each $V_i^*,\ i > 2$, etc. The resulting sequence of vertices, $v_1, v_2, \ldots, v_k$, induces a complete subgraph in G. This completes the proof of Theorem 2.1 in the special case when H is a complete graph.

3 Geometric consequences and problems

Given a family $\mathcal{F}$ of arcwise connected sets in the plane, define its *intersection graph* $G(\mathcal{F})$ as a graph whose vertex set is $\mathcal{F}$ and in which two vertices are connected by an edge if and only if the corresponding sets have a nonempty intersection.

It is well known and easy to see [EET76], [PS00] that, as k tends to infinity, almost all graphs with k vertices *cannot* be obtained as (an induced subgraph of) the intersection graph of a family $\mathcal{F}$ of arcwise connected sets in the plane. Therefore, Theorem 1 immediately implies

Corollary 3.1. *There exists a constant $\varepsilon > 0$ such that every family $\mathcal{F}$ of arcwise connected sets in the plane has two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$ with at least n^ε members such that either every member of $\mathcal{F}_1$ intersects all members of $\mathcal{F}_2$ or no member of $\mathcal{F}_1$ intersects any member of $\mathcal{F}_2$.*

Note that in the special case when $\mathcal{F}$ consists of straight-line segments, the expression n^ε in the last statement can be replaced by εn (see [PS00]).

Fix an orthogonal (x, y, z) coordinate system in 3-space. A straight line is called *vertical* if it is parallel to the z-axis. Given two non-vertical skew lines, whose projections to the (x, y)-plane are not parallel, we can determine which one passes *above* the other. A family of pairwise skew, non-vertical lines is said to be in *general position* if among their projections to the (x, y)-plane no two are parallel.

Problem 3.2. Does there exist a positive constant ε such that every family $\mathcal{L}$ of n straight lines in general position in 3-space has $k \geq n^\varepsilon$ members, $l_1, l_2, \ldots, l_k$, such that l_i passes above l_j for all $i < j$?

Theorem 3 implies a somewhat weaker result.

Corollary 3.3. *There exists a positive constant ε such that every family $\mathcal{L}$ of n straight lines in general position in 3-space has two subfamilies $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{L}$ with at least n^ε members such that every member of $\mathcal{L}_1$ passes above all members of $\mathcal{L}_2$.*

Proof (sketch): Letting $\mathcal{L} = \{l_1, l_2, \ldots, l_n\}$, construct a tournament T on the vertex set $\mathcal{L}$ by drawing a directed edge from l_i to l_j if and only if l_i passes above l_j. It follows from a theorem of Erdős and Szekeres [ES35], [HM94] that there exists a function $f(k)$ tending to infinity such that any set of k lines in general position in the plane has an $f(k)$-element subset forming a convex chain (i.e., bounding an infinite convex polygon).

It is shown in [PPW93] that there exists no *weaving pattern* of 5 lines. That is, if e.g. the projections of $l_1, l_2, \ldots, l_5$ to the (x, y)-plane form a convex chain in this order, then they cannot induce an *ordered* tournament S_1 corresponding to the situation where each line passes alternately above and below the other 4 lines. Now we can apply a result of [APS99] to find another tournament, S_2, with the property that no matter how we order its vertices, it always has an ordered subtournament isomorphic to S_1. Finally, using a probabilistic argument, we can construct a tournament S with the property that every $f(|V(S)|)$-element subtournament of S contains a subsubtournament isomorphic to S_2.

It follows from the definitions that T is S-free. Therefore, we can apply Theorem 3 to finish the proof. $\square$

Problem 3.4. Does Corollary 3.3 remain true if we replace n^ε by εn?

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