

On the size of k -cross-free families

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Abstract

Two subsets A, B of an n -element ground set X are said to be *crossing*, if none of the four sets $A \cap B$, $A \setminus B$, $B \setminus A$ and $X \setminus (A \cup B)$ are empty. It was conjectured by Karzanov and Lomonosov forty years ago that if a family \mathcal{F} of subsets of X does not contain k pairwise crossing elements, then $|\mathcal{F}| = O_k(n)$. For $k = 2$ and 3 , the conjecture is true, but for larger values of k the best known upper bound, due to Lomonosov, is $|\mathcal{F}| = O_k(n \log n)$. In this paper, we improve this bound by showing that $|\mathcal{F}| = O_k(n \log^* n)$ holds, where \log^* denotes the iterated logarithm function.

1 Introduction

As usual, denote $[n] := \{1, \dots, n\}$ and let $2^{[n]}$ be the family of all subsets of $[n]$. Two sets $A, B \in 2^{[n]}$ are said to be *crossing*, if $A \setminus B$, $B \setminus A$, $A \cap B$ and $[n] \setminus (A \cup B)$ are all non-empty.

We say that a family $\mathcal{F} \subset 2^{[n]}$ is *k -cross-free* if it does not contain k pairwise crossing sets. The following conjecture was made by Karzanov and Lomonosov [12], [11] and later by Pevzner [14]; see also Conjecture 3 in [4], Section 9.

Conjecture 1. *Let $k \geq 2$ and n be positive integers, and let $\mathcal{F} \subset 2^{[n]}$ be a k -cross-free family. Then $|\mathcal{F}| = O_k(n)$.*

Here and in the rest of this paper, $f(n) = O_k(n)$ means that $f(n) \leq c_k n$ for a suitable constant $c_k > 0$, which may depend on the parameter k .

It was shown by Edmonds and Giles [8] that every 2-cross-free family $\mathcal{F} \subset 2^{[n]}$ has at most $4n - 2$ members. Pevzner [14] proved that every 3-cross-free family on an n -element underlying set has at most $6n$ elements, and Fleiner [9] established the weaker bound $10n$, using a simpler argument. For $k > 3$, Conjecture 1 remains open. The best known general upper bound for the size of a k -cross-free family is $O_k(n \log n)$, which can be obtained by the following elegant argument, due to Lomonosov.

Let $\mathcal{F} \subset 2^{[n]}$ be a maximal k -cross-free family. Notice that for any set $A \in \mathcal{F}$, the complement of A also belongs to \mathcal{F} . Thus, the subfamily

$$\mathcal{F}' = \{A \in \mathcal{F} : |A| < n/2\} \cup \{A \in \mathcal{F} : |A| = n/2 \text{ and } 1 \in A\}$$

contains precisely half of the members of \mathcal{F} . For every s , $1 \leq s \leq n/2$, any two s -element members of \mathcal{F}' that have a point in common, are crossing. Since \mathcal{F}' has no k pairwise crossing members, every element of $[n]$ is contained in at most $k - 1$ members of \mathcal{F}' of size s . Thus, the number of s -element members is at most $(k - 1)n/s$, and

$$|\mathcal{F}'| = |\mathcal{F}|/2 \leq 1 + \sum_{s=1}^{n/2} (k - 1)n/s = O_k(n \log n).$$

The main result of the present note gives the first improvement on this 40 years old bound. Let $\log_{(i)} n$ denote the function $\log \dots \log n$, where the \log is iterated i times, and let $\log^* n$ denote the *iterated logarithm* of n , that is, the largest positive integer i such that $\log_{(i)} n > 1$.

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Theorem 2. *Let $k \geq 2$ and n be positive integers, and let $\mathcal{F} \subset 2^{[n]}$ be a k -cross-free family. Then $|\mathcal{F}| = O_k(n \log^* n)$.*

Conjecture 1 has been proved in the following special case. Let \mathcal{F} be a k -cross-free family consisting of contiguous subintervals of the cyclic sequence $1, 2, \dots, n$. It was shown by Capoleas and Pach [5] that in this case

$$|\mathcal{F}| \leq 4(k-1)n - 2 \binom{2k-1}{2},$$

provided that $n \geq 2k - 1$. This bound cannot be improved.

A *geometric graph* G is a graph drawn in the plane so that its vertices are represented by points in general position in the plane and its edges are represented by straight-line segments between these points. Two edges of G are said to be *crossing* if the segments representing them intersect in an interior point.

Conjecture 3. *Let $k \geq 2$ and n be positive integers, and let G be a geometric graph with n vertices, containing no k pairwise crossing edges. Then the number of edges of G is $O_k(n)$.*

The result of Capoleas and Pach mentioned above implies that Conjecture 3 holds for geometric graphs G , where the points representing the vertices of G form the vertex set of a convex n -gon. It is also known to be true for $k \leq 4$; see [3], [1], [2]. For $k > 4$, it was proved by Valtr [16] that if a geometric graph on n vertices contains no k pairwise crossing edges then its number of edges is $O_k(n \log n)$ edges.

A bipartite variant of Conjecture 1 was proved by Suk [15]. He showed that if $\mathcal{F} \subset 2^{[n]}$ does not contain $2k$ sets A_1, \dots, A_k and B_1, \dots, B_k such that A_i and B_j are crossing for all $i, j \in [k]$, then $|\mathcal{F}| \leq (2k-1)^2 n$.

The notion of k -cross-free families was first introduced by Karzanov [11] in the context of multicommodity flow problems. Let $G = (V, E)$ be a graph, $X \subset V$. A *multiflow* f is a fractional packing of paths in G . We say that f locks a subset $A \subset X$ in G if the total value of all paths between A and $X \setminus A$ is equal to the minimum number of edges separating A from $X \setminus A$ in G . A family \mathcal{F} of subsets of X is called *lockable* if for every graph G with the above property there exists a multiflow f that locks every member $A \in \mathcal{F}$. The celebrated locking theorem of Karzanov and Lomonosov [12] states that a set family is lockable if and only if it is *3-cross-free*. This is a useful extension of the Ford-Fulkerson theorem for network flows, and it generalizes some previous results of Cherkasky [6] and Lovász [13]; see also [10].

2 The proof of Theorem 2

In this section, we prove our main theorem. Throughout the proof, floors and ceilings are omitted whenever they are not crucial, and \log stands for the base 2 logarithm. Also, for convenience, we shall use the following extended definition of binomial coefficients: if x is a real number and k is a positive integer,

$$\binom{x}{k} = \begin{cases} \frac{x(x-1)\dots(x-k+1)}{k!} & \text{if } x \geq k-1 \\ 0 & \text{if } x < k-1. \end{cases}$$

Let us remark that the function $f(x) = \binom{x}{k}$ is monotone increasing and convex.

A pair of sets, $A, B \in 2^{[n]}$, are said to be *weakly crossing*, if $A \setminus B$, $B \setminus A$ and $A \cap B$ are all non-empty. Clearly, if A and B are crossing, then A and B are weakly crossing as well. We call a set family $\mathcal{F} \subset 2^{[n]}$ *weakly k -cross-free* if it does not contain k pairwise weakly crossing sets.

As our first step of the proof, we show that if $\mathcal{F} \subset 2^{[n]}$ is a k -cross-free family, then we can pass to a weakly k -cross-free family $\mathcal{F}' \subset 2^{[n]}$ by losing a factor of at most 2 in the cardinality.

Lemma 4. *Let $\mathcal{F} \subset 2^{[n]}$ be a k -cross-free family. Then there exists a weakly k -cross-free family $\mathcal{F}' \subset 2^{[n]}$ such that $|\mathcal{F}'| \geq |\mathcal{F}|/2$.*

Proof. Let

$$\mathcal{F}' = \{A \in \mathcal{F} : 1 \notin A\} \cup \{[n] \setminus A : A \in \mathcal{F}, 1 \in A\}.$$

Clearly, we have $|\mathcal{F}'| \geq |\mathcal{F}|/2$.

Note that two sets $A, B \in [n]$ are crossing if and only if A and $[n] \setminus B$ are crossing. Hence, \mathcal{F}' does not contain k pairwise crossing sets. But no set in \mathcal{F}' contains 1, so we cannot have $A \cup B = [n]$ for any $A, B \in \mathcal{F}'$. Thus, $A, B \in \mathcal{F}'$ are crossing if and only if A and B are weakly crossing. Hence, \mathcal{F}' satisfies the conditions of the lemma. \square

Now Theorem 2 follows trivially from the combination of Lemma 4 and the following theorem.

Theorem 5. *Let $k \geq 2$ and n be positive integers and let $\mathcal{F} \subset 2^{[n]}$ be a weakly k -cross-free family. Then $|\mathcal{F}| = O_k(n \log^* n)$.*

The rest of this section is devoted to the proof of this theorem. Let us briefly sketch the idea of the proof while introducing some of the main notation.

Let \mathcal{F} be a weakly k -cross-free family. First, we shall divide the elements of \mathcal{F} into $\log n$ parts according to their sizes: for $i = 0, \dots, \log n$, let $\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \leq 2^{i+1}\}$. We might refer to the families \mathcal{F}_i as *blocks*. Next, we show that, as the block \mathcal{F}_i is weakly k -cross-free, it must have the following property: a positive proportion of \mathcal{F}_i can be covered by a collection of chains Γ_i with the maximal elements of these chains forming an antichain. These chains are going to be the objects of main interest in our proof.

We show that if \mathcal{F} has too many elements, then we can find k chains $\mathcal{C}_1 \subset \Gamma_{i_1}, \dots, \mathcal{C}_k \subset \Gamma_{i_k}$ for some $i_1 < \dots < i_k$, and k elements $C_{j,1} \subset \dots \subset C_{j,k}$ in each chain \mathcal{C}_j such that $C_{j,l} \subset C_{j',l'}$ if $j \leq j'$ and $l \leq l'$, and $C_{j,l}$ and $C_{j',l'}$ are weakly crossing otherwise. But then we arrive to a contradiction since the k sets $C_{1,k}, C_{2,k-1}, \dots, C_{k,1}$ are pairwise weakly crossing.

Now let us show how to execute this argument precisely.

Proof of Theorem 5. Without loss of generality, we can assume that \mathcal{F} does not contain the empty set and 1-element sets, since by deleting them we decrease the size of \mathcal{F} by at most $n + 1$.

Let us remind the reader of the definition of blocks: for $i = 0, 1, \dots, \log n - 1$, we have

$$\mathcal{F}_i := \{X \in \mathcal{F} : 2^i < |X| \leq 2^{i+1}\}.$$

The next claim gives an upper bound on the size of an antichain in \mathcal{F}_i .

Claim 6. *If $\mathcal{A} \subset \mathcal{F}_i$ is an antichain, then*

$$|\mathcal{A}| \leq \frac{(k-1)n}{2^i}.$$

Proof. Suppose that there exists $x \in [n]$ such that x is contained in k sets from \mathcal{A} . Then these k sets are pairwise weakly crossing. Hence, every element of $[n]$ is contained in at most $k - 1$ of the sets in \mathcal{A} , which implies that

$$(k-1)n \geq \sum_{A \in \mathcal{A}} |A| \geq |\mathcal{A}| 2^i.$$

\square

In the next claim, we show that a positive proportion of \mathcal{F}_i can be covered by chains whose maximal elements form an antichain. We shall use the following notation concerning chains. If \mathcal{C} is a chain of size l , then we denote its elements by $\mathcal{C}(1) \subset \dots \subset \mathcal{C}(l)$. Accordingly, let $\min \mathcal{C} = \mathcal{C}(1)$ and $\max \mathcal{C} = \mathcal{C}(l)$.

Claim 7. *For every $i \geq 0$, there exists a collection Γ_i of chains in \mathcal{F}_i such that $\{\max \mathcal{C} : \mathcal{C} \in \Gamma_i\}$ is an antichain and*

$$\sum_{\mathcal{C} \in \Gamma_i} |\mathcal{C}| \geq \frac{|\mathcal{F}_i|}{k-1}.$$

Proof. Let \mathcal{M} be the family of maximal elements of \mathcal{F}_i with respect to containment. For each $M \in \mathcal{M}$, let $\mathcal{H}_M \subset \mathcal{F}_i$ be a family of sets contained in M such that the system $\{\mathcal{H}_M\}_{M \in \mathcal{M}}$ forms a partition of \mathcal{F}_i .

Note that any two sets in \mathcal{H}_M intersect, since every $A \in \mathcal{H}_M$ satisfies $A \subset M$ and $|A| > |M|/2$. Hence, \mathcal{H}_M cannot contain an antichain of size k , otherwise, these k sets would be pairwise weakly crossing. Therefore, by Dilworth's theorem [7], \mathcal{H}_M contains a chain \mathcal{C}_M of size at least $|\mathcal{H}_M|/(k-1)$. The collection $\Gamma_i = \{\mathcal{C}_M : M \in \mathcal{M}\}$ meets the requirements of the claim. \square

Let Γ_i be a collection of chains in \mathcal{F}_i guaranteed by Claim 7. Since the maximal elements of the chains in Γ_i form an antichain, Claim 6 gives the following upper bound on the size of Γ_i :

$$|\Gamma_i| \leq \frac{(k-1)n}{2^i}. \quad (1)$$

From now on, fix some positive real numbers a, b with $a \leq b < \log n$ and consider the union of blocks $\mathcal{F}_{a,b} = \bigcup_{a < i \leq b} \mathcal{F}_i$. Analogously, let $\Gamma_{a,b} = \bigcup_{a < i \leq b} \Gamma_i$. Allowing a and b to be not necessarily integers will serve as a slight convenience. In what follows, we bound the size of $\mathcal{F}_{a,b}$.

For each chain $\mathcal{C} \in \Gamma_{a,b}$, define a set $Y(\mathcal{C})$ by picking an arbitrary element from each of the difference sets $\mathcal{C}(j+1) \setminus \mathcal{C}(j)$ for $j = 1, \dots, |\mathcal{C}| - 1$, and from \mathcal{C}_1 , as well. Clearly, we have $|Y(\mathcal{C})| = |\mathcal{C}|$. For every $y \in [n]$, let $d(y)$ be the number of chains \mathcal{C} in $\Gamma_{a,b}$ such that $y \in Y(\mathcal{C})$. Note that

$$\sum_{y \in [n]} d(y) = \sum_{\mathcal{C} \in \Gamma_{a,b}} |Y(\mathcal{C})| = \sum_{\mathcal{C} \in \Gamma_{a,b}} |\mathcal{C}| \geq \frac{|\mathcal{F}_{a,b}|}{k-1}, \quad (2)$$

where the last inequality holds by Claim 7.

We will bound the size of $\mathcal{F}_{a,b}$ by arguing that one cannot have k different elements of $[n]$ appearing in $Y(\mathcal{C})$ for many different chains $\mathcal{C} \in \Gamma_{a,b}$ without violating the condition that \mathcal{F} is weakly k -cross-free. Thus, $\sum_{y \in [n]} d(y)$ must be small. For this, we need the following definition.

Definition 8. Let $y \in [n]$. Consider a k -tuple of chains $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ in $\Gamma_{a,b}$, where $\mathcal{C}_i \in \Gamma_{j_i}$ for a strictly increasing sequence $j_1 < \dots < j_k$. We say that $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is good for y if

- (i) $y \in Y(\mathcal{C}_i)$ for $i \in [k]$,
- (ii) if $C_i \in \mathcal{C}_i$ is the smallest set such that $y \in C_i$, then $C_1 \subset \dots \subset C_k$.

Next, we show that if $d(y)$ is large, then y is good for many k -tuples of chains. Let $g(y)$ denote the number of good k -tuples for y .

Claim 9. For every $y \in [n]$, we have

$$g(y) \geq \binom{d(y)/(k-1)^2}{k}.$$

Proof. Let $d = d(y)$ and let $\mathcal{C}_1, \dots, \mathcal{C}_d \in \Gamma_{a,b}$ be the chains such that $y \in Y(\mathcal{C}_i)$. Also, for $i = 1, \dots, d$, let C_i be the smallest set in \mathcal{C}_i containing y , and let $\mathcal{H} = \{C_1, \dots, C_d\}$.

The family \mathcal{H} is intersecting. Therefore, it cannot contain an antichain of size k , as any two elements of such an antichain are weakly crossing. Applying Dilworth's theorem [7], we obtain that \mathcal{H} contains a chain of size at least $s = \lceil d/(k-1) \rceil$. Without loss of generality, let $C_1 \subset \dots \subset C_s$ be such a chain.

For any $a \leq i \leq b$, \mathcal{F}_i contains at most $k-1$ members of the sequence C_1, \dots, C_s . Otherwise, if $\mathcal{C}_{j_1}, \dots, \mathcal{C}_{j_k} \in \Gamma_i$ for some $1 \leq j_1 < \dots < j_k \leq s$, the maximal elements $\max \mathcal{C}_{j_1}, \dots, \max \mathcal{C}_{j_k}$ are pairwise weakly crossing, because these sets form an antichain and contain y .

This implies that the sets C_1, \dots, C_s are contained in at least $r = \lceil s/(k-1) \rceil \geq d/(k-1)^2$ different blocks. Thus, we can assume that there exist $i_1 < \dots < i_r$ and $j_1 < \dots < j_r$ such that $C_{i_l} \in \mathcal{F}_{j_l}$ for $l \in [r]$.

Then any k -element subset of $\{C_{i_1}, \dots, C_{i_r}\}$ is a good k -tuple for y , resulting in at least

$$\binom{r}{k} \geq \binom{d(y)/(k-1)^2}{k}$$

good k -tuples for y . □

Now we give an upper bound on the total number of k -tuples that may be good for some $y \in [n]$. A k -tuple of chains in $\Gamma_{a,b}$ is called *nice* if it is good for some $y \in [n]$. Let N be the number of nice k -tuples.

Claim 10. We have

$$N < \frac{2(k-1)^k n}{2^a} \binom{b}{k-1}.$$

Proof. Let $\mathcal{C} \in \Gamma_{a,b}$. Let us count the number of nice k -tuples $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ for which $\mathcal{C} = \mathcal{C}_1$. Note that in a nice k -tuple $(\mathcal{C}_1, \dots, \mathcal{C}_k)$, the set $\min \mathcal{C}_1$ is contained in $\max \mathcal{C}_1, \dots, \max \mathcal{C}_k$.

But then, for any positive integer i satisfying $a < i \leq b$, there are at most $k - 1$ chains in Γ_i that can belong to a nice k -tuple with first element \mathcal{C} . Indeed, suppose that there exist k chains $\mathcal{D}_1, \dots, \mathcal{D}_k$ in Γ_i that all appear in a nice k -tuple with their first element being \mathcal{C} . Then $\{\max \mathcal{D}_1, \dots, \max \mathcal{D}_k\}$ is an intersecting antichain: it is intersecting because $\max \mathcal{D}_j$ contains $\min \mathcal{C}$ for $j \in [k]$, and it is an antichain, by the definition of Γ_i . Thus, any two sets among $\max \mathcal{D}_1, \dots, \max \mathcal{D}_k$ are weakly crossing, a contradiction.

Hence, the number of nice k -tuples $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ for which $\mathcal{C}_1 = \mathcal{C}$ is at most $\binom{b}{k-1} (k-1)^{k-1}$, as there are at most $\binom{b}{k-1}$ choices for $j_2 < \dots < j_k \leq b$ such that $\mathcal{C}_l \in \Gamma_{j_l}$ for $l = 2, \dots, k$, and there are at most $k - 1$ further choices for each chain \mathcal{C}_l in Γ_{j_l} .

Clearly, the number of choices for $\mathcal{C} = \mathcal{C}_1$ is at most the size of $\Gamma_{a,b}$, which is

$$|\Gamma_{a,b}| = \sum_{a < i \leq b} |\Gamma_i| \leq \sum_{a < i \leq b} \frac{(k-1)n}{2^i} < \frac{2(k-1)n}{2^a};$$

see (1) for the first inequality. Hence, the total number of nice k -tuples is at most

$$\frac{2(k-1)^k n}{2^a} \binom{b}{k-1}.$$

□

The next claim is the key observation in our proof. It tells us that a k -tuple of chains cannot be good for k different elements of $[n]$.

Claim 11. *There are no k different elements $y_1, \dots, y_k \in [n]$ and a k -tuple $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ such that $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is good for y_1, \dots, y_k .*

Proof. Suppose that there exist such a k -tuple $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ and k elements y_1, \dots, y_k . For $i, j \in [k]$, let $C_{i,j}$ be the smallest set in \mathcal{C}_i that contains y_j . By the definition of a good k -tuple, we have $C_{1,j} \subset \dots \subset C_{k,j}$ for $j \in [k]$. Also, the sets $C_{1,1}, \dots, C_{1,k}$ are distinct elements of the chain \mathcal{C}_1 , so, without loss of generality, we can assume that $C_{1,1} \subset C_{1,2} \subset \dots \subset C_{1,k}$.

First, we show that this assumption forces $C_{i,1} \subset \dots \subset C_{i,k}$ for all $i \in [k]$, as well. To this end, it is enough to prove that we cannot have $C_{i,j'} \subset C_{i,j}$ for some $1 \leq j < j' \leq k$. Indeed, suppose that $C_{i,j'} \subset C_{i,j}$. Then $y_j \in C_{i,j}$, but $y_j \notin C_{i,j'}$. However, $y_j \in C_{1,j}$ and $C_{1,j} \subset C_{1,j'} \subset C_{i,j'}$, contradiction.

Next, we show that any two sets in the family

$$\mathcal{H} := \{C_{i,k+1-i} : i \in [k]\}$$

are weakly crossing. Every element of \mathcal{H} contains $C_{1,1}$, so \mathcal{H} is an intersecting family. Our task is reduced to showing that \mathcal{H} is an antichain. Take $C_{i,k+1-i}$ and $C_{i',k+1-i'}$ for some $i, i' \in [k]$, $i < i'$. Then $|C_{i,k+1-i}| < |C_{i',k+1-i'}|$, since $C_{i,k+1-i} \in \mathcal{F}_{j_i}$ and $C_{i',k+1-i'} \in \mathcal{F}_{j_{i'}}$ for some $j_i < j_{i'}$. Thus, $C_{i,k+1-i} \not\subset C_{i',k+1-i'}$. On the other hand, we have $y_{k+1-i} \in C_{i,k+1-i}$ and $y_{k+1-i} \notin C_{i',k+1-i'}$, so $C_{i,k+1-i} \not\subset C_{i',k+1-i'}$.

Therefore, any two sets of the k -element family \mathcal{H} are weakly crossing, which is a contradiction. □

Let M be the number of pairs $(y, (\mathcal{C}_1, \dots, \mathcal{C}_k))$ such that $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is a good k -tuple for $y \in [n]$. Let us double count M .

On one hand, Claim 11 implies that $M \leq (k-1)N$. Plugging in our upper bound of Claim 10 for N , we get

$$M \leq (k-1)N < \frac{2(k-1)^{k+1}n}{2^a} \binom{b}{k-1} \leq \frac{2n(k-1)^{k+1}b^{k-1}}{2^a(k-1)!}.$$

For simplicity, write $c_1(k) = 2(k-1)^{k+1}/(k-1)!$, then our inequality becomes

$$M \leq \frac{c_1(k)nb^{k-1}}{2^a}. \quad (3)$$

On the other hand, we have

$$M = \sum_{y \in [n]} g(y),$$

where $g(y)$, as before, stands for the number of good k -tuples for y . Applying Claim 9, we can bound the right-hand side from below, as follows.

$$\sum_{y \in [n]} g(y) \geq \sum_{y \in [n]} \binom{d(y)/(k-1)^2}{k}.$$

Exploiting the convexity of the function $\binom{x}{k}$, Jensen's inequality implies that the right-hand side is at least

$$n \binom{\sum_{y \in [n]} d(y)/(k-1)^2}{k}.$$

Finally, using (2), we obtain

$$M \geq n \binom{|\mathcal{F}_{a,b}|/(k-1)^3 n}{k}. \quad (4)$$

Suppose that $|\mathcal{F}_{a,b}| > 2k(k-1)^3 n$. In this case, we have

$$\binom{|\mathcal{F}_{a,b}|/(k-1)^3 n}{k} > \left(\frac{|\mathcal{F}_{a,b}|}{2(k-1)^3 n} \right)^k \frac{1}{k!}.$$

Writing $c_2(k) = 1/2^k(k-1)^{3k}k!$, we can further bound the right-hand side of (4) and arrive at the inequality

$$M > \frac{c_2(k) |\mathcal{F}_{a,b}|^k}{n^{k-1}}. \quad (5)$$

Comparing (3) and (5), we obtain

$$\frac{c_1(k)nb^{k-1}}{2^a} > \frac{c_2(k) |\mathcal{F}_{a,b}|^k}{n^{k-1}},$$

which yields the following upper bound for the size of $\mathcal{F}_{a,b}$:

$$|\mathcal{F}_{a,b}| < n \left(\frac{c_1(k)}{c_2(k)} \right)^{1/k} \frac{b^{(k-1)/k}}{2^{a/k}}.$$

Recall that (5) and the last inequality hold under the assumption that $|\mathcal{F}_{a,b}| > 2k(k-1)^3 n$. Hence, writing $c_3(k) = (c_1(k)/c_2(k))^{1/k}$, we get that

$$|\mathcal{F}_{a,b}| < \max \left\{ 2k(k-1)^3 n, \frac{c_3(k)nb^{(k-1)/k}}{2^{a/k}} \right\} \quad (6)$$

holds without any assumption.

We finish the proof by choosing an appropriate sequence $\{a_i\}_{i=0}^s$ and applying the bound (6) for the families $\mathcal{F}_{a_i, a_{i+1}}$.

Define the sequence $\{a_i\}_{i=0,1,\dots}$ such that $a_0 = 0$, $a_1 = k^2$ and $a_{i+1} = 2^{a_i/(k-1)}$ for $i = 1, 2, \dots$. Let s be the smallest positive integer such that $a_s > \log n$. Clearly, we have $s = O_k(\log^*(n))$. Also,

$$|\mathcal{F}_{a_0, a_1}| = |\mathcal{F}_{0, k^2}| \leq \sum_{l=1}^{2^{k^2}} \frac{(k-1)n}{l} = O_k(n),$$

as \mathcal{F} has at most $(k-1)n/l$ elements of size l for $l \in [n]$, by the weakly k -cross-free property. Finally, for $i = 1, \dots, s-1$, (6) yields that

$$|\mathcal{F}_{a_i, a_{i+1}}| < \max \left\{ 2k(k-1)^3 n, \frac{c_3(k)na_{i+1}^{(k-1)/k}}{2^{a_i/k}} \right\} = \max\{2k(k-1)^3, c_3(k)\}n.$$

The proof of Theorem 5 can be completed by noting that

$$|\mathcal{F}| = \sum_{i=0}^{s-1} |\mathcal{F}_{a_i, a_{i+1}}| \leq O_k(n) + s \max\{2k(k-1)^3, c_3(k)\}n = O_k(n \log^* n).$$

□

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$$g(y) \geq (k-1)^k \binom{d(y)/(k-1)^2}{k}.$$

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