Geometric Graphs and Hypergraphs

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Abstract

A *geometric hypergraph* H is a collection of i-dimensional simplices called *hyperedges* or, simply, *edges*, induced by some $(i+1)$-tuples of a *vertex set* V in general position in d-space. In the case $d = 2, i+1$, H is called a *geometric graph.* We survey some extremal problems on geometric graphs and hypergraphs.

1 From thrackles to geometric graphs

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and assume that it has no loops or multiple edges. A *drawing* of G is a representation of G in the plane such that every vertex corresponds to a point, and every edge is represented by a Jordan arc connecting the corresponding two points without passing through any other vertex. Two edges (arcs) are said to *cross* each other if they have an interior point p in common. For simplicity, we always assume that no three edges cross at the same point. A crossing p is called *proper* if in a small neighborhood of p one edge passes from one side of the other edge to the other side.

There are many well-known unsolved problems for graph drawings. For example, even for some very simple graphs G, we do not how to find the *crossing number* of G, i.e., the minimum number of crossing pairs of edges in a planar drawing of G. In the case when G is a complete bipartite graph, this is Turán's brick factory problem [26],[14]. The determination of the crossing number is known to be NP–complete [10].

Another well-known open problem that illustrates our ignorance about graph drawings was raised by Conway more than thirty years ago. He defined a *thrackle* as a drawing of a graph G with the property that any two distinct edges either

(i) share an endpoint, and then they do not have any other point in common; or

(ii) do not share an endpoint, in which case they meet exactly once and determine a proper crossing.

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Conjecture 1 *(Conway) The number of edges of a thrackle cannot exceed the number of its vertices.*

A graph that can be drawn as a thrackle is said to be *thrackleable*. Assuming that the above conjecture is true, Woodall [27] characterized all thrackleable graphs. With this assumption, a finite graph is thrackleable if and only if it has at most one odd cycle, it has no cycle of length four, and each of its connected components contains at most one cycle. Note that it is quite straightforward to check the necessity of these conditions. Using a construction suggested by Conway, the thrackle conjecture can be reduced to the following statement: If a graph G consists of two even cycles meeting in a single vertex then G is not thrackleable (cf. [27],[24]).

The best known upper bound on the number of edges of a thrackle is the following.

Theorem 1.1 *[19] Every thrackle of n vertices has at most $2n - 3$ edges.*

The proof is based on the following result.

Theorem 1.2 *[19] Every thrackleable bipartite graph is planar.*

It has been known for a long time that the thrackle conjecture is true for *geometric graphs*, i.e., for drawings where every edge is represented by a segment [15],[9],[21].

2 Turán-type results for geometric graphs

A graph drawn in the plane by possibly crossing straight-line segments is called a *geometric graph*. More precisely, a geometric graph G consists of a set of points (vertices) $V(G)$ in general position in the plane and a set of closed segments (edges) $E(G)$ whose endpoints belong to $V(G)$. If V is the vertex set of a convex n-gon, then G is called a *convex geometric graph*. We say that H is a *geometric subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The systematic study of geometric graphs was initiated by P. Erdős, Y. Kupitz [17], and M. Perles.

In the spirit of Turán's classical theorem for abstract graphs [25], one can raise the following general question. Given a class $\mathcal{H}$ of so-called *forbidden geometric subgraphs*, what is the maximum number of edges that a geometric graph of n vertices can have without containing a geometric subgraph belonging to $\mathcal{H}$? The result quoted in the last paragraph of the preceding section provides an answer to this question in the special case when the forbidden configuration consists of two disjoint edges, i.e., $\mathcal{H}$ is the class of all 4-point geometric graphs with two disjoint edges (that cannot share even an endpoint).

Theorem 2.1 *[23] If a geometric graph contains no k pairwise disjoint edges, then its number of edges cannot exceed $(k-1)^4 n$.*

It is an interesting problem to determine the best possible bound on the number of edges in the above theorem, at least for $k = 3$ (see [12]).

Encouraged by 2.1, one is tempted to conjecture that for every fixed k, the maximum number of edges of a geometric graph of n vertices, which does not contain k pairwise *crossing* edges, is $O(n)$. For $k = 2$, this follows from the fact that every *planar* graph with $n \geq 3$ vertices has at most $3n - 6$ edges. For $k = 3$, the conjecture has been verified in [1]. For larger values of k, the best known result is the following.

Theorem 2.2 *[22],[1] Let $k \geq 3$ be fixed. If a geometric graph of n vertices does not contain k pairwise crossing edges, then its number of edges is $O(n(\log n)^{2k-6})$.*

The analogous problem for *convex* geometric graphs has been completely settled in [4].

Theorem 2.3 *Let $n \geq 2k$. The maximum number of edges that a convex geometric graph of n vertices can have without containing k pairwise crossing edges, is $2(k-1)n - \binom{2k-1}{2}$.*

A non-crossing tree is called a *caterpillar* if it contains no three disjoint paths of length two starting from the same vertex. We close this section with an unpublished result of Perles.

Theorem 2.4 *If a geometric graph with n vertices has more than $\lfloor n(k-2)/2 \rfloor$ edges, then it contains a k-vertex caterpillar of every possible type.*

3 Ramsey-type results for geometric graphs

The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices. In classical Ramsey–theory, we want to find large monochromatic subgraphs in a complete graph whose edges are colored with several colors [3], [13]. Most questions of this type can be generalized to complete geometric graphs, where the monochromatic subgraphs are required to satisfy certain geometric conditions.

The following two statements were conjectured by A. Bialostocki and P. Dierker.

Theorem 3.1 *[16] If the edges of a finite complete geometric graph are colored by two colors, there exists a non-selfintersecting spanning tree, all of whose edges are of the same color.*

Theorem 3.2 *[16] If the edges of a complete geometric graph with $3n-1$ vertices are colored by two colors, there exist n pairwise disjoint edges of the same color.*

The analogues of Theorems 3.1 and 3.2 for *abstract* graphs, i. e., when the geometric constraints are ignored, were noticed by Erdős–Rado (see [8]) and Gerencsér–Gyárfás [11], respectively. In fact, Gerencsér and Gyárfás proved the stronger result that for any 2-coloring of the edges of a complete graph with $3n - 1$ vertices, there exists a monochromatic path of length $2n - 1$. This statement, as well as Theorem 3.2, is best possible, as is shown by the following example. Take the disjoint union of a complete graph of $n-1$ vertices and a complete graph of $2n-1$ vertices, all of whose edges are red and blue, respectively, and color all edges between the two parts red.

It seems that a key to obtaining strong Ramsey-type results for geometric graphs is the introduction of suitable ordering relations which store a lot of information about the geometric structure. The proofs of the last two theorems in this section are based on the following wellknown (and very easy) lemma of Dilworth.

Lemma 3.3 *[6] Any partially ordered set of size n^2+1 either has a totally ordered subset of size $n+1$ or contains $n+1$ pairwise incomparable elements.*

Theorem 3.4 *[16] If the edges of a complete geometric graph of n^2+1 vertices are colored by red and blue, one can find either a non-selfintersecting red path of length n or a non-selfintersecting blue path of length n.*

Theorem 3.5 *[18], [16] Let $r=r(n)$ denote the smallest positive integer with the property that any family of r closed segments in general position in the plane has n members that are either pairwise disjoint or pairwise crossing. We have*

$$n^{2.322} \leq r(n) \leq n^5.$$

4 Geometric hypergraphs

For many applications of geometric graph theory, we have to find the right generalizations of the above results to systems of surfaces or surface patches in d-space. For simplicity, we will only discuss the case when these surface patches are flat (simplices).

Definition 4.1 *A d-*dimensional geometric *r-*hypergraph *H_r^d is a pair (V,E), where V is a set of points in general position in $\Re^d$, and E is a set of closed $(r-1)$-dimensional simplices induced by some r-tuples of V. The sets V and E are called the* vertex set *and* edge set *of H_r^d, respectively.*

Akiyama and Alon [2] proved the following theorem. Let $V=V_1\cup\ldots\cup V_d$ ($|V_1|=\ldots=|V_d|=n$) be a dn-element set in general position in $\Re^d$, and let E consist of all $(d-1)$-dimensional simplices having exactly one vertex in each V_i. Then E contains n disjoint simplices. Combining this with a result of Erdős [7], we obtain a non-trivial upper bound for the number of edges of a d-dimensional geometric d-hypergraph of n vertices that contains no k pairwise disjoint edges.

If we want to exclude *crossings* rather than disjoint edges, we face the following problem. Even if we restrict our attention to systems of triangles induced by 3-dimensional point sets in general position, it is not completely clear how a "crossing" should be defined. If two segments cross, they do not share an endpoint. Should this remain true for triangles? We have to clarify the terminology.

Definition 4.2 *Two simplices are said to have a* non-trivial intersection, *if their relative interiors have a point in common. If, in addition, the two simplices are vertex disjoint, then they are said to* cross.

More generally, k simplices are said to have a non-trivial intersection, *if their relative interiors have a point in common. If, in addition, all simplices are vertex disjoint, then they are said to* cross.

Consider k simplices. It is important to note that the fact that *every pair* of them has a non-trivial intersection does not imply that *all* of them do. To emphasize that this stronger condition is satisfied, we often say that the simplices have a *non-trivial intersection in the strong sense*, or simply that they *strongly intersect*. Similarly, a set of *pairwise* crossing simplices is not necessarily *crossing*. If want to emphasize that they *all* cross, we will say that they *cross in the strong sense*, or shortly that they *strongly cross*.

Generalizing the basic problem of Section 2, one can try to determine the maximum number of edges that a d-dimensional geometric r-hypergraph H_r^d of n vertices can have without containing some fixed crossing pattern. The following results are taken from [5].

Theorem 4.1 *For any fixed $k > 1$, one can select at most $O(n^{\lceil d/2 \rceil})$ d-dimensional simplices induced by n points in d-space with the property that no k of them share a common interior point. This bound is asymptotically tight.*

Theorem 4.2 *Let E be any set of d-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$. If E has no two crossing elements, then $|E| = O(n^d)$, and this bound is asymptotically tight.*

Theorem 4.3 *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$ such that E has no k members with pairwise non-trivial intersections $(d, k > 1)$. Then, for $k = 2$ and 3, we have $|E| = O(n^{d-1})$. Otherwise, $|E| = O(n^{d-1} \log^{2k-6} n)$.*

It is an outstanding open problem to decide whether the order of magnitude of the above bound can be improved e.g. for $d = 4, k = 2$. However, one can show that the following related result is asymptotically tight.

Theorem 4.4 *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$. If E has no two crossing members, then $|E| = O(n^{d-1})$, and this bound cannot be improved.*

Theorem 4.5 *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$, where $d, k > 1$. If E has no k pairwise crossing members, then $|E| = O(n^{d-(1/d)^{k-2}})$.*

We close this paper with a generalization of 3.2.

Theorem 4.6 *Let us color with two colors all $(d-1)$-dimensional simplices induced by $(d+1)n - 1$ points in general position in $\Re^d$. Then one can always find n disjoint simplices of the same color. This result cannot be improved.*

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