Halving lines and perfect cross-matchings

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Abstract

It is shown that a set P of $2n$ points in general position in the plane admits a perfect matching with pairwise crossing segments if and only if it has precisely n halving lines. As a consequence, one can give a $O(n \log n)$-time algorithm which decides whether there exists such a matching in P and, if so, finds it.

1 Preliminaries

Let $P = \{p_1, p_2, \ldots, p_{2n}\}$ be a set of $2n$ points in the plane in general position, i.e., no three points are collinear. A line p_ip_j is said to be a *halving line* of P if both open half-planes bounded by p_ip_j contain precisely $n-1$ points. The number of halving lines of P is denoted by $h(P)$.

Taking an arbitrary line through any point of P and turning it around by at most 180 degrees, it always arrives at a position where it becomes a halving line. Thus, we have $h(P) \geq n$, and equality holds, e.g., when P is the vertex set of a convex $2n$-gon.

It is an intriguing open problem to determine the asymptotic behavior of $h(n) = \max_P h(P)$, where the maximum is taken over all $2n$-element sets in general position in the plane. It is known that

$$c_1 n \log n \leq h(n) \leq c_2 n^{4/3}$$

for suitable constants $c_1, c_2 > 0$ (see [L], [EL], [D]). This function plays an important role in the analysis of many algorithms in computational geometry (cf. [E]).

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We say that two segments *cross* if they have an interior point in common. Let $c(P)$ denote the maximum number of pairwise crossing segments $\overline{p_i p_j}$ whose endpoints belong to P. Obviously, $c(P) \leq n$ holds for every $2n$-element set P. If $c(P) = n$, we say that P has a *perfect cross-matching*. This is the case, for example, when P is the vertex set of a convex $2n$-gon.

Let $c(n) = \min_P c(P)$, where the minimum is taken over all $2n$-element sets in general position in the plane. We have

$$c(n) \geq c_3\sqrt{n},$$

for some positive constant c_3, but there is no sublinear upper bound known for $c(n)$ (see [A],[P]). In fact, in [A] it was shown that every $2n$-element set in general position has an at least $c_3\sqrt{n}$-element subset which not only admits a perfect cross-matching, but also satisfies a much stronger condition. In this strong sense the result is best possible [V]. It looked difficult to improve the lower bound on $c(n)$, because we had no good characterization of perfectly cross-matchable sets.

The aim of this note is to give such a good characterization and to design an efficient algorithm which decides whether a set admits a perfect cross-matching.

2 Characterization of perfectly cross-matchable sets

In this section, we would like to point out a simple relation between $c(P)$ and $h(P)$: the first quantity attains its maximum if and only if the second attains its minimum. More precisely, we have the following.

Theorem 1. *A set of $2n$ points in general position in the plane admits a perfect cross-matching if and only if it has precisely n halving lines.*

Proof: Suppose first that $P = \{p_1, p_2, \ldots, p_{2n}\}$ has a perfect cross-matching (i.e., n pairwise crossing segments) $\overline{p_{2i-1}p_{2i}}$, $1 \leq i \leq n$. The extension of each of these segments is a halving line, because each of them separates the two endpoints of all other segments $\overline{p_{2i-1}p_{2i}}$. We will show that P has no other halving lines.

Assume, in order to obtain a contradiction, that (say) p_1p_3 is also a halving line. We may suppose without loss of generality that p_1p_2 is horizontal, p_2 is to the right of p_1, and that p_{2i} is below and p_{2i-1} is above p_1p_2, for every $2 \leq i \leq n$. Since each segment $\overline{p_{2i-1}p_{2i}}$ $(3 \leq i \leq n)$ has to cross $\overline{p_1p_2}$, if $\overline{p_{2i-1}p_{2i}}$ has an endpoint to the left of p_1p_3, then its other endpoint must lie to the right of p_1p_3. However, both p_2 and p_4 are on the right-hand side of p_1p_3. This implies that the number of points to the right of p_1p_3 exceeds by at least 2 the number of points to the left of it, contradicting our assumption that p_1p_3 is a halving line.

Suppose next that P has precisely n halving lines. Since there is *at least* one halving line through every point p_k, we obtain that there must be *exactly* one. Thus, we can assume without loss of generality that the complete list of halving lines is $p_{2i-1}p_{2i}$ $(1 \leq i \leq n)$. We will show that the segments $\overline{p_{2i-1}p_{2i}}$ $(1 \leq i \leq n)$ are pairwise crossing.

Assume, for contradiction, that $\overline{p_1p_2}$ and $\overline{p_3p_4}$ have no interior point in common. By renumbering the points if necessary, we may also suppose that p_1p_2 is horizontal, p_2 is to the right of p_1, $\overline{p_3p_4}$ is entirely above the line p_1p_2, and that p_3 is closer to p_1p_2 than p_4 is. Notice that a minor counter-clockwise turn around p_3 will bring the line $\ell = p_3p_4$ into a position, where there are precisely n points on its right-hand side. (Indeed, p_4 will be added to the set of points to the right of p_3p_4.) If we continue to turn ℓ around p_3 in the counter-clockwise direction, we arrive at a position where ℓ becomes parallel to p_1p_2, i.e., it becomes horizontal. At that moment, there are at most $n-2$ points above ℓ (these points form a subset of the set of all points different from p_3 which lie above the halving line p_1p_2). Hence, there is an intermediate position $\ell = p_3p_k$ for some $k \neq 4$, in which the number of points on the right-hand side of ℓ is precisely $n-1$. This means that p_3p_k is a halving line which does not appear in the complete list of halving lines, $p_{2i-1}p_{2i}$ $(1 \leq i \leq n)$. Contradiction. □

Actually, this argument also yields the uniqueness of the perfect cross-matching.

Theorem 2. *Any set of points in general position in the plane admits at most one perfect cross-matching.*

Proof: As we have shown, every perfect cross-matching of P consists of exactly those segments between two points of P, whose extensions are *halving lines* of P. □

3 Algorithm

The above characterization of perfectly cross-matchable sets allows us to design an $O(n \log n)$-time algoritm which decides whether a set of $2n$ points satisfies this property and, if so, finds a perfect cross-matching for it.

Let P be a $2n$-element point set in general position in the plane, which is the union of two n-element sets, P_1 and P_2, separated by a straight line (say, by the y-axis). For any non-vertical line ℓ, let $P_i(\ell^+)$ (resp. $P_i(\ell^-)$)

denote the set of points in P_i lying above (resp. below) ℓ. A line ℓ not passing through any point of P is called a *ham-sandwich cut* for P, if

$$|P_1(\ell^+)| = |P_2(\ell^-)| = \lfloor n/2 \rfloor.$$

It was shown by Megiddo [M] that one can always find such a line ℓ in $O(n)$ steps (see also [LM]).

Any matching of P that has a segment to the left of the y-axis, has another one to the right of it, and these two segments cannot cross. Thus, if there exists a perfect cross-matching for P, then all of its segments must cross the y-axis and, similarly, they must also cross the ham-sandwich cut ℓ. Consequently, a perfect cross-matching M for P is the union of a perfect cross-matching M_1 for $P_1(\ell^+) \cup P_2(\ell^-)$ and a perfect cross-matching M_2 for $P_1(\ell^-) \cup P_2(\ell^+)$. Let M_i^+ and M_i^- denote the *upper envelope* and the *lower envelope* (i.e., the pointwise maximum and pointwise minimum) of the lines supporting the segments of M_i, respectively ($i = 1, 2$). Clearly, M_i^+ and M_i^- are unbounded convex polygonal paths, with at most $\lfloor n/2 \rfloor$ vertices each. (See Figure 1.)

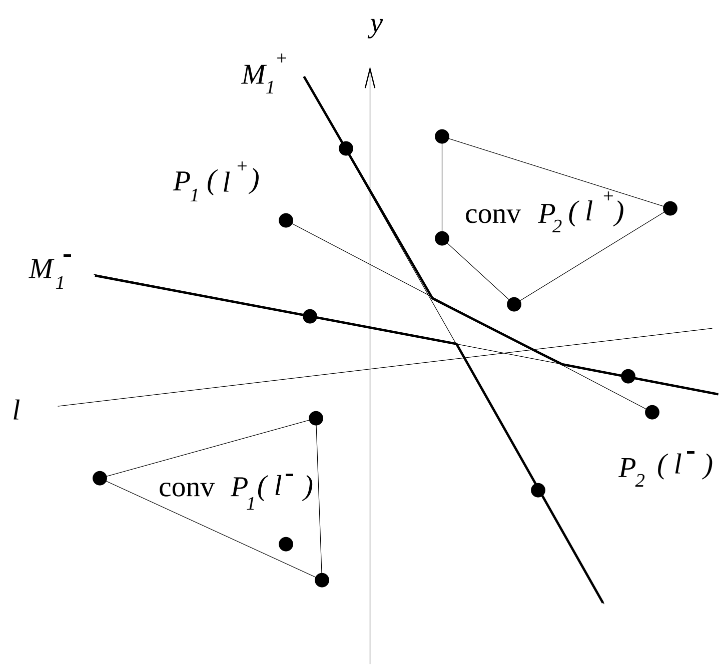

Figure 1.

We need the following corollary of Theorem 2.

Claim. The set P admits a perfect cross-matching M if and only if the following conditions are satisfied.

(1) $P_1(\ell^+) \cup P_2(\ell^-)$ admits a perfect cross-matching M_1 and $P_1(\ell^-) \cup P_2(\ell^+)$ admits a perfect cross-matching M_2.

(2) The convex hull $\text{conv}P_2(\ell^+)$ is above the polygonal path M_1^+, and $\text{conv}P_1(\ell^-)$ is below M_1^-. Similarly, $\text{conv}P_1(\ell^+)$ is above M_2^+, and $\text{conv}P_2(\ell^-)$ is below M_2^-.

Then, we have $M = M_1 \cup M_2$.

Proof. We have seen before that if P admits a perfect cross-matching M, then it satisfies condition (1) and $M = M_1 \cup M_2$ holds. By Theorem 2, M_1 and M_2 are uniquely determined. To see that (2) is necessary, too, assume that (say) $P_2(\ell^+)$ has a point p below M_1^+. Then p lies below the supporting line of at least one segment $\overline{qq'} \in M_1$. Let p' denote the element of $P_1(\ell^-)$ connected to p in M_2. Then $\overline{pp'} \cap \overline{qq'} = \emptyset$, contradicting our assumption that any two segments of M cross.

Suppose next that conditions (1) and (2) are satisfied. Then $M = M_1 \cup M_2$ is a perfect cross-matching for P. Indeed, if there were two disjoint segments, $\overline{pp'} \in M_1$ and $\overline{qq'} \in M_2$, such that (say) $\overline{qq'}$ is below (resp. above) the line pp', then $\text{conv}P_2(\ell^+)$ would not lie above the polygonal path M_1^+ (resp. $\text{conv}P_1(\ell^-)$ would not lie below M_1^-), contradicting condition (2). □

Let M^+ and M^- denote the upper and the lower envelope of all lines supporting the segments of $M = M_1 \cup M_2$, respectively. Clearly, M^+ can be obtained as the upper envelope of M_1^+ and M_2^+, and M^- can be obtained as the lower envelope of M_1^- and M_2^-.

It is well known that one can compute the union and the intersection of two convex polygons of at most n sides in time $O(n)$ ([PH], [S]). Thus, if we know $\text{conv}P_i(\ell^+), \text{conv}P_i(\ell^-), M_i^+$, and M_i^- for $i = 1, 2$, then in linear time we can determine $\text{conv}P_i (i = 1, 2), M^+$ and M^-. If any of the conditions of the Claim is not satisfied, we conclude that P does not admit a perfect cross-matching.

So one can use a divide-and-conquer algorithm to decide whether $P = P_1 \cup P_2$ admits a perfect cross-matching and, if yes, to compute it simultaneously with $\text{conv}P_i \ \ (i = 1, 2), M^+$ and M^-. At each stage it takes linear time to find a ham-sandwich cut ℓ and to do the merge step.

We obtained the following.

Theorem 3. *There is an $O(n \log n)$ time, $O(n)$ space algorithm which decides whether a set of $2n$ points in general position in the plane admits a perfect cross-matching and, if so, computes it.*

Clearly, any decision tree that determines the perfect cross-matching of a planar point set of $2n$ points (if it exists) has height $\Omega(n \log n)$. In this sense Theorem 3 is asymptotically tight.

References

[A] B. Aronov, P. Erdős, W. Goddard, D.J. Kleitman, M. Klugerman, J. Pach, and L.J. Schulman, Crossing families, *Combinatorica* **14** (1994), 127–134.

[D] T. Dey, Improved bounds for k-sets, *Discrete and Computational Geometry,* to appear.

[E] H. Edelsbrunner, *Algorithms in Combinatorial Geometry,* Springer-Verlag, New York, 1987.

[EL] P. Erdős, L. Lovász, A. Simmons, and E. Straus, Dissection graphs of planar point sets, in: *A Survey of Combinatorial Theory* (G. Srivastava, ed.), North-Holland, Amsterdam, 1973, 139–149.

[LM] Chi-Yuan Lo, J. Matoušek, and W. Steiger, Algorithms for ham-sandwich cuts, *Discrete and Computational Geometry* **11** (1994), 433–452.

[L] L. Lovász, On the number of halving lines, *Annales Universitatis Scientarium Budapest, Eötvös, Sectio Mathematica* **14** (1971), 107–108.

[M] N. Megiddo, Partitioning with two lines in the plane, *J. Algorithms* **6** (1985), 430–433.

[P] J. Pach and P.K. Agarwal, *Combinatorial Geometry,* J. Wiley & Sons, New York, 1995.

[PH] F.P. Preparata and S.J. Hong, Convex hulls of finite sets of points in two and three dimensions, *Communications ACM* **20** (1977), 87–93.

[S] M.I. Shamos, *Computational Geometry, Ph.D. Thesis,* UMI No. 7819047, Yale University, New Haven, CT, 1978.

[V] P. Valtr, On mutually avoiding sets, in: *The Mathematics of Paul Erdős, II (R. L. Graham and J. Nešetřil, eds.), Algorithms and Combin.* **14**, Springer, Berlin, 1997, 324–332.