Directions in Combinatorial Geometry

János Pach[*]
Courant Institute, New York University
and Hungarian Academy of Sciences
pach@cims.nyu.edu

By indirections find directions out.
(William Shakespeare: Hamlet)

Abstract

This mini-survey concentrates on some recent developments in combinatorial geometry related to the distribution of directions determined by a finite point set. It is based on the material of my invited address at the Jahrestagung der Deutschen Mathematiker Vereinigung in Rostock on September 19, 2003.

1 Introduction, apology, directions, incidences

I would not dare to hazard any judgment or prediction concerning the most important *directions* of research in combinatorial geometry. During the past couple of decades the subject has gone through a growth spurt that is far from being over. It it very difficult to identify the most important trends. Many of the changes have been stimulated by the "geometrization" of other parts of mathematics and by the theoretical and practical demands of computer science and industry (including computer graphics, robotics, computer-aided design).

I will concentrate on a few open problems in discrete geometry related to the concept of "direction", used as a technical term. The *direction* determined by a pair of points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in the (affine or Euclidean) plane is the ratio $\frac{y_2-y_1}{x_2-x_1}$, that is, the slope of the line p_1p_2. Two pairs determine the same direction if the corresponding ratios coincide.

We get another possible interpretation of this concept, by completing the plane with a "line at infinity," ℓ_∞, and saying that two point pairs determine the same direction if

[*]Supported by NSF grant CR-00-98246, a PSC-CUNY Research Award, grants from OTKA and BSF.

their connecting lines intersect ℓ_∞ at the same point. In this latter context, it is apparent that the directions determined by a point set depend only on the structure of *incidences* between points and the lines. Problems of this type have been extensively investigated ever since Euclid proposed his system of axioms based entirely on these notions. Although the parallel postulate was scrutinized for well over two thousand years, and by the end of the nineteenth century projective geometry had become one of the most developed mathematical disciplines, a number of exciting simple questions concerning incidences were completely overlooked. One such question that Euclid would have certainly liked was asked by Sylvester [46] in 1893: is it true that any finite set P of points, not all in a line, determines at least one *ordinary line*, that is, a line passing through precisely two elements of P? Forty years later the question was rediscovered by Erdős and shortly thereafter answered by Gallai [25].

Sylvester–Gallai theorem. *Every noncollinear set of n points in the plane determines an ordinary line.*

In fact, the minimum number of ordinary lines determined by such a point set is known to be at least $\lceil \frac{6}{13}n \rceil$, for $n > 7$, but the conjectured minimum is $\lceil \frac{1}{2}n \rceil$, if n is sufficiently large [15], [13], [8].

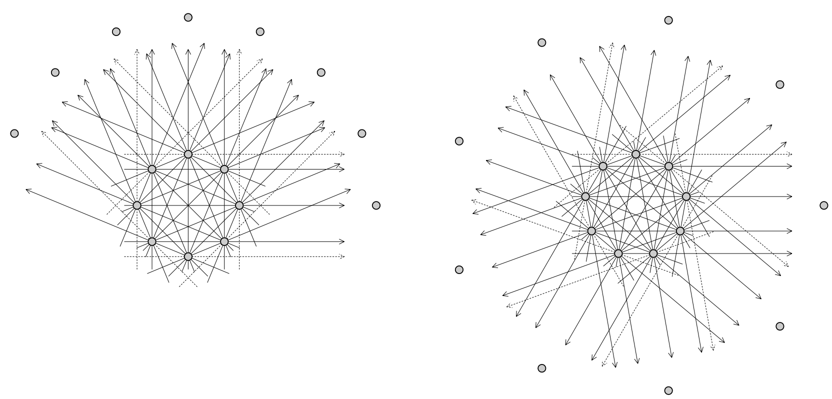

Fig. 1. Sets of 16 and 18 points with 8 and 9 ordinary lines

Erdős pointed out the following immediate corollary of the Sylvester–Gallai theorem.

Corollary 1.1. *Any set of n noncollinear points in the plane determines at least n distinct connecting lines. Equality is attained if and only if all but one of the points are collinear.*

We can argue by induction. The corollary is trivially true for $n = 3$. Suppose that we have already verified it for $(n-1)$-element sets, where $n > 3$. Consider a noncollinear set P of n points. Let pq be an ordinary line, $p, q \in P$. At least one of the sets $P \setminus \{p\}$ or $P \setminus \{q\}$ is not collinear. Applying the induction hypothesis to this set, we conclude that it determines at least $n-1$ connecting lines, and all of them are different from pq. The cases of equality can be obtained by a similar argument.

The question naturally arises: can the corollary be strengthened to guarantee the existence of n connecting lines with distinct slopes? The answer is yes if n is even, as was conjectured by Scott [44] in 1970 and proved by Ungar [49] twelve years later.

Ungar theorem. *The minimum number of different directions assumed by the connecting lines of $n \geq 4$ noncollinear points in the plane is $2\lfloor n/2 \rfloor$.*

In contrast to Corollary 1.1, here there is an overwhelming diversity of extremal configurations, for which equality is attained. Four infinite families and more than one hundred sporadic configurations were catalogued by Jamison and Hill [35] (see also [34] for an excellent survey).

The main difficulty in studying the distribution of directions determined by a finite point set is that, although the problem is invariant under affine transformations of the plane, it seems likely that one has to analyze the *algebraic* relations between the slopes of the connecting lines. This would "smuggle" some *metric* elements into our investigations—and perhaps Euclid would be not so enthusiastic about such a development. We mention some algebraic aspects of these problems in Section 4 of this paper.

Ungar's brilliant proof uses the method of *allowable sequences*, invented by Goodman and Pollack [26], [27], for coding the angular information by a sequence of permutations. This enables him to translate the problem into a combinatorial one, and solve it in an elegant and much more general setting, for "pseudolines." This approach, suggested independently by Goodman and Pollack and by Cordovil [12], is outlined at the beginning of Section 2. In the rest of the section, we discuss a number of generalizations of Ungar's theorem, including a recent three-dimensional version, found by Pinchasi, Sharir, and myself [39], [40]. Section 3 contains some related results and open problems on repeated angles.

Over the years Erdős [19] raised a number of innocent looking questions on incidences between points and lines (or other curves) that turned out to be notoriously difficult. One of the first significant accomplishments in this respect was the proof of the following result conjectured by Erdős.

Szemerédi–Trotter theorem [47]. *The maximum number of incidences n points and l lines in the plane is $O(n^{2/3}l^{2/3}+n+l)$. The order of magnitude of this bound cannot be improved.*

The Szemerédi–Trotter theorem is one of the very few asymptotically tight results in this field. One may wonder why such a "natural" question on incidences did not occur to anyone, say, in the nineteenth century? I believe that the explanation is simple: no matter how natural these problems may sound today, they must have appeared quite "exotic" to "mainstream" mathematicians a hundred years ago, before combinatorial optimization became a separate subject.

In the past two decades research in this field has gained considerable momentum. The Szemerédi–Trotter theorem has found several applications in additive number theory [16], [17], [45], in Fourier analysis [32], [33], and in measure theory [2], [5], [51]. It is also related to Kakeya's problem [50]: A *Kakeya set* (or *Besicovitch set*) is a subset of $\mathbf{R}^d$ that contains a unit segment in every direction. Besicovitch was the first to construct such sets with zero measure. Kakeya's problem is to decide whether the Hausdorff dimension of a Kakeya set is always at least d. The planar version of this question was answered in the affirmative by Davies [14] and, in a stronger form, by Córdoba [12] and by Bourgain [6]. For $d \geq 3$, this is a major unsolved problem.

2 Allowable sequences, Ungar-type theorems

Fix a noncollinear set P of n points in the plane such that no two points have the same x-coordinate. Label the elements of P by $1, 2, \ldots, n$ in the order of increasing x-coordinates. Following Goodman and Pollack [26], [27], we define a circular sequence of permutations. We take a horizontal line ℓ and start turning it in the counterclockwise direction. In each position, we record the order of the orthogonal projections of the elements of P into ℓ. The original order is represented by the permutation $\pi = 12 \ldots n$. As we turn ℓ, changes occur in this permutation if and only if ℓ passes through a position perpendicular to one of the slopes determined by two (or more) points of P. In such a case, we obtain a new permutation π' that can be obtained from π by "flipping" some of its substrings: namely those corresponding to subsets of elements lying on parallel lines orthogonal to ℓ. Thus, as we turn ℓ through 180 degrees, the number of changes in the permutation will be equal to the number of different slopes determined by point pairs in P. Finally, we end up with the permutation $n, n-1, n-2, \ldots, 1$. If we continue turning ℓ, we obtain the same sequence of permutations as before, except that now each of them is reversed. After a full turn, we get back $\pi = 12 \ldots n$.

Ungar's idea was the following. Suppose n is even, and mark the middle of each permutation by an imaginary barrier separating the first $\frac{n}{2}$ elements from the last $\frac{n}{2}$. To estimate the number of permutations in the sequence, Ungar first classified the "moves" transforming one permutation into the next one. If a move involves flipping a string containing (resp. touching) the barrier, he called it a *crossing* (resp. *touching*) move. If a move is neither crossing nor touching, it is called *ordinary*. The basic observation is that between any two crossing moves there must be a touching one. Indeed, in a crossing move the order of the two elements on opposite sides of the barrier will change, and if the next nonordinary move is again a crossing move, then the order of these two elements would change back. However, as we turn ℓ through 180 degrees, the order of any two points can (and must) reverse only once. Another elegant argument allows us to give a lower bound on the number of ordinary moves between a touching move and a crossing move, leading to a proof of Ungar's theorem. In fact, the proof applies to a more general situation. Suppose that we have a sequence of permutations starting with $1, 2, \ldots, n$ and ending with $n, n-1, \ldots, 1$, with the property that the order of any two elements changes precisely once. In each move" we are allowed to flip a collection of nonoverlapping proper subsegments of the permutation. A sequence of permutations satisfying this condition is called an *allowable sequence*. It follows that the length of any allowable sequence on n elements is at least $2\lfloor \frac{n}{2} \rfloor$.

Scott [44] also conjectured that in three-dimensional space the minimum number of different directions assumed by the connecting lines of n points, not all in a plane, is $2n - O(1)$. For instance, if n is odd, consider the set obtained from the vertex set of a regular $(n-3)$-gon P_{n-3} (or from any other centrally symmetric extremal configuration for Ungar's theorem) by adding its center c and two other points whose midpoint is c and whose connecting line is orthogonal to the plane of P_{n-3}.

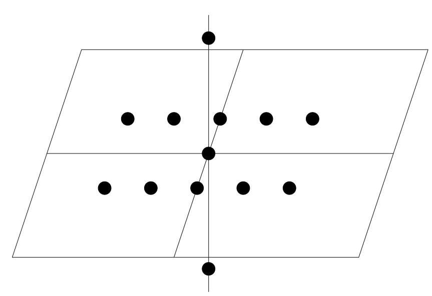

Fig. 2. n noncoplanar points in 3-space with $2n-5$ directions

At first glance it appears that Ungar's approach is doomed to fail in higher dimensions, because it is based on the linear (or rather the circular) ordering of all critical directions. This may well be the case in higher dimensions. However, somewhat surprizingly, Scott's three-dimensional conjecture can be settled by reducing it to a planar statement, which is a far-reaching generalization of Ungar's theorem.

Theorem 2.1 [40]. *Any noncoplanar set of $n \geq 6$ points in $\mathbf{R}^3$ determines at least $2n-5$ different directions if n is odd and at least $2n-7$ different directions if n is even. This bound is sharp for every odd n.*

Ungar's theorem can be rephrased as follows: from all closed segments whose endpoints belong to a noncollinear set of n points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ such that no two of them are parallel. To formulate our generalization of Ungar's result, we need to relax the condition of two segments being *parallel*.

Two *closed* segments in the plane (or in $\mathbf{R}^d$) are called *convergent* if

(1) they do not belong to the same line, and

(2) their supporting lines intersect, and their intersection point does not belong to either of the segments.

An alternative definition is that two segments are convergent if and only if they are disjoint and their convex hull is a nondegenerate planar quadrilateral. (Two parallel segments that lie on distinct lines are also considered convergent, by regarding their lines to meet at infinity.)

Theorem 2.2 [39]. *From all closed segments determined by a set of n noncollinear points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ pairwise nonconvergent ones, lying in distinct lines.*

It is easier to handle the d-dimensional problems ($d \geq 4$) under the assumption that no three points of the set are collinear. For this case, Blokhuis and Seress [4] conjectured that any set of n points determines at least $(d-1)n - d(d-2)$ distinct directions. For $d = 4$, this conjecture was verified in [40] up to an additive constant. Perhaps asymptotically the same bound holds under the weaker assumption that not all of the points lie in the same hyperplane.

Ungar's theorem states that every noncollinear point set in the plane determines many directions. Dirac [15] asked whether one can always find a *point* belonging to at least roughly $\frac{n}{2}$ connecting lines of distinct slopes.

Dirac's conjecture. *There is a constant c such that any set P of n points, not all on a line, has an element incident to at least $\frac{n}{2} - c$ lines spanned by P.*

Putting the same number of points on two lines shows that this bound, if true, is asymptotically tight. Many small examples listed by Grünbaum [28] show that the conjecture is false with $c = 0$. An infinite family of counterexamples was constructed by Felsner (personal communication). The "weak Dirac conjecture," first proved by Beck [3], states that there exists $\epsilon > 0$ such that one can always find a point incident to at least ϵn lines spanned by P. This statement also follows from the Szemerédi–Trotter theorem (see Section 1).

According to a beautiful result of Motzkin [37], Rabin, and Chakerian [10], any set of n noncollinear points in the plane, colored with two colors, *red* and *green*, determines a monochromatic line. Motzkin and Grünbaum [29] initiated the investigation of *biased* colorings, i.e., colorings without monochromatic red lines. Their motivation was to justify the intuitive feeling that if there are many red points in such a coloring and not all of them are collinear, then the number of green points must also be rather large. Denoting the sets of red and green points by R and G, respectively, it is a challenging unsolved question to decide whether the "surplus" $|R| - |G|$ of the coloring can be arbitrarily large. We do not know any example where this quantity exceeds six [30]. It is another important ingredient of the proof of Theorem 2.1 that under some special restrictions the surplus is indeed bounded.

The problem of biased colorings was rediscovered by Erdős and Purdy [22], who formulated it as follows. What is the smallest number $m(n)$ of points necessary to represent (i.e., stab) all lines spanned by n noncollinear points in the plane, if the generating points cannot be used? An $\Omega(n)$ lower bound follows imediately from the weak Dirac conjecture.

3 Repeated angles

In an important paper [18] published in the *American Mathematical Monthly*, Erd Hos asked the following twin questions. Consider a set P of n points in the plane (or in a higher-dimensional space).

(1) At most how many point pairs $\{p, q\} \subset P$ can determine the same distance?

(2) At least how many distinct distances must be determined by the point pairs in P?

In the same spirit, one can raise a number of interesting questions for *triples* of points. This line of research was initiated by Erdős and Purdy [20], [21].

(1') At most how many triples $(p, q, r) \subset P$ can determine the same angle?

(2') At least how many distinct angles must be determined by triples of points in P?

Concerning question (1'), Pach and Sharir [41] proved the following result.

Theorem 3.1. *For any $\gamma \in (0, \pi)$, there are at most $O(n^2 \log n)$ triples among n points in the plane that determine angle γ. Moreover, this order of magnitude is attained for a dense set of angles.*

We do not know whether this order of magnitude can indeed be reached for every γ. In three-dimensional space, Apfelbaum and Sharir [1] showed that among n points the same angle can occur at most $O(n^{\frac{7}{3}})$ times and that for right angles this bound can be attained. In this case, it is not even clear whether there exists any other angle for which the bound is asymptotically tight.

Purdy [42] noticed that in four-dimensional space the right angle can occur $\Theta(n^3)$ times, since the points $p_x = (\cos x, \sin x, 0, 0)$, $q_y = (1, 0, y, 0)$, and $r_z = (-1, 0, 0, z)$ always determine a right angle at p_x. For all other angles, there is an upper bound of $O(n^{\frac{5}{2}}\beta(n))$, where $\beta(n)$ is an extremely slowly growing function related to the inverse Ackermann function [1]. However, the best known lower bound for angles different from $\frac{\pi}{2}$ is the same as in the plane: $\Omega(n^2)$ and $\Omega(n^2 \log n)$ for some special values.

In spaces of dimensions six and higher, any given angle can be represented by $\Theta(n^3)$ triples taken from an n-element set. According to a well known construction of Lenz (see e.g. [8], [38]), the number of mutually congruent triangles with an angle γ can be $\Omega(n^3)$. The analogous statement in five-dimensional space is not known to be true.

Problem 3.2. *Can every angle $0 < \gamma < \pi$ different from $\frac{\pi}{2}$ occur $\Omega(n^3)$ times among n points in five-dimensional space?*

Almost nothing is known about problem (2').

Corrádi–Erdős–Hajnal conjecture [23]. *Given n points in the plane, not all on a line, they always determine at least $n-2$ distinct angles in $[0, \pi)$.*

The number of distinct angles determined by a regular n-gon is precisely $n-2$, but there are several other configurations for which the conjectured lower bound is tight. It easily follows from the "weak Dirac conjecture" (mentioned in the previous section) that there is a constant $c > 0$ such that any noncollinear set of n points in the plane determines at least cn distinct angles.

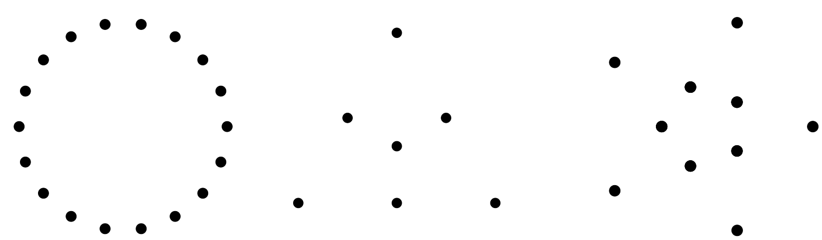

Fig. 2. Sets of n points with $n-2$ distinct angles

4 Finite planes, algebraic aspects

So far most of the results concerning directions, slopes, angles, and incidences have been established using combinatorial arguments. But the use of certain algebraic tools may turn out to be inevitable.

The following classical result, which is a far-reaching generalization of Corollary 1.1, can be obtained by an elegant application of the so-called "linear algebra method." This was the starting point of many investigations in the theory of block designs and finite projective planes.

De Bruijn–Erdős theorem [9]. *Let $\mathcal{L} = \{L_1, L_2, \ldots\}$ be a family of proper subsets of an n-element set with the property that each pair $\{p, q\} \subset P$ belongs to precisely one member of $\mathcal{L}$. Then we have $|\mathcal{L}| \geq n$ with equality if and only if (1) one of the sets contains all but one elements of P and the others are two-element sets containing the remaining element; or (2) $\mathcal{L}$ is the system of lines of a finite projective plane defined on P.*

Rédei [43] used lacunary polynomials to prove an analogue of Ungar's theorem for finite affine planes.

Rédei–Megyesi theorem. *Let p be an odd prime. Then any noncollinear set of p points in the affine plane $AG(2,p)$ determines at least $\frac{p+3}{2}$ different directions.*

Rédei's analysis was completed by Lovász and Schrijver [36], who characterized the extremal configurations. These considerations turned out to be intimately related to the structure of blocking sets in finite projective planes. (A *blocking set* is a set of points intersecting every line.) See [48], for a survey.

As we have mentioned in Section 1, the Szemerédi–Trotter theorem has some exciting number-theoretic consequences.

Erdős–Szemerédi theorem [24]. *There exists $\varepsilon > 0$ such that for any set A of n reals either the set of sums $A+A = \{a+b \mid a, b \in A\}$ or the set of products $A \cdot A = \{ab \mid a, b \in A\}$ has at least $\Omega(n^{1+\varepsilon})$ elements.*

The best known value of ε (roughly $\frac{3}{11}$) was established by Solymosi [45], but it is conjectured that the theorem remains true for every $\varepsilon < 1$.

The following elegant argument due to Elekes [16] proves that the result holds with $\varepsilon = \frac{1}{4}$. Apply the Szemerédi–Trotter theorem to the set of points $P = (A+A) \times (A \cdot A) \subseteq \mathbf{R}^2$ and to the set $\mathcal{L}$ of n^2 lines of the form $y = a(x-b)$, where $a, b \in A$. Observe that the line $y = a(x-b)$ passes through at least n elements of P, namely, all points of the form $(c+b, ac)$ for $c \in A$. Therefore, the number of incidences between the elements of P and $\mathcal{L}$ is at least n^3. On the other hand, this quantity is at most $O(|P|^{2/3}|\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|) = O(|P|^{2/3}n^{4/3} + |P| + n^2)$. Comparing these two bounds, we obtain $|P| = |A+A| \times |A \cdot A| = \Omega(n^{5/2})$, as required.

According to the above results, any finite subset A of the field of real numbers is very far from being closed either under addition or under multiplication. The same question can be asked for other fields F. If F has a subfield A, then we cannot expect such a result. However, for finite fields F of prime order, we have the following.

Bourgain–Katz–Tao theorem [7]. *Let F be a finite field of prime order. For any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that, whenever $|F|^{\delta} < |A| < |F|^{1-\delta}$, we have*

$$\max\{|A+A|, |A \cdot A|\} = \Omega(|A|^{1+\varepsilon}).$$

The proof is based on the following Szemerédi–Trotter-type result. Let $F^2 = F \times F$ be a finite field plane, where $F = \mathbf{Z}/p\mathbf{Z}$ and p is a prime. For any $0 < \alpha < 2$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that the number of incidences between $n \leq p^{\alpha}$ points and $l \leq p^{\alpha}$ lines in F^2 is at most $O(n^{\frac{3}{2}-\varepsilon})$.

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