DIMACS Technical Report 95-54
December 1995

On a Metric Generalization of Ramsey's Theorem

by

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DIMACS is a cooperative project of Rutgers University, Princeton University, AT&T Bell Laboratories and Bellcore.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

An increasing sequence of reals $x = \langle x_i : i < \omega \rangle$ is *simple* if all "gaps" $x_{i+1} - x_i$ are different. Two simple sequences x and y are *distance similar* if $x_{i+1} - x_i < x_{j+1} - x_j$ if and only if $y_{i+1} - y_i < y_{j+1} - y_j$ for all i and j. Given any bounded simple sequence x and any coloring of the pairs of rational numbers by a finite number of colors, we prove that there is a sequence y distance similar to x all of whose pairs are of the same color. We also consider many related problems and generalizations.

0. Introduction.

Let $x = \langle x_0, \dots, x_{k-1} \rangle, y = \langle y_0, \dots, y_{k-1} \rangle$ be two increasing sequences of integers. We call x and y *distance similar* and write $x \overset{\mathrm{d}}{\sim} y$ if the set of consecutive distances of x is ordered the same way as that of y, i.e.,

$$x_i - x_{i-1} < x_j - x_{j-1} \Longleftrightarrow y_i - y_{i-1} < y_j - y_{j-1} \text{ for } i, j < k.$$

Let us call a sequence x *simple* if the consecutive distances are all different. In this paper we will only make claims about simple sequences.

The starting point of our investigations is the following "metric" generalization of Ramsey's theorem, which was also proved independently by Noga Alon. (For a quantitative version of this result, see a forthcoming paper of Shelah [S95].)

Theorem 0. *For every simple sequence $x = \langle x_i : i < k \rangle$ and for every integer l there is an $n = n(k, l)$ such that for every coloring $f : [n]^2 \to l$ of the pairs of n with l colors there is a sequence $y = \langle y_i : i < k \rangle, y_i \in n$, which is distance similar to x and all of its pairs are colored the same.*

A little meditation shows that nothing like this is true if we try to restrict distance similarity by considering all possible distances $\{x_i - x_j : j < i < k\}$.

In this paper, we generalize Theorem 0 to (simple) sequences of reals $x = \langle x_n : n < \omega \rangle$ or even $x = \langle x_\beta : \beta < \alpha \rangle$ where α is a countable ordinal and the sequences are increasing or decreasing. The notion of distance similarity $x \overset{\mathrm{d}}{\sim} y$ can be easily extended to such sequences.

The very first question is: What can be said if the underlying set is ω? The first facts are discouraging. Define $f : [\omega]^2 \to 2$ by $f(\{n, m\}) = 0$ iff $m < \frac{n}{2}$ for $m < n < \omega$. In color 0 there is no homogeneous triangle $\{u, v, w\}$ with $u < v < w$ and $v - u > w - v$, while there is no infinite homogeneous set for color 1. Pursuing the matter further, much to our surprise we found quite a few positive results already in case the underlying set is $\mathbb{Q}$, the set of rationals, and even stronger results in case we work with $\mathbb{R}$.

In the next three sections we present the proofs of these results and some further counter-examples. Finally, we will mention some striking problems. However, to be able to state the results in the language of partition calculus, we need more notation.

1. Statement of the results.

1.1 Definition. For $X \subseteq \mathbb{R}$ and $\alpha < \omega_1$, let $(X)^\alpha$ denote the set of increasing simple sequences of type α, i.e.,

$$f(X)^\alpha = \{x : x = \langle x_i : i < \alpha \rangle \wedge x_0 < \ldots < x_i < \ldots \wedge x_i \in X \\ \wedge\ x_{i+1} - x_i \neq x_{j+1} - x_j \text{ for } i \neq j < \alpha\}$$

and

$$(X)^{\alpha,*} = \{x : x = \langle x_i : i < \alpha \rangle \wedge x_0 > \ldots > x_i > \ldots \wedge x_i \in X \\ \wedge\ x_{i+1} - x_i \neq x_{j+1} - x_j \text{ for } i \neq j < \alpha\}\,.$$

1.2 Definition. For $x \in (\mathbb{R})^\alpha \cup (\mathbb{R})^{\alpha,*}$, let $\underline{x}$ denote the equivalence class of x for the relation $x \overset{\mathrm{d}}{\sim} y$.

1.3 Definition. For $x \in (\mathbb{R})^\alpha$, let $\underline{x}^*$ denote the equivalence class of a $y \in (\mathbb{R})^{\alpha,*}$ such that

$$x_{i+1} - x_i < x_{j+1} - x_j \Leftrightarrow y_i - y_{i+1} < y_j - y_{j+1}.$$

For $x \in (\mathbb{R})^{\alpha,*}$, define $\underline{x}^* \in (\mathbb{R})^\alpha$ similarly.

1.4 Main Definition. Let $X \subset \mathbb{R}$, let ν be an ordinal and let $\langle \Theta_\mu : \mu < \nu \rangle$ be a sequence such that for each $\mu < \nu$, Θ_μ is either a cardinal, or an order type, or the equivalence class of an $x \in (\mathbb{R})^\alpha \cup (\mathbb{R})^{\alpha,*}$ for some $\alpha < \omega_1$. The *partition relation* $X \to (\Theta_\mu)^2_{\mu<\nu}$ is said to hold if for every 2-partition $f : [X]^2 \to \nu$ of X into ν parts, there are a $\mu < \nu$ and a $Y \subset X$ such that Y is homogeneous for f in class μ (i.e., f is constant on the pairs of Y) and $|Y| = \Theta_\mu$ in cases Θ_μ is a cardinal, type $Y(<) = \Theta_\mu$ if Θ_μ is an order type, and $Y = \{y_i : i < \alpha\}$ otherwise, where $y = \langle y_i : i < \alpha \rangle \in (\mathbb{R})^\alpha \cup (\mathbb{R})^{\alpha,*}$ satisfies

$$\underline{y} = \Theta_\mu.$$

We write $\not\to$ to indicate the negation of this statement.

We are going to use all the – more or less self-explanatory – shorthand notations and alternatives introduced in the book [EHMR84]. For example, if $X \to (\Theta_\mu)^2_{\mu<\nu}$ holds for every $X \subset \mathbb{R}$ with type $X(<) = \Theta$, we write

$$\Theta \to (\Theta_\mu)^2_{\mu<\nu}.$$

In this notation, Theorem 0 says that for every $k, l \in \omega$ and for every $x \in (\mathbb{R})^k$ there is an $n \in \omega$ with

$$n \to (\underline{x})^2_l.$$

As we have already mentioned, there are strong positive relations for simple sequences $x \in (\mathbb{R})^\alpha$ with distance sequences tending to 0, provided the underlying set X contains a set dense in itself, i.e., type $X(<) \geq \eta$. (As usual, η denotes the order type of $\mathbb{Q}$, the set of rational numbers.)

Note that bounded sequences have distance sequences tending to 0, but neither the boundedness nor this property is invariant under distance similarity.

In §2, we first prove a sequence of preparatory lemmas, which enable us to show that

$$\eta \to (\eta, \underline{x} \vee \underline{x}^*)^2 \text{ and } \eta \to (\underline{x})^2_k \quad \text{for all } k < \omega$$

hold for every simple sequence $x \in (\mathbb{R})^\omega$, whose distance sequence tends to 0. (See Theorem 2.10.) Note that the first of these statements is self-strengthening and implies $\eta \to (\eta, (\underline{x} \vee \underline{x}^*)_k)^2$ for $k < \omega$. In fact, it implies the second as well. It also implies Theorem 0, by compactness.

Using the methods described in §2, we prove in §3 (see Theorem 3.2) that for $X \subset \mathbb{R}, |X| > \omega$,

$$X \to (\eta, \underline{x} \vee \underline{x}^*)^2 \text{ and } X \to (\underline{x})^2_2$$

hold for every $\alpha < \omega_1$ and for every $x \in (\mathbb{R})^\alpha$ with distance sequence tending to 0. Here the first statement is not self-strengthening and does not seem to imply the second. The first is a generalization of a theorem of Hajnal [H60] claiming that $X \to (\eta, \alpha \vee \alpha^*)^2$, the second is a strengthening of a theorem of Galvin [G75], according to which $X \to (\alpha)^2_2$.

We do not know if

$$X \to (\eta, (\underline{x} \vee \underline{x}^*)_k)^2,$$

and, as a corollary, if

$$X \to (\underline{x})^2_k \text{ for } 3 \leq k < \omega.$$

The second statement would be a generalization of the corresponding instance of the Baumgartner–Hajnal theorem [BH73]: $\Phi \to (\omega)^1_\omega \Rightarrow \Phi \to (\alpha)^2_k$, the first would be analogous to a stronger result of [G75]: $\Phi \to (\eta)^1_\omega \Rightarrow \Phi \to (\eta, (\alpha \vee \alpha^*)_k)^2$. However, the proofs of these results do not seem to generalize.

In the last section, we discuss some open problems where the underlying set X is ω.

2. Lemmas. The case of a countable underlying set.

The following lemma and its generalizations are crucial for all our positive results. The lemma may be well-known but it was new to us. After we formulated it, 2.1 was first proved by M. Laczkovich.

2.1 Embedding Lemma. *Let $x \in (\mathbb{R})^\alpha \cup (\mathbb{R})^{\alpha,*}$ $(\alpha < w_1)$ be a simple sequence with distance sequence tending to 0, $A \subset \mathbb{R}$, $tpA(<) \geq \eta$. Then A contains a sequence y which is distance similar to x.*

Note that if $x = \langle 0, 5, 7, 11 \rangle$ and C is the triadical Cantor set, then x does not embed into C maintaining the order of *all* distances. This is one of the reasons why we restrict ourselves to the investigation of consecutive distances.

In the partition theorems we need homogeneous Y's with $x \overset{\mathrm{d}}{\sim} y$, hence we need a general procedure yielding a great many Y's of the above kind.

In what follows, $\mathcal{J} \subset \mathbb{P}(\mathbb{R})$ is a proper ideal of subsets of $\mathbb{R}$ satisfying the following property:

2.2. For all $B \notin \mathcal{J}$ there are $C, D \notin \mathcal{J}$ such that $C, D \subset B$ and $C < D$.

In this section we will use the results for $\mathcal{J}_0 = \{A \subset \mathbb{R} :\ \text{type } A(<) \not\geq \eta\}$, and in the next one we consider

$$\mathcal{J}_1 = \{A \subset \mathbb{R} : |A| \leq \aleph_0\} \text{ and}$$
$$\mathcal{J}_2 = \{A \subset \mathbb{R} : A \text{ is meager}\}.$$

All these ideals have property 2.2. As usual, $\mathcal{J}^+$ denotes the complement of $\mathcal{J}$.

2.3 Strong Embedding Lemma. *Assume that $\mathcal{J}$ is an ideal, $\mathcal{J} \subset \mathbb{P}(\mathbb{R}), A \subset \mathbb{R}, A \in \mathcal{J}; \alpha < \omega_1$. Let $x \in (\mathbb{R})^\alpha$ be a simple sequence whose distance sequence tends to 0. Assume further that $T = T_{\mathcal{J}} \subset \mathcal{J}^+ \times \mathcal{J}^+$ satisfies the following properties:*

(i) T is downward closed;

(ii) For every $B \subset A, B \in \mathcal{J}^+$, there is a pair $\langle C, D\rangle \in T$ with $C, D \subset B$ and $C < D$.

Then there is a sequence $\{A_\beta : \beta < \alpha\} \subset \mathcal{J}^+$ of subsets of A such that

(1) $A_0 < \ldots < A_\beta < \ldots$;

(2) For every $\alpha > \beta > \gamma$, $\langle A_\beta, A_\gamma\rangle \in T$;

(3) For every $y \in (A)^\alpha$ with $y_\beta \in A_\beta$ ($\beta < \alpha$), we have $x \overset{\mathrm{d}}{\sim} y$.

Proof: To simplify the notation, we may assume $\alpha \geq \omega$. Let $D = \{x_{\beta+1} - x_\beta : \beta < \alpha\}$. By the assumption, $D = \{d_n : n < \omega\}$ for a monotone decreasing sequence $\langle d_n : n < \omega\rangle$ tending to 0. Since x is simple, there is a one-to-one enumeration $\alpha = \{\beta_n : n < \omega\}$ with $d_n = x_{\beta_n+1} - x_{\beta_n}$. We may assume $\beta_0 = 0$.

We are going to define a set-matrix

$$\{A_i^n : i \leq n\} \subset \mathcal{J}^+$$

by induction on $n \in \omega$. Let $A_0^0 = A$. Assume that $n > 0$ and that the sets $\{A_i^{n-1} : i \leq n-1\} \subset \mathcal{J}^+$ have already been defined. Let $i(n)$ be the $i \leq n-1$ for which

$$\beta_i = \max\{\beta_j : j \leq n-1 \wedge \beta_j < \beta_n\}.$$

Choose sets $C, D \in A^n_{i(n)}, \langle C, D\rangle \in T$ such that

$$(*) \qquad d(C), d(D) \leq d(C, D) \leq \min\{d(A_i^{n-1}) : i \leq n-1\}.$$

Here $d(X)$ and $d(X, Y)$ are the diameter of X and the distance of X and Y, respectively.

Define

$$A_j^n = \begin{cases} A_j^{n-1} & \text{for } j < i(n), \\ C & \text{for } j = i(n), \\ D & \text{for } j = i(n)+1, \\ A_{j-1}^{n-1} & \text{for } i(n)+1 < j \leq n+1. \end{cases}$$

This defines the set-matrix. It is easy to see, by induction on n, that for $i < j \leq n$, $A_i^n < A_j^n$ iff $\beta_i < \beta_j$ iff $x_{\beta_i} < x_{\beta_j}$ and also that the sets $\{A_i^n : n \in \omega, i \leq n\}$ form a tree by reverse inclusion.

We now claim that $i(n) \to \infty$ if $n \to \infty$. Indeed, if $i(n) = i(m)$ for $m < n$, then $\beta_m > \beta_n$, otherwise $\beta_{i(m)} < \beta_m \leq \beta_{i(n)} = \beta_{i(m)}$, hence $i(n) = k$ can not hold for infinitely many n. Thus, for every fixed i, the sequence A_i^n stabilizes. That is, there exists n_i such that $A_i^n = A_{\beta_i}$ for $n > n_i$. Choosing $n \geq \max(n_i, n_j)$, we see that $A_{\beta_j} < A_{\beta_i}$ iff $\beta_i < \beta_j$, i.e., (1) holds.

To see that (2) is true, assume $\beta_i < \beta_j$ and $n \geq \max(n_i, n_j)$. There is a minimal $m \leq n$ such that A_{β_i} and A_{β_j} are contained in different elements of $\{A_k^m : k \leq m\}$. But then, because of the tree structure, $A_{\beta_i} \subset A^m_{i(m)}$, $A_{\beta_j} \subset A^m_{i(m)+1}$, and $\langle A_{\beta_i}, A_{\beta_j}\rangle \in T$, by (i). Finally, (3) follows trivially from (*). $\square$

The following definitions and Lemmas 2.4–7 have their origin in the early papers [H60] and [G70]. In all of them it is tacitly assumed that the ideals $\mathcal{J}$ satisfy (2.2). For a graph G, we denote by $\overline{G}$ the complement of G, and for every vertex u, let $G(u) = \{v : \{u,v\} \in G\}$.

2.4 Definition. (i) Assume $B < C, \langle B, C\rangle \in \mathcal{J}^+ \times \mathcal{J}^+$. The pair $\langle B, C\rangle$ is *left good* for G if for all $B' \subset B, C' \subset C$, $\langle B', C'\rangle \in \mathcal{J}^+ \times \mathcal{J}^+$, $\exists\, x \in B'$ with $G(x) \cap C' \in \mathcal{J}^+$.

(ii) Assume $B < C, \langle B, C\rangle \in \mathcal{J}^+ \times \mathcal{J}^+$. The pair $\langle B, C\rangle$ is *left very good* for G if for all $x \in B$, $\overline{G}(x) \cap C \in \mathcal{J}$.

Right good and *right very good* can be defined similarly. *Good* means that the pair is both left and right good.

2.5 Lemma. *Assume $\langle B, C\rangle \in \mathcal{J}^+ \times \mathcal{J}^+$. Then either $\langle B, C\rangle$ is left good for G or there are $B' \subset B, C' \subset C$, $\langle B', C'\rangle \in \mathcal{J}^+ \times \mathcal{J}^+$ such that $\langle B', C'\rangle$ is left very good for $\overline{G}$.* □

2.6 Lemma. *Assume $A \in \mathcal{J}^+$, and G is a graph on A. For easier readability, let us assume that A', A'', B, C run over sets in $\mathcal{J}^+$. Then one of the following statements (i), (ii), (iii) is true:*

(i) $\exists A' \subset A\ \forall A'' \subset A'\ \exists B, C \subset A''$ such that $\langle B, C\rangle$ is left very good for $\overline{G}$;

(ii) $\exists A' \subset A\ \forall A'' \subset A'\ \exists B, C \subset A''$ such that $\langle B, C\rangle$ is right very good for $\overline{G}$;

(iii) $\exists A' \subset A\ \forall A'' \subset A'\ \exists B, C \subset A''$ such that $\langle B, C\rangle$ is good for G.

Proof: Indeed, if (i) and (ii) both fail, then we can choose A'' showing the failure of both. By (2.2), we can choose $B, C \subset A''$ such that $B < C$ and $B, C \in \mathcal{J}^+$. By the previous lemma, $\langle B, C\rangle$ must be good for G. □

2.7 Lemma. *Assume $A \in \mathcal{J}^+, [A]^2 = \bigcup_{j<k} I_j$, $k < \omega$. Then there are $j_0, j_1 < k$ and $A' \subset A, A' \in \mathcal{J}^+$ such that for $\forall A'' \subset A'$, $A'' \in \mathcal{J}^+$ $\exists B, C \subset A''$ where $B, C \in \mathcal{J}^+$, $B < C$ with $\langle B, C\rangle$ left good for I_{j_0} and right good for I_{j_1}.*

Proof: Assume indirectly that no such pair j_0, j_1 exists. Applying the indirect assumption for all pairs repeatedly, we get an $A'' \subset A$ where $A'' \notin \mathcal{J}$. By property (2.2), we can choose $B, C \subset A''$, $B, C \notin \mathcal{J}$, $B < C$. There must be a $B' \subset B, C' \subset C$ where $B', C' \notin \mathcal{J}$ and a $j_0 < k$ such that $\langle B', C'\rangle$ is left good for I_{j_0}. Similarly for j_1. □

We need

2.8 Galvin's "Hopping Around Lemma". [G70] *Assume that condition (iii) of Lemma 2.6 holds. Then G contains a complete subgraph of type η.* □

Though the proof of this is just as simple as the proof of the other lemmas, to save space we omit it.

2.9 Corollary. [G70] If we color the pairs of rational numbers by $k < \omega$ colors, then we can find a set of type η all of whose pairs have just two of the colors. In notation, $\eta \to [\eta]^2_{k,2}$ *for* $k < \omega$.

Proof: Assume $[\mathbb{Q}]^2 \subset \bigcup_{j<k} I_j$. By (2.7), there exist $A \notin \mathcal{J}_0$, $j_0, j_1 < k$ with the property that for all $A'' \subset A'$, $A'' \notin \mathcal{J}$ there are $B, C \subset A''$ such that $B, C \notin \mathcal{J}$ and $\langle B, C\rangle$ is good for $I_{j_0} \cup I_{j_1}$. Hence the claim follows from (2.8). $\square$

2.10 Theorem. *Assume that $x \in (\mathbb{R})^\omega$ is a simple sequence with distance sequence tending to 0, and let $k < \omega$.*

Then $\eta \to (\eta, \underline{x} \vee \underline{x}^*)^2$. Consequently, $\eta \to (\eta, (\underline{x} \vee \underline{x}^*)_k)^2$ and $\eta \to (\underline{x})^2_k$.

Proof. We work with the ideal $\mathcal{J} = \mathcal{J}_0 = \{A \subset \mathbb{R} : \eta \not\leq \text{ type } A(<)\}$.

To prove the main clause of the theorem, let $A \subset \mathbb{R}$, $A \notin \mathcal{J}$, and $[A]^2 = I_0 \cup I_1$. If (iii) of (2.6) holds with $G = I_0$, then (2.8) implies the existence of a homogeneous η for the class 0. By (2.6), we may assume that (i) or (ii) hold. First assume that (i) holds. Let T denote the set of pairs $\langle B, C\rangle$ such that $B, C \in \mathcal{J}^+$ and $\langle B, C\rangle$ is left very good for $\overline{I}_0 = I_1$. The Strong Embedding Lemma yields a sequence $A_0 < \ldots < A_n < \ldots$ such that $\langle A_n, A_m\rangle$ are left very good for I_1 for $n < m$, and for every y with $y_n \in A_n$ for all $n < \omega$, we have $x \overset{\mathrm{d}}{\sim} y$. Define the sequence $y_n \in A_n$ by induction on $n < \omega$. Assume that $y_i \in A_i$ has already been defined for $i < n$. Then $\bigcap_{i<n} I_1(n) \cap A_n \notin \mathcal{J}$, hence it is nonempty. Let y_n be an element of this set. $y = \langle y_n : n < \omega\rangle$ is distance similar to x, by the Strong Embedding Lemma, and $Y = \{y_n : n < \omega\}$ is homogeneous for color 1. If case (ii) holds, we obtain a decreasing 1-homogeneous sequence y distance similar to x, analogously.

$\eta \to (\eta, \underline{x} \vee \underline{x}^*)^2$ obviously implies $\eta \to (\eta, (\underline{x} \vee \underline{x}^*)_k)^2$.

Finally, let us remark that $\Phi \to (\omega, (\Theta_\mu \vee \Theta^*_\mu)_{1 \leq \mu < v})^2 \Rightarrow \Phi \to (\Theta_\mu)^2_{1 \leq \mu < v}$, by a result of Galvin [G75]. He states this implication only in the case when the Θ_μ's are ordinals, but the argument works in our case, too. For any ordered set $\langle S, \prec\rangle$ of type Φ and partition $[S]^2 = \bigcup_{1 \leq \mu < \nu} I_\mu$, we can take a well-ordering $\prec_0$ of S, and define $I'_0 = \{\{x, y\} \in [S]^2 : x \prec y \wedge x \prec_0 y\}$, $I'_\mu = I_\mu \setminus I_0$ for $1 \leq \mu < \nu$, and apply the assumption for the partition $[S]^2 = \bigcup_{\mu<\nu} I'_\mu$. $\square$

To finish this section, first we note that a classical remark of Erdős and Rado gives $\eta \not\to (\omega, \omega + 1)^2$, hence we can not generalize Theorem 2.10 for α-sequences with $\alpha > \omega$.

Our second remark is concerning the entry "η" in the first claim of the theorem. One may think at first glance that this statement remains true for any set which is dense in an interval of the underlying set. A case in point is another classical Erdős-Rado theorem, $\eta \to (\eta, \aleph_0)^2$, which is in fact a corollary of the following stronger result:

$$\mathbb{Q} \to (\text{dense in an interval}, \aleph_0)^2.$$

We claim that this is false in our case.

2.11 Definition. Let $\Theta[\text{small, large}] = \Theta[s, l]$ denote the equivalence class of say $x = \langle 0, 1, 3\rangle$.

2.12 Proposition. $\mathbb{Q} \not\to (\text{dense in an interval}, \Theta[s, l])^2$.

Proof: Let $\mathbb{Q} = \{r_i : i < \omega\}$ be a one-to-one enumeration of $\mathbb{Q}$. Define $f : [\mathbb{Q}]^2 \to 2$ by letting $f(\{r_i, r\}) = 1$ for $r_i < r \in \mathbb{Q}$ iff

$$r \in \left(r_i + \frac{1}{i+1}, r_i + \frac{1}{2i+2}\right).$$

Clearly, there is no homogeneous $\Theta[s, l]$ for color 1. Assume $A \subset \mathbb{Q}$ is dense in an interval I of $\mathbb{Q}$. For some $i \in \omega$, we have $r_i \in A$ and $\left(r_i + \frac{1}{i+1}, r_i + \frac{1}{2i+2}\right) \subset I$. Then for some $r \in A \cap \left(r_i + \frac{1}{i+1}, r_i + \frac{1}{2i+2}\right)$, $f(\{r_i, r\}) = 1$ holds. □

3. The uncountable case.

What we can prove does not require too many new ideas, hence we will only outline the proofs.

3.1. Theorem. *Assuming the continuum hypothesis,*

$$\mathbb{R} \not\to (\aleph_1, \Theta[\text{small, large}])^2.$$

(For Θ[small, large] see Definition 2.11.)

Proof (Sketch): Solving a problem of the senior authors, Hechler constructed from CH a 2-partition $[\mathbb{R}]^2 = I_0 \cup I_1$ such that there is no uncountable homogeneous set for the class I_0 and for every $x \in \mathbb{R}$, $I_1(x) \cap (-\infty, x)$ is merely a sequence converging to x. That is,

$$I_1(x) \cap (-\infty, x) = \{x_n : \ n < \omega\}$$

for some monotone increasing sequence $\langle x_n : n < \omega\rangle$ tending to x. Now, it is easy to modify his construction to satisfy

$$x - x_{n+1} < x_{n+1} - x_n \quad \text{for } n < \omega,$$

and then this partition meets the requirements of our theorem. □

3.2 Theorem. *Assume that $X \subset \mathbb{R}, |X| \geq \aleph_1, \alpha < \omega_1$, and let $x \in (\mathbb{R})^\alpha$ be a simple sequence with distance sequence tending to 0. Then*

(1) $$X \to (\eta, \underline{x} \vee \underline{x}^*)^2 \quad \text{and}$$

(2) $$X \to (\underline{x})^2_2.$$

If X is non-meager, then η in (1) can be replaced by a set dense in an interval.

Proof (sketch): Assume that $[X]^2 = I_0 \cup I_1$. We work with the ideal $\mathcal{J} = \mathcal{J}_1$. We may assume that (iii) of 2.6 is false for $G = I_0$. To prove (1), just like in the proof of Theorem 2.10, we obtain a sequence $A_0 < \ldots < A_\beta < \ldots$ or $A_0 > \ldots > A_\beta > \ldots$ with $A_\beta \in \mathcal{J}^+$ for $\beta < \alpha$, depending on whether (i) or (ii) of (2.6) holds, respectively. We can pick $y_\beta \in A_\beta$ $(\beta < \alpha)$ by induction on β, homogeneous for the class 1, because the ideal $\mathcal{J}$ is σ-complete.

Assume for contradiction that (2) is false. Similarly as in the first proof, this implies that (ii) of (2.6) is false for both $G = I_0$ and $G = I_1$. By (2.5), we may then assume that for all $A \subset X$, $A \notin \mathcal{J}$ there are $B, C \notin \mathcal{J}$, $B < C$, $B, C \subset A$ such that $\langle B, C\rangle$ is left good for both I_0 and I_1. Now, invoking (2.7) we may assume that there is a $j < 2$ such that the above $\langle B, C\rangle$ is right good for I_j. Then, by Galvin's Lemma 2.8, there is an η-set homogeneous for I_j, and by the Embedding Lemma this contradicts the assumption.

To prove the final clause, using the Strong Embedding Lemma we may assume that both (i) and (ii) are false for the ideal $\mathcal{J} = \mathcal{J}_2$ and for $G = I_1$. By (2.6), we may assume that every pair $\langle B, C\rangle$ with $B < C$, $B, C \notin \mathcal{J}$, $B, C \subset X$ is good for I_0. It is easy to see that some interval (a, b) has the property that $X \cap (a', b') \notin \mathcal{J}$ for every subinterval (a', b') of (a, b). Let $\{(a_n, b_n) : n < \omega\}$ be an enumeration of all subintervals of (a, b) with rational endpoints, and let $A_n = X \cap (a_n, b_n)$. Now the Galvin lemma can be replaced with the following claim, much easier to prove: There is a transversal for $\{A_n : n \in \omega\}$ homogeneous for the class I_0. The $a_n \in A_n$ can be easily picked using that $\mathcal{J}$ is σ-complete. $\square$

4. Open problems and remarks.

We do not intend to analyze in general the case when $x \in (\mathbb{R})^\omega$ with distance sequence not tending to 0. Our remarks about $(\omega)^\omega$ will shed some light on the general situation.

As we have already remarked in the introduction,

$$\omega \not\to (\Theta[\text{large, small}], \omega)^2.$$

(See Definition 2.11.)

In 2.12 and 3.1, Θ[small, large] and Θ[large, small] played symmetric roles, but this is no longer true here. It is very easy to confirm

4.1 Proposition. $\omega \to (\Theta[\text{small, large}], \omega)^2$ $\square$

This only implies automatically that $\omega \to (\Theta[\text{small, large}], \underline{n})^2$ for every $n \in (\omega)^\omega$ with a monotone increasing distance sequence. We can prove a little more.

4.2 Proposition. *Let $n \in (\omega)^\omega$ be a simple sequence whose distance sequence is monotone increasing from a certain point on. Then $\omega \to (\Theta[\text{small, large}], \underline{n})^2$.*

Proof: Let $n|k = \langle n_i : i < k\rangle$ be an initial segment of n with the property that

$$n_i - n_{i-1} < n_j - n_{j-1} \quad \text{for all } j \geq k \text{ and } i < j.$$

We may also assume without loss of generality that $n_1 - n_0 < n_2 - n_1$. Let $G \subset [\omega]^2$ be a graph on the vertex set ω, and let $\mathcal{J}$ denote the ideal consisting of all subsets of ω with upper density zero. Assume that G has no triangle of type [small, large], i.e., there are no three integers $x < y < z$ with $\{x, y\}, \{y, z\}, \{x, z\} \in G$ and $y - x < z - y$.

Suppose first that $G(x) \in \mathcal{J}$ for all $x \in \omega$. By Theorem 0, there exists $m = \langle m_i : i < k \rangle$ distance similar to $n|k$ such that $\{m_i : i < k\}$ is an independent set of G. We can recursively select $m_j \in \omega, j \geq k$ such that $m_i - m_{i-1} < m_j - m_{j-1}$ for all $i < j$, using the fact that $\cup_{i<j} G(m_i) \in \mathcal{J}$.

Suppose next that $G(x) \notin \mathcal{J}$ for some $x \in \omega$. Szemerédi's theorem [SZ75] now implies that there are arbitrarily long arithmetic progressions in $G(x)$ and, just like in the previous case, we can find an independent set $\{m_i : i < k\} \subset G(x)$ which is distance similar to $n|k$. Since G has no [small, large] triangle, for every $j \geq k$ one can recursively select m_j meeting the requirements, because no sufficiently large element of $G(x)$ is connected to any element of $\{m_i : i < j\}$. □

In fact, a straightforward generalization of the above argument yields the following slightly stronger result. Let X be a subset of ω with positive upper density, and let m be a simple finite sequence whose distance sequence is monotone increasing. Then

$$X \to (\underline{m}, \underline{n})^2$$

for every monotone increasing simple sequence $n = \langle n_i : i < \omega \rangle$ with the property that one can find $i_0 < i_1 < i_2 < \dots$ such that

$$n_\mu - n_{\mu-1} < n_\nu - n_{\nu-1} \text{ whenever } \mu \leq i_k < \nu \text{ for some } k < \omega.$$

It is quite possible that the last arrow relation holds for all $n \in (\omega)^\omega$.

This is related to Hindman's theorem [H74]. For a subset $A \subset \omega$, let $\Sigma(A)$ denote the set of all non-repetitious non-empty sums formed from elements of A, and let us call a set $B \subset \omega$ a *Hindman set* if $\Sigma(A) \subset B$ for some infinite subset A of ω. Hindman's theorem states that

$$\omega \to (\text{Hindman set})^1_k \quad \text{for } k < \omega.$$

In other words, the set system $\mathcal{J} = \{B \subset \omega : B \text{ is not a Hindman set}\}$ is a proper ideal.

Now the following easy embedding lemma makes the connection.

4.3 Lemma. *Assume that $B \subset \omega$ is a Hindman set and let $n \in (\omega)^\omega$. Then there is an $m = \langle m_i : i < \omega \rangle \in (\omega)^\omega$ which is distance similar to n and $\{m_i : i < \omega\} \subset B$.*

Proof: Let $A = \{k_j : j < \omega\}$ be an increasing enumeration of a set A with $\Sigma(A) \subset B$. Assume that $n_{i+1} - n_i$ is the $j(i)$-th member of the distance set $\{n_{i+1} - n_i : i < \omega\}$ in its increasing enumeration. Then $m_0 = k_0,\ m_{i+1} = k_0 + \sum_{\nu=0}^{i} k_{j(\nu)+1}$, $i < \omega$ satisfy the requirements. □

By Lemma 4.3, Hindman's theorem implies that

$$\omega \to (\underline{n})^1_k \quad \text{for } k < \omega,\ n \in (\omega)^\omega,$$

and an affirmative answer to the following question would imply that Proposition 4.2 is true for all simple sequences $n \in (\omega)^\omega$.

4.4 Problem. Does $\omega \to (\Theta[\text{small, large}], \text{Hindman set})^2$ hold?

Indeed, we could not even decide if

$$\omega \to (3, \text{Hindman set})^2$$

is true.

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