Saturated simple topological graphs

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Abstract

A *simple topological graph* G is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing. G is called *saturated* if no further edge can be added without violating this condition. We construct saturated simple topological graphs with n vertices and $O(n)$ edges. These constructions are nearly optimal: it is shown that every saturated simple topological graph with n vertices has at least cn edges for some constant $c \geq 1.5$. Several related problems are also considered.

1 Introduction

Saturation problems in graph theory have been studied at length, ever since the paper of Erdős, Hajnal, and Moon [2]. Given a graph H, a graph G is H-saturated if G does not contain H as a subgraph, but the addition of any edge joining two non-adjacent vertices of G creates a copy of H. The saturation number of H, $\text{sat}(n, H)$, is the minimum number of edges in an H-saturated graph on n vertices. The saturation number for complete graphs was determined in [2]. A systematic study by Kászonyi and Tuza [7] found the best known general upper bound for $\text{sat}(n, H)$ in terms of the independence number of H. The saturation number is now known, often precisely, for many graphs; for these results and related problems in graph theory we refer the reader to the thorough survey of J. Faudree, R. Faudree, and Schmitt [3]. It is worth noting that $\text{sat}(n, H) = O(n)$, quite unlike the Turán function $\text{ex}(n, H)$, that is often superlinear.

In this paper, we study a saturation problem for *drawings* of graphs. In a drawing of a simple undirected graph G in the plane, every vertex is represented by a point, and every edge is represented by a curve between the points that correspond to its endpoints. If it does not lead to confusion, these points and curves are also called *vertices* and *edges*. We assume that in a drawing no edge passes through a vertex and no two edges are tangent to each other. A graph, together with its drawing, is called a *simple topological graph* if any two edges have at most one point in common, which is either their common endpoint or a proper crossing.

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Our motivation comes from the following problem: At least how many pairwise disjoint edges can one find in every simple topological graph with n vertices and m edges [8]? (Note that the simplicity condition is essential here, as there are complete topological graphs on n vertices and no *two* disjoint edges, in which every pair of edges intersect at most twice [9].) For *complete* simple topological graphs, i.e., when $m = \binom{n}{2}$, Pach and Tóth conjectured ([1], page 398) that one can always find $\Omega(n^\delta)$ disjoint edges for a suitable constant $\delta > 0$. This was shown by Suk [11] with $\delta = 1/3$; see [4] for an alternative proof. Recently, Ruiz-Vargas [10] has improved this bound to $\Omega\left(\sqrt{\frac{n}{\log n}}\right)$. Unfortunately, all known proofs break down for *non-complete* simple topological graphs. For *dense* graphs, i.e., when $m \geq \varepsilon n^2$ for some $\varepsilon > 0$, Fox and Sudakov [5] established the existence of $\Omega(\log^{1+\gamma} n)$ pairwise disjoint edges, with $\gamma \approx 1/50$. However, if $m \ll n^2$, the best known lower bound, due to Pach and Tóth [9], is only $\Omega\left((\log m - \log n)/\log\log n\right)$.

We know a great deal about the structure of complete simple topological graphs, but in the non-complete case our knowledge is rather lacunary. We may try to extend a simple topological graph to a complete one by adding extra edges and then explore the structural information we have for complete graphs. The results in the present note suggest that this approach is not likely to succeed: there exist very sparse simple topological graphs to which no edge can be added without violating the simplicity condition.

A simple topological graph is *saturated* if we cannot add any further edge such that the resulting drawing is still a simple topological graph. In other words, if we connect any two non-adjacent vertices by a curve, it will have at least two common points with one of the existing edges.

Consider the simple topological graph G_1 with eight vertices, depicted in Figure 1. It is easy to verify that the vertices x and y cannot be joined by a new edge so that the resulting topological graph remains simple. Indeed, every edge of G_1 is incident either to x or to y, and any curve joining x and y must cross at least one edge. On the other hand, G_1 can be extended to a (saturated) simple topological graph in which every pair of vertices except x and y are connected by an edge.

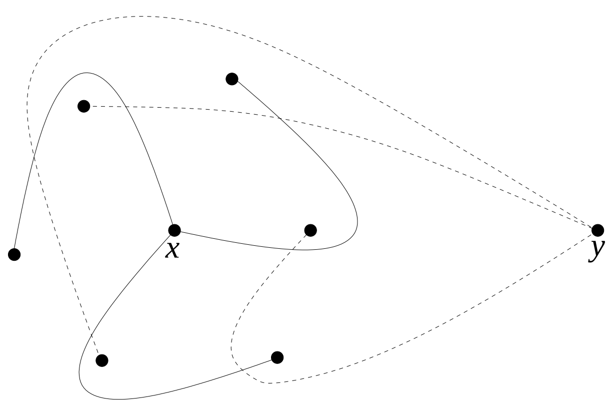

Figure 1: Topological graph G_1: edge $\{x, y\}$ can not be added.

Another example was found independently by Kynčl [6]: The simple topological graph G_2 depicted in Figure 2 has only six vertices, from which x and y cannot be joined by an edge without intersecting one of the original edges at least twice. Again, G_2 can be extended to a simple topological graph, in which every pair of vertices except x and y are connected by an edge. Although G_2 has fewer vertices than G_1, in the sequel it will be more convenient to use G_1 as a basis for more general constructions.

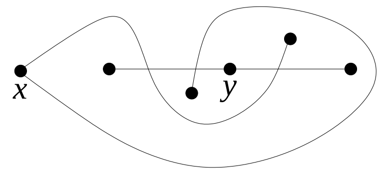

Figure 2: Topological graph G_2: edge $\{x, y\}$ cannot be added.

Let $s(n)$ denote the minimum number of edges that a saturated simple topological graph on n vertices can have. In view of the fact that the graphs shown in Figures 1 and 2 can be extended to nearly complete simple topological graphs, it is a natural question to ask whether $s(n) = \Omega(n^2)$. Our next theorem shows that $s(n) = \Theta(n)$.

Theorem 1. *Let $s(n)$ be the minimum number of edges that a saturated simple topological graph on n vertices can have, $n \geq 4$. Then we have*

$$c_1 n \leq s(n) \leq c_2 n,$$

for suitable constants $c_1 \geq 1.5$ and $c_2 \leq 17.5$.

The paper is organized as follows. In Section 2, we establish the upper bound in Theorem 1. In Section 3, we show that every vertex of a saturated simple topological graph has degree at least three, a fact from which the lower bound in Theorem 1 follows immediately. In Section 4, we discuss some related questions and prove the following theorem.

Theorem 2. *Let k be a positive integer, and let G be a (non-complete) topological graph, in which any two edges have at most k points in common. If two vertices are not adjacent, then they can be connected by a curve which has at most $2k$ points in common with any edge of G.*

2 Proof of Theorem 1: The Upper Bound

To construct saturated simple topological graphs with an arbitrarily large number, n, of vertices and $O(n)$ edges, we first need to modify G_1. Consider the edges of G_1 incident to x, and modify them in a small neighborhood of x so that the resulting edges have distinct endpoints, they pairwise cross each other, and their union encloses a *region* X (i.e., a connected component X of the complement of the union of the edges) which contains x. Analogously, modify the other three edges of G_1 in a small neighborhood of y. Let Y be the region that contains y and is enclosed by the modified edges. The resulting simple topological graph G has 12 vertices and 6 edges; see Figure 3. The points $x, y \in V(G_1)$ do not belong to $V(G)$.

Lemma 1. *Let x and y be any pair of points belonging to the regions X and Y in G, respectively. Then any curve joining x and y will meet at least one of the edges of G at least twice.*

Proof of Lemma 1: We prove the claim by contradiction. Let a_1, a_2, a_3, b_1, b_2, b_3 denote the edges of G. They divide the plane into eight regions, X, Y, A_1, A_2, A_3, B_1, B_2, B_3; see Figure 3. Suppose

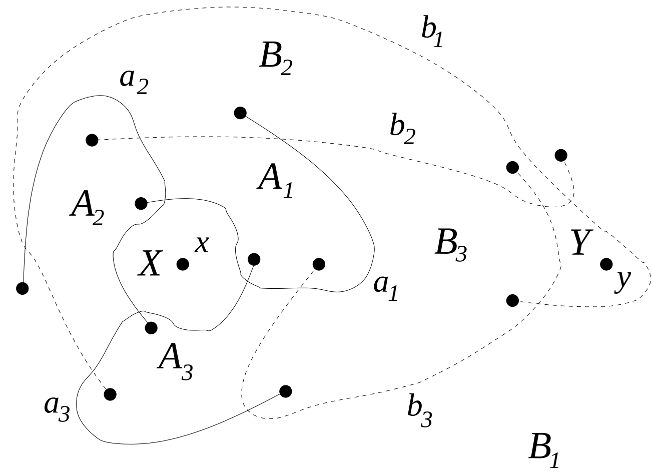

Figure 3: Topological graph G: edge $\{x, y\}$ can not be added.

there exists an oriented curve from x to y that crosses every edge of G at most once. Let γ be such a curve with the smallest number of crossings with the edges of G. Let $c_1, c_2, \ldots, c_{m-1}$ be the crossings between γ and the edges of G, ordered according to the orientation of γ. They divide γ into intervals $I_1, I_2, \ldots, I_m$, ordered again according to the orientation of γ. The first interval I_1 lies in region X, and the last one, I_m, is in region Y. Observe that no other interval can belong to X or to Y, because in this case we could simplify γ and obtain a curve with a smaller number of crossings. By symmetry, we can assume that the first crossing, c_1, is a crossing between γ and edge a_1. Then I_2 belongs to A_1. The following property holds.

Property $\mathcal{P}$: If for some $j \geq 2$, interval I_j belongs to region A_i (respectively, B_i), then one of the points $c_1, c_2, \ldots, c_{j-1}$ is a crossing between γ and the edge a_i (respectively, b_i).

We prove property $\mathcal{P}$ by induction on j. Clearly, the property holds for $j = 2$. Assume that I_{j-1} is in region A_i (respectively, B_i) and one of $c_1, c_2, \ldots, c_{j-2}$ is a crossing between γ and a_i (respectively, b_i). For simplicity, assume that I_{j-1} belongs to the region A_1 and that one of the points $c_1, c_2, \ldots, c_{j-2}$ is a crossing between γ and a_1; the other cases are analogous. Since c_{j-1} cannot belong to a_1, it must be a crossing between γ and either a_2 or b_2. In the first case, I_j belongs to A_2, in the second to B_2. In either case, Property $\mathcal{P}$ is preserved.

Now, we can complete the proof of Lemma 1. Consider the interval I_{m-1}. Since I_m lies in region Y, for some i, interval I_{m-1} must lie in B_i. Suppose for simplicity that I_{m-1} lies in B_1. By Property $\mathcal{P}$ (with $j = m - 1$, $m \geq 3$), one of the points $c_1, c_2, \ldots, c_{m-2}$ must be a crossing between γ and b_1. However, using that I_m is in Y, c_{m-1} must be another crossing between γ and b_1. Thus, γ crosses b_1 twice, which is a contradiction. $\square$

Now, we return to the proof of the upper bound in Theorem 1. Modify the drawing of G in Figure 3 so that region Y becomes unbounded, and let H be the resulting topological graph. Denote by Y (respectively, by X) the *outer region* (respectively, the *inner region*) of H; see Figure 4.

For every $n \geq 1$, construct a saturated simple topological graph F_n, as follows. Let $k = \lfloor n/12 \rfloor$. Take a disjoint union of k scaled and translated copies of H, denoted by H^1, H^2, $\ldots$, H^k, such that for any i, $1 < i \leq k$, let H^i lie entirely in the inner region of H^{i-1}. For $1 \leq i \leq k$, let V_i be the vertex set of H^i. Finally, place $n - 12k$ additional vertices in the inner region of H^k, and let

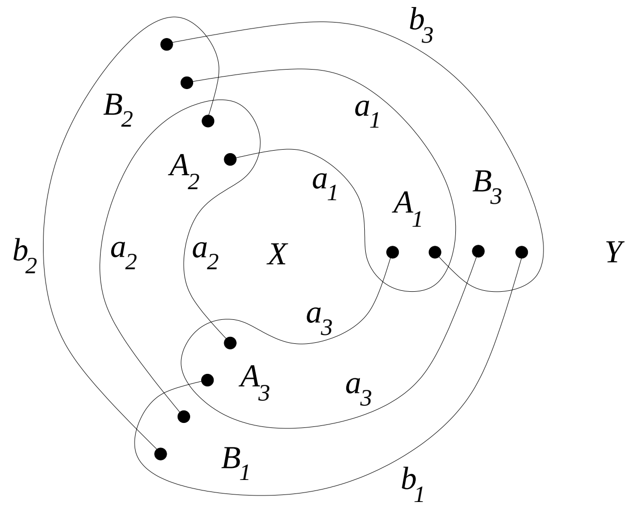

Figure 4: Topological graph H, a modification of G.

V_{k+1} denote the set of these vertices (see Figure 5). Obviously, we have $|V_{k+1}| < 12$.

Add to this topological graph all possible missing edges one by one, as long as it remains simple. We end up with a saturated simple topological graph F_n with n vertices. Observe that for every i and j with $1 \leq i < j - 1 < k$, V_i lies in the outer region of H^{i+1}, while V_j is in the inner region of H^{i+1}. By Lemma 1 (applied with $G = H^{i+1}$, $x \in V_j$, $y \in V_i$), no edge of F_n runs between V_i and V_j. Hence, every vertex in V_i can be adjacent to at most 35 other vertices; namely, to the elements of $V_{i-1} \cup V_i \cup V_{i+1}$. Therefore, F_n is a saturated simple topological graph with n vertices and at most $17.5n$ edges.

3 Proof of Theorem 1: The Lower Bound

We will need the following lemma. A vertex of a (topological) graph is *isolated* if its degree is zero. A triangle in a (topological) graph is called *isolated* if its vertices are not incident to any edges other than the edges of the triangle.

Lemma 2. *A saturated simple topological graph on at least four vertices cannot contain*

(i) any isolated triangle,

(ii) any isolated vertex,

(iii) any vertex of degree one.

Proof of Lemma 2: We prove only (i); the proofs of (ii) and (iii) are almost identical.

Let G be a saturated simple topological graph, and suppose for contradiction that T is an isolated triangle with vertices x, y, and z. By definition, the edges of T cannot cross one another.

We distinguish two cases.

Case 1. The edges of T do not participate in any crossing.

The edges of G divide the plane into regions. Let R denote a region bounded by the edges of T and at least one other curve C. Since G is saturated, C cannot consist of a single isolated

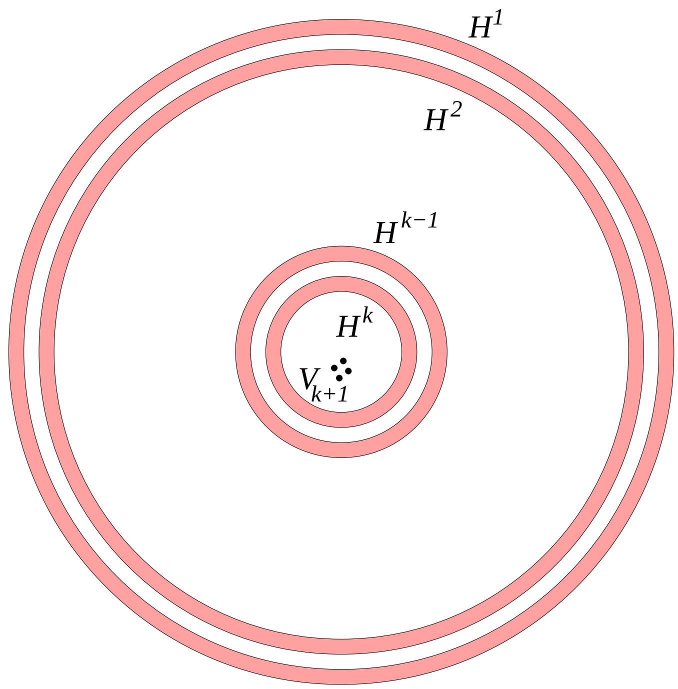

Figure 5: Saturated simple topological graph F_n.

vertex. Let $e = \{u, v\}$ be an edge that contributes to C, and let p be a point on e that belongs to the boundary of R; see Figure 6. Choose a point p' inside of R, very close to p. Let β be a curve running inside R that connects a vertex of T, say x, to p'. Let β' be a curve joining p' and u, and running very close to the edge e. Adjoining β and β' at p', we obtain a curve γ that connects x and u, two previously non-adjacent vertices of G. Curve γ is not crossed by any edge of T and by any edge of G incident to u. Since β is crossing-free, all crossings between γ and the edges of G must lie on β' and, hence, must correspond to crossings along the edge e. Therefore, every edge of G can cross γ only at most once. Consequently, γ can be added to G as an extra edge so that it remains simple. This contradicts the assumption that G was saturated.

Case 2. There is an edge of T which participates in a crossing.

Assume without loss of generality that $e = \{x, y\}$ is crossed by another edge of G. Let p denote the crossing on e *closest* to x, and suppose that p is a crossing between e and another edge $f = \{u, v\}$. Point p divides f into two parts. At least one of them, say, part up, is not crossed by edge $\{x, z\}$ of T. Choose a point p' in a very small neighborhood of p. Let β be a curve connecting x and p', running very close to e. Let β' be a curve between p' and u, running very close to f. Adjoining β and β' at p' we obtain a curve γ connecting x and u, two vertices that were not adjacent in G; see Figure 6. Just like in the previous case, add γ to G as an extra edge. Curve γ is not crossed by any edge incident to x or u. Since the portion xp of e is crossing-free, β must be crossing-free, too. Therefore, all possible crossings between γ and the edges of G must lie on β' and, hence, correspond to crossings along f. Thus, every edge of G crosses γ at most once, contradicting our assumption that G was saturated. □

The lower bound in Theorem 1 immediately follows from the statement below.

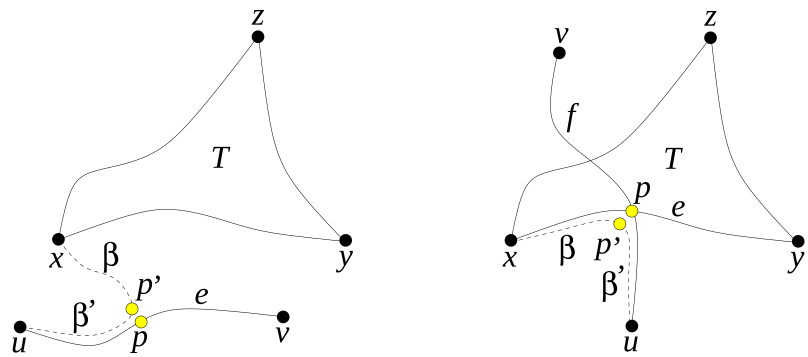

Figure 6: Case 1 and Case 2 of Lemma 2.

Lemma 3. *In every saturated simple topological graph with at least four vertices, every vertex has degree at least 3.*

Proof of Lemma 3: We prove the claim by contradiction. Let G be a saturated simple topological graph, and let x be a vertex of degree two in G. (By Lemma 2, the degree of x cannot be 0 or 1.) Let y and z denote the neighbors of x. We distinguish two cases.

Case 1. Edges $\{x, y\}$ and $\{x, z\}$ do not participate in any crossing.

By Lemma 2, y and z both have degree at least two, and x, y, and z do not span an isolated triangle. Hence, at least one of these three vertices, say, y, has a neighbor, w, different from x and z. Let γ be a curve connecting x to w, which runs very close to the edge $\{x, y\}$ from x to a point in a small neighborhood of y, and from that point all the way to w runs very close to edge $\{y, w\}$. We can assume that γ does not cross $\{x, y\}$ and $\{y, w\}$. Add γ to G as an extra edge. Clearly, γ does not cross any edge incident to x or w. As in the proof of Lemma 2, it is easy to verify that no other edge of G can cross γ twice. This contradicts the assumption that G was saturated.

Case 2. At least one of the edges $\{x, y\}$ and $\{x, z\}$ participates in a crossing.

Assume without loss of generality that $e = \{x, y\}$ crosses by another edge of G. Let p be the crossing on e *closest* to x, and suppose that the other edge passing through p is $f = \{u, v\}$. Vertex z cannot be identical to both u and v. Assume, for example, that $z \neq u$. Point p divides f into two pieces, at least one of which, say, up, does not cross the edge $\{x, z\}$. This remains true even if z is identical to v, since in this case f does not cross by $\{x, z\}$, due to the simplicity of G. Let γ be a curve connecting x and u, which follows e very closely from x to a point in a small neighborhood of p, and then, from that point all the way to u, closely follows f. We can assume that γ does not cross e and f. Add γ to G as an extra edge. As in Case 2 of the proof of Lemma 2, it is easy to show that this new edge does not meet any original edge of G more than once. Again, this contradicts the assumption that G was saturated. □

4 Concluding Remarks

For any positive integer k, call a topological graph *k-simple* if any two of edges have at most k points in common, each of which is either a common endpoint or a proper crossing (at which one of the edges passes from one side of the first edge to the other). A k-simple topological graph G

is *saturated* if we cannot add any further edge such that it remains k-simple. In other words, G is saturated if no matter how we connect two non-adjacent vertices by a curve, it has at least $k+1$ points in common with at least one edge of G. The drawings of K_6 minus an edge and K_8 minus an edge, described in Section 1, as well as the saturated simple topological graphs F_n constructed in Section 2, are examples of such graphs with $k=1$.

Problem 1. *Do there exist non-complete saturated k-simple topological graphs for every integer $k>1$?*

It would be interesting to find even one such example with $k=2$.

For any positive integers k and ℓ, $k<\ell$, a topological graph G *together with* a pair of non-adjacent vertices $\{u,v\}$ is called a *(k,ℓ)-construction* if G is k-simple and any curve joining u and v has at least ℓ points in common with at least one edge of G. Using this terminology, every saturated non-complete k-simple topological graph together with any pair of non-adjacent vertices is a $(k,k+1)$-construction.

Problem 2. *For which pairs of positive integers (k,ℓ) with $k<\ell$ do there exist (k,ℓ)-constructions?*

We do not know any examples with $(k,l)\neq(1,2)$. Theorem 2 can be rephrased as follows.

Theorem 2'. *There is no (k,ℓ)-construction with $\ell>2k$.*

Proof of Theorem 2: Let G be a non-complete k-simple topological graph, and let u and v be two non-adjacent vertices of G. We prove that u and v can be connected by a curve that has at most $2k$ points in common with any edge of G.

Place a new vertex at each crossing of G and subdivide the edges accordingly. Let G' denote the resulting topological graph. Choose an arbitrary path α in G', connecting u and v. We distinguish two types of vertices on α. A vertex x of G' that lies on α is called a *passing vertex* if the two edges of α incident to x belong to the same edge of G. A vertex x of G' that lies on α is a *turning vertex* if it is not a passing vertex, that is, if the two edges of α meeting at x belong to distinct edges of G.

Assign to α a unique *code*, denoted by $c(\alpha)$, as follows. Suppose that α contains r turning vertices for some $r\geq 0$. These vertices divide α into $r+1$ intervals, $I_1^\alpha, I_2^\alpha, \ldots, I_{r+1}^\alpha$, ordered from u to v. Set $p_0^\alpha=r$ and for any i, $1\leq i\leq r+1$, let p_i^α denote the number of passing vertices on I_i^α. Let $c(\alpha)=(p_0^\alpha, p_1^\alpha, p_2^\alpha, \ldots, p_{r+1}^\alpha)$; see Figure 7.

Order the codes of all (u,v)-paths lexicographically: if α and β are two (u,v)-paths in G', with codes $c(\alpha)=(p_0^\alpha=r, p_1^\alpha, p_2^\alpha, \ldots, p_{r+1}^\alpha)$ and $c(\beta)=(p_0^\beta=s, p_1^\beta, p_2^\beta, \ldots, p_{s+1}^\beta)$, respectively, then let $c(\alpha)\prec_{\text{lex}} c(\beta)$ if and only if $c(\alpha)\neq c(\beta)$ and for the smallest index i such that $p_i\neq q_i$ we have $p_i<q_i$.

Finally, define a *partial ordering* $\prec$ on the set of all the (u,v)-paths in G': for any two (u,v)-paths, α and β, let $\alpha\prec\beta$ if and only if $c(\alpha)\prec_{\text{lex}} c(\beta)$.

Let γ be a *minimal* element with respect to $\prec$. Suppose that γ has r turning vertices, $t_1, t_2, \ldots, t_r$, $r\geq 0$, that divide γ into intervals $I_1^\gamma, I_2^\gamma, \ldots, I_{r+1}^\gamma$, ordered from u to v. Consider the intervals as half-closed, that is, for every i, $0\leq i\leq r$, let t_i belong to I_{i+1}^γ.

Next we establish some simple properties of the intersections of γ with the edges of G.

Lemma 4. *Let e be an edge of G that has only finitely many points in common with γ. Then all of these points belong to two consecutive intervals of γ.*

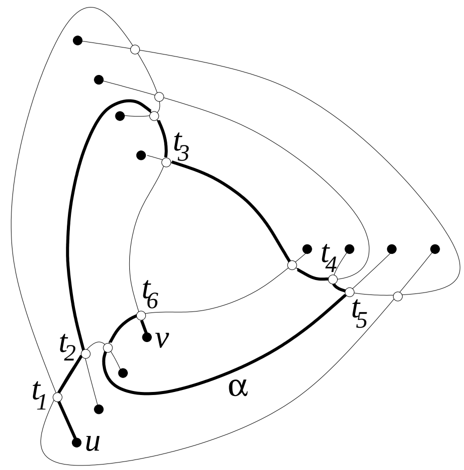

Figure 7: A (u,v)-path α (in bold) with $c(\alpha) = (6,0,0,1,1,0,1,0)$ and its turning vertices t_i.

Proof of Lemma 4: Suppose for contradiction that e has nonempty intersection with at least two non-consecutive intervals of γ. Let x (and y) denote the crossing of e and γ, closest to (respectively, farthest from) u along γ. Let x belong to I_i^γ, let y belong to I_j^γ, where $i < j-1$.

Let γ' be another (u,v)-path, which is identical to γ between u to x, identical to e from x to y, and finally identical to γ from y to v; see Figure 8. If $i < j-2$, then it is evident that $c(\gamma') \prec_{\text{lex}} c(\gamma)$, since γ' has fewer turning vertices than γ. If $i = j-2$, then γ and γ' have the same number of turning vertices, but $I_i^{\gamma'}$ contains fewer passing vertices than I_i^γ (hence $p_i^{\gamma'} < p_i^\gamma$), and we have $c(\gamma') \prec_{\text{lex}} c(\gamma)$. In both cases we obtain that $\gamma' \prec \gamma$, contradicting the minimality of γ. $\square$

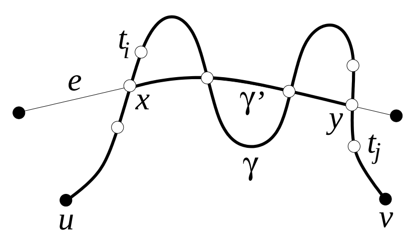

Figure 8: Two (u,v)-paths γ and γ' (both in bold) in the proof of Lemma 4.

Lemma 5. *Let e be an edge of G that has only finitely many points in common with γ.*

(i) If none of the common points is a vertex of e, then e crosses γ at most $2k$ times.

(ii) If one of the common points is a vertex of e, then e crosses γ at most $2k-1$ times.

Proof of Lemma 5: First, suppose that no vertex of e lies on γ. By Lemma 4, e crosses at most two consecutive intervals of γ. Each interval is a part of some edge of G and hence crosses e at most k times. This proves (i).

Suppose next that one of the vertices of e lies on γ. Observe that such a vertex must be a turning vertex of γ, say t_i. Again, by Lemma 4, e crosses at most two consecutive intervals of γ.

Each interval is a part of some edge of G. Moreover, one of them has a common endpoint with e. Therefore, e crosses one of the intervals at most k times and the other at most $k-1$ times. This proves (ii). $\square$

Note that no edge e of G that has only finitely many points in common with γ can have both of its endpoints on γ. Otherwise, both endpoints must be turning vertices of γ, say t_i and t_j for some $i < j$. Since the underlying abstract graph G is simple (that is, G has no multiple edges), the edge of G that contains I_{i+1}^{γ} must be different from the edge that contains I_j^{γ}. Hence, there is at least one turning vertex between t_i and t_j on γ. Now consider another (u,v)-path γ' that is identical to γ from u to t_i, identical to e from t_i to t_j, and finally identical to γ from t_j to v. The turning vertices t_i and t_j of γ are also turning vertices on γ'. Since the turning vertices of γ that lie between t_i and t_j are not among the turning vertices of γ', γ' has fewer turning vertices than γ. Therefore, we have $c(\gamma') \prec_{\text{lex}} c(\gamma)$, contradicting the minimality of γ.

Lemma 6. *Let e be an edge of G that contains an interval I_i^{γ} of γ. Then e and γ have at most k points in common outside of I_i^{γ}. Furthermore, one of these points is t_i, the endpoint of I_i^{γ}.*

Proof of Lemma 6: Since I_i^{γ} and I_{i+1}^{γ} are separated by t_i, and I_i^{γ} is contained in e, it follows that e cannot contain I_{i+1}^{γ}. Similarly, e cannot contain I_{i-1}^{γ}.

If e has a point p in I_j^{γ} with $j < i$, consider another (u,v)-path γ' that is identical to γ from u to p, identical to e from p to t_{i-1}, and finally identical to γ from t_{i-1} to v; see Figure 9. If $j < i-1$, the turning vertices t_j and t_{i-1} of γ are not among the turning vertices of γ'. Although p was a passing vertex of γ and is now a turning vertex of γ', still γ' has fewer turning vertices than γ. Therefore, $c(\gamma') \prec_{\text{lex}} c(\gamma)$. If $j = i-1$, the turning vertex t_i of γ is not a turning vertex of γ'. Again, p was a passing vertex of γ and is now a turning vertex of γ'. So, γ and γ' have the same number of turning vertices. Since p is not a passing vertex of γ', $I_{i-1}^{\gamma'}$ has fewer passing vertices than I_{i-1}^{γ} (hence $p_{i-1}^{\gamma'} < p_{i-1}^{\gamma}$), and we have that $c(\gamma') \prec_{\text{lex}} c(\gamma)$. In all of the above cases, we obtain that $\gamma' \prec \gamma$, contradicting the minimality of γ.

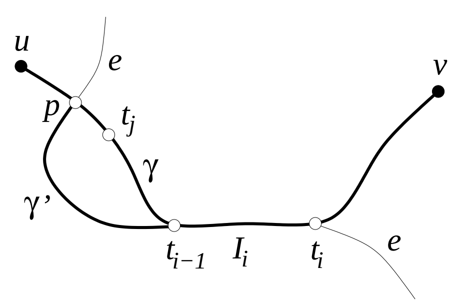

Figure 9: Two u,v-paths γ and γ' (both in bold) in the proof of Lemma 6; $j < i-1$.

Similarly, if e has a point p in I_j^{γ} with $j > i+1$, consider another (u,v)-path γ' that is identical to γ from u to t_i, identical to e from t_i to p, and finally identical to γ from p to v. The turning vertices t_i and t_{j-1} of γ are not among the turning vertices of γ'. Although p was a passing vertex of γ and is a turning vertex of γ', still γ' has fewer turning vertices than γ. Therefore, $c(\gamma') \prec_{\text{lex}} c(\gamma)$, contradicting the minimality of γ.

Note that the case $j = i+1$ cannot be settled in the same way as the previous cases, since the number of passing vertices on e between t_i and p may not be smaller than the number of passing vertices on γ between t_i and p. Nevertheless, we can conclude that no interval of γ other than I_i^γ is contained in e. Furthermore, the only interval of γ other than I_i^γ that can share some points with e is I_{i+1}^γ. Let f be the edge of G that contains I_{i+1}^γ. Since e and f have at most k points in common, e and I_{i+1}^γ can have at most k points in common, too. The point t_i, the common endpoint of I_i^γ and I_{i+1}^γ, is one of these points. □

Now we are in a position to complete the proof of Theorem 2. Join u and v by a curve β that runs very close to γ.

We claim that any edge e of G has at most $2k$ points in common with β. If e has only finitely many points in common with γ and none of them is a vertex of e, then every crossing between e and β corresponds to a crossing between e and γ. Therefore, by Lemma 5(i), e and β cross each other at most $2k$ times. If e has only finitely many points in common with γ, but one of them is a vertex of e, then each crossing between e and β corresponds to a crossing between e and γ, and there may be an additional crossing near the vertex of e on γ. Again, by Lemma 5(ii), there are at most $2k$ crossings between e and β. Finally, if e contains a whole interval I_i^γ of γ, then each crossing between e and β corresponds to a crossing between e and γ, or to a vertex of e on γ. There may be an additional crossing near the endpoint t_i of I_i^γ. Thus, there are at most $k+1$ crossings.
□

There is a slightly different further generalization of the concept of saturated k-simple topological graphs. For any positive integers k and ℓ, $k < \ell$, a topological graph G is called a (k,ℓ)-*saturated* if G is k-simple and any curve joining *any pair* of non-adjacent vertices has at least ℓ points in common with at least one edge of G. Obviously, every saturated k-simple topological graph is $(k, k+1)$-saturated.

Problem 3. *For which pairs of positive integers (k,ℓ) with $k < \ell$ do there exist (non-complete) (k,ℓ)-saturated topological graphs?*

We do not have a single example for (k,ℓ)-saturated topological graphs with $(k,\ell) \neq (1,2)$.

Clearly, every (k,ℓ)-saturated topological graph together with any pair of its non-adjacent vertices is a (k,ℓ)-construction. However, for $\ell > k+1$, the existence of a (k,ℓ)-construction does not necessarily imply the existence of a (k,ℓ)-saturated topological graph. It may help in proving the existence of one, as we have seen in the case $(k,\ell) = (1,2)$.

The constant 17.5 in Theorem 1 can almost certainly be improved. It is unlikely that every vertex in V_i, for most values of i, can be connected to all 35 possible neighbors (that is, all other vertices that belong to $V_{i-1} \cup V_i \cup V_{i+1}$), while preserving the simplicity of F_n.

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