Opaque Sets

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Abstract. The problem of finding "small" sets that meet every straight-line which intersects a given convex region was initiated by Mazurkiewicz in 1916. We call such a set an *opaque set* or a *barrier* for that region. We consider the problem of computing the shortest barrier for a given convex polygon with n vertices. No exact algorithm is currently known even for the simplest instances such as a square or an equilateral triangle. For general barriers, we present a $O(n)$ time approximation algorithm with ratio $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$. For connected barriers, we can achieve the approximation ratio $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ again in $O(n)$ time. We also show that if the barrier is restricted to the interior and the boundary of the input polygon, then the problem admits a fully polynomial-time approximation scheme for the connected case and a quadratic-time exact algorithm for the single-arc case. These are the first approximation algorithms obtained for this problem.

1 Introduction

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916 [27]; see also [3, 18]. Let C be a convex body in the plane. Following Bagemihl [3], we call a set B an *opaque set* or a *barrier* for C, if it meets all lines that intersect C. A *rectifiable curve* (or arc) is a curve with finite length. A barrier may consist of one or more rectifiable arcs. It does not need to be connected and its portions may lie anywhere in the plane, including the exterior of C; see [3], [5].

What is the length of the shortest barrier for a given convex body C? In spite of considerable efforts, the answer to this question is not known even for the simplest instances of C, such as a square, a disk, or an equilateral triangle; see [6], [7, Problem A30], [10], [12], [13], [16, Section 8.11]. The three-dimensional analogue of this problem was raised by Martin Gardner [17]; see also [2, 5]. Some

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entertaining variants of the problem appeared in different forms [20, 23, 24], for instance: What should a swimmer at sea do in a thick fog if he knows that he is within a mile of a straight shoreline? The shortest known solution resembles the shortest known single-arc barrier for a disk of radius one mile; see [7, Problem A30].

A barrier blocks any line of sight across the region C or detects any ray that passes through it. Motivated by potential applications in guarding and surveillance, the problem of short barriers has been studied by several research communities. Recently, it circulated in internal publications at the Lawrence Livermore National Laboratory. The shortest barrier known for the square is illustrated in Figure 1(right). It is conjectured to be optimal. The best lower bound we know is 2, established by Jones [19].

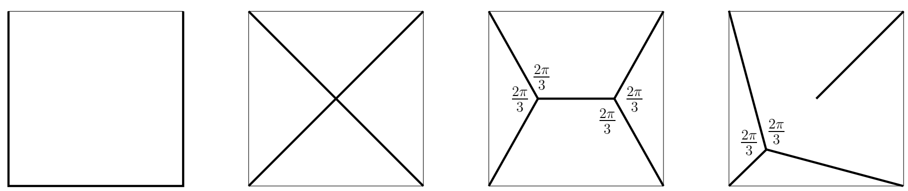

Fig. 1: Four barriers for the unit square. From left to right: 1: single-arc; 2–3: connected; 4: disconnected. The first three from the left have lengths 3, $2\sqrt{2} = 2.8284\ldots$, and $1+\sqrt{3} = 2.7320\ldots$. Right: The diagonal segment $[(1/2, 1/2), (1, 1)]$ together with three segments connecting the corners $(0, 1)$, $(0, 0)$, $(1, 0)$ to the point $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6})$ yield a barrier of length $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.639\ldots$.

Related work. The type of curve barriers considered may vary: the most restricted are barriers made from single continuous arcs, then connected barriers, and lastly, arbitrary (possibly disconnected) barriers. For the unit square, the shortest known in these three categories have lengths 3, $1+\sqrt{3} = 2.7320\ldots$ and $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$, respectively. They are depicted in Figure 1. Interestingly, it has been shown by Kawohl [21] that the barrier in Figure 1(right) is optimal in the class of curves with at most two components (there seems to be an additional implicit assumption that the barrier is restricted to the interior of the square). For the unit disk, the shortest known barrier consists of three arcs. See also [12, 16].

If instead of curve barriers, we want to find *discrete* barriers consisting of as few points as possible with the property that every line intersecting C gets closer than $\varepsilon > 0$ to at least one of them in some fixed norm, we arrive at a problem raised by László Fejes Tóth [14, 15]. The problem has been later coined suggestively as the "point goalie problem" [31]. For instance, if C is an axis-parallel unit square, and we consider the *maximum norm*, the problem was studied by Bárány and Füredi [4], Kern and Wanka [22], Valtr [35], and

Richardson and Shepp [31]. Makai and Pach [26] considered another variant of the question, in which we have a larger class of functions to block.

The problem of short barriers has attracted many other researchers and has been studied at length; see also [6, 11, 25]. Obtaining lower bounds for many of these problems appears to be notoriously hard. For instance in the point goalie problem for the unit disk (with the Euclidean norm), while the trivial lower bound is $1/\varepsilon$, as given by the opaqueness condition in any one direction, the best lower bound known is only $1.001/\varepsilon$ as established in [31] via a complicated proof.

Our Results. Even we have so little control on the shape or length of optimal barriers, for any convex polygon, barriers whose lengths are somewhat longer can computed efficiently. Let P be a given convex polygon with n vertices.

1. A (possibly disconnected) segment barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times the optimal, can be computed in $O(n)$ time.
2. A connected polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times the optimal can be also computed in $O(n)$ time.
3. For interior single-arc barriers we present an algorithm that finds an optimal barrier in $O(n^2)$ time.
4. For interior connected barriers we present an algorithm that finds a barrier whose length is at most $(1+\varepsilon)$ times the optimal in polynomial time.

It might be worth mentioning to avoid any confusion: the approximation ratios are for each barrier class, that is, the length of the barrier computed is compared to the optimal length in the corresponding class; and of course these optimal lengths might differ. For instance the connected barrier computed by the the approximation algorithm with ratio $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ is *not* necessarily shorter than the (possibly disconnected) barrier computed by the the approximation algorithm with the larger ratio $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$.

2 Preliminaries

Definitions and notations. For a polygonal curve γ, let $|\gamma|$ denote the length (or weight) of γ. Similarly, if Γ is a set of polygonal curves, let $|\Gamma|$ denote the total length of the curves in Γ. As usual, when there is no danger of confusion, we also denote by $|A|$ the cardinality of a set A. We call a barrier consisting of segments (or polygonal lines) a *segment barrier*. In order to be able to speak of the *length* $\ell(B)$ of a barrier B, we restrict our attention to barriers that can be obtained as the union of finitely many simple rectifiable curves. We first show (Lemma 1) that the shortest segment barrier is not much longer than the shortest rectifiable one. Due to space limitations we omit the proof of Lemma 1.

Lemma 1. *Let B be a barrier of length $\ell(B) < \infty$ for a convex body C in the plane. Then, for any $\varepsilon > 0$, there exists a segment barrier B_ε for C, consisting of finitely many straight-line segments, such that $\ell(B_\varepsilon) \le \ell(B) + \varepsilon$.*

Denote by $\mathrm{per}(C)$ the perimeter of a convex body C in the plane. The following lemma providing a lower bound on the length of an optimal barrier for C in terms of $\mathrm{per}(C)$, is used in the analysis of our approximation algorithms. Its proof is folklore; see e.g. [13].

Lemma 2. *Let C be a convex body in the plane and let B be a barrier for C. Then the length of B is at least $\frac{1}{2} \cdot \mathrm{per}(C)$.*

Proof. By Lemma 1, we can assume w.l.o.g. that B is a segment barrier. Let $B = \{s_1, \ldots, s_n\}$ consist of n segments of lengths $\ell_i = |s_i|$, where $L = |B| = \sum_{i=1}^{n} \ell_i$. Let $\alpha_i \in [0, \pi)$ be the angle made by s_i with the x-axis. For each direction $\alpha \in [0, \pi)$, the blocking (opaqueness) condition for a convex body C can be written as

$$\sum_{i=1}^{n} \ell_i |\cos(\alpha - \alpha_i)| \geq W(\alpha), \tag{1}$$

where $W(\alpha)$ is the width of C in direction α. By integrating this inequality over the interval $[0, \pi]$, one gets:

$$\sum_{i=1}^{n} \ell_i \int_0^{\pi} |\cos(\alpha - \alpha_i)| \, \mathrm{d}\alpha \geq \int_0^{\pi} W(\alpha) \, \mathrm{d}\alpha. \tag{2}$$

According to Cauchy's surface area formula [28, pp. 283–284], for any planar convex body C, we have

$$\int_0^{\pi} W(\alpha) \, \mathrm{d}\alpha = \mathrm{per}(C). \tag{3}$$

Since

$$\int_0^{\pi} |\cos(\alpha - \alpha_i)| \, \mathrm{d}\alpha = 2,$$

we get

$$2L = \sum_{i=1}^{n} 2\ell_i \geq \mathrm{per}(C) \;\Rightarrow\; L \geq \frac{1}{2} \cdot \mathrm{per}(C), \tag{4}$$

as required. ☐

For instance, for the square, $\mathrm{per}(C) = 4$, and Lemma 2 immediately gives $L \geq 2$, the lower bound of Jones [19]).

A key fact in the analysis of the approximation algorithm is the following lemma. This inequality is implicit in [36]; another proof can be found in [9].

Lemma 3. *Let P be a convex polygon. Then the minimum-perimeter rectangle R containing P satisfies $\mathrm{per}(R) \leq \frac{4}{\pi}\, \mathrm{per}(P)$.*

Let P be a convex polygon with n vertices. Let $\mathrm{OPT}_{\mathrm{arb}}(P)$, $\mathrm{OPT}_{\mathrm{conn}}(P)$ and $\mathrm{OPT}_{\mathrm{arc}}(P)$ denote optimal barrier lengths of the types arbitrary, connected, and single-arc. Let us observe the following inequalities:

$$\mathrm{OPT}_{\mathrm{arb}}(P) \leq \mathrm{OPT}_{\mathrm{conn}}(P) \leq \mathrm{OPT}_{\mathrm{arc}}(P). \tag{5}$$

We first deal with connected barriers, and then with arbitrary (i.e., possibly disconnected) barriers.

3 Connected Barriers

Theorem 1. *Given a convex polygon P with n vertices, a connected polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times longer than the optimal can be computed in $O(n)$ time.*

Proof. Consider the following algorithm **A1** that computes a connected barrier consisting of a single-arc; refer to Figure 2. First compute a parallel strip of

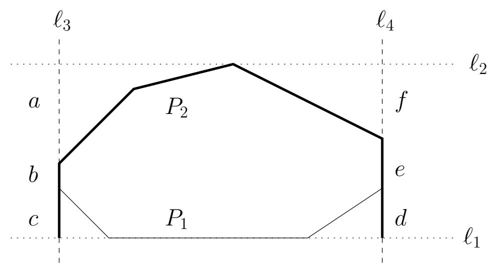

Fig. 2: The approximation algorithm **A1** returns B_2 (in bold lines).

minimum width enclosing P. Assume w.l.o.g. that the strip is bounded by the two horizontal lines ℓ_1 and ℓ_2. Second, compute a minimal orthogonal (i.e., vertical) strip enclosing P, bounded by the two vertical lines ℓ_3 and ℓ_4. Let a, b, c, d, e, f be the six segments on ℓ_3 and ℓ_4 as shown in the figure; here b and e are the two (possibly degenerate) segments on the boundary of P. Let P_1 be the polygonal path (on P's boundary) between the lower vertices of b and e. Let P_2 be the polygonal path (on P's boundary) between the top vertices of b and e. Consider the following two barriers for P: B_1 consists of the polygonal path P_1 extended upward at both ends until they reach ℓ_2. B_2 consists of the polygonal path P_2 extended downwards at both ends until they reach ℓ_1. The algorithm returns the shorter of the two.

Let p, w, and r, respectively, be the perimeter, the width, and the in-radius of P. Clearly

$$|P_1| + |P_2| + |b| + |e| = p.$$

We have the following equalities:

$$|B_1| = |a| + |b| + |P_1| + |e| + |f|,$$
$$|B_2| = |c| + |b| + |P_2| + |e| + |d|.$$

By adding them up we get

$$|B_1| + |B_2| = |P_1| + |P_2| + |b| + |e| + 2w = p + 2w.$$

Hence

$$\min\{|B_1|, |B_2|\} \le p/2 + w.$$

By Blaschke's Theorem (see e.g. [32]), every planar convex body of width w contains a disk of radius $w/3$. Thus $r \geq w/3$. According to a result of Eggleston [10], the optimal connected barrier for a disk of radius r has length $(\pi+2)r$. It follows that the optimal connected barrier for P has length at least $(\pi+2)w/3$. By Lemma 2, $p/2$ is another lower bound on the optimal solution. Thus the approximation ratio of the algorithm **A1** is at most

$$\begin{aligned}\frac{p/2+w}{\max\{(\pi+2)w/3,\, p/2\}} &= \min\left\{\frac{p/2+w}{(\pi+2)w/3}, \frac{p/2+w}{p/2}\right\} \\ &= \min\left\{\frac{3}{2(\pi+2)}\cdot\frac{p}{w}+\frac{3}{\pi+2}, 1+2\cdot\frac{w}{p}\right\}.\end{aligned}$$

The equation

$$\frac{3x}{2(\pi+2)}+\frac{3}{\pi+2}=1+\frac{2}{x}$$

has one positive real root $x_0 = \frac{2(\pi+2)}{3}$. Consequently, the approximation ratio of the algorithm **A1** is at most $1+\frac{3}{\pi+2} = \frac{\pi+5}{\pi+2} = 1.5834\ldots$. The algorithm takes $O(n)$ time, since computing the width of P takes $O(n)$ time; see [29, 34]. $\square$

4 Single-arc Barriers

Since **A1** computes a single-arc barrier, and we have $\mathrm{OPT}_{\mathrm{conn}}(P) \leq \mathrm{OPT}_{\mathrm{arc}}(P)$, we immediately get an approximation algorithm with the same ratio $1.5834\ldots$ for computing single-arc barriers. One may ask whether this single arc barrier computed by **A1** is optimal (in the class of single arc barriers). We show that this is not the case:

Consider a Reuleaux triangle T of (constant) width 1, with three vertices a, b, c. Now slightly shave the two corners of T at b and c to obtain a convex body T' of (minimum) width $1-\varepsilon$ along bc. The algorithm **A1** would return a curve of length close to $\pi/2+1 = 2.57\ldots$, while the optimal curve has length at most $2\pi/3 + 2(1-\sqrt{3}/2) = 2\pi/3+2-\sqrt{3} = 2.36\ldots$. This example shows a lower bound of $1.088\ldots$ on the approximation ratio of the algorithm **A1**. Moreover, we believe that the approximation ratio of **A1** is much closer to this lower bound than to $1.5834\ldots$.

We next present an improved version **B1** of our algorithm **A1** that computes the shortest single-arc barrier of the form shown in Figure 2; see below for details.

Let P be a convex polygon with n sides, and let ℓ be a line tangent to the polygon, i.e., $P \cap \ell$ consists of a vertex of P or a side of P. For simplicity assume that ℓ is the x-axis, and P lies in the closed halfplane $y \geq 0$ above ℓ. Let $T = (\ell_1, \ell_2)$ be a minimal vertical strip enclosing P. Let $p_1 \in \ell_1 \cap P$ and $p_2 \in \ell_2 \cap P$, be the two points of P of minimum y-coordinates on the two vertical lines defining the strip. Let $q_1 \in \ell_1$ and $q_2 \in \ell_2$ be the projections of p_1 and p_2, respectively, on ℓ. Let $\mathrm{arc}(p_1, p_2) \subset \partial(\mathrm{conv}(P))$ be the polygonal arc connecting p_1 and p_2 on the top boundary of P . The U-curve corresponding to P and ℓ, denoted $U(P, \ell)$ is the polygonal curve obtained by concatenating q_1p_1, arc(p_1, p_2), and p_2q_2, in this order. Obviously, for any line ℓ, the curve $U(P, \ell)$ is a single-arc barrier for P. Let $U_{\min}(P)$ be the U-curve of minimum length over all directions $\alpha \in [0, \pi)$ (i.e., lines ℓ of direction α).

We next show that given P, the curve $U_{\min}(P)$ can be computed in $O(n)$ time. The algorithm **B1** is very simple: instead of rotating a line ℓ around P, we fix ℓ to be horizontal, and rotate P over ℓ by one full rotation (of angle 2π). We only compute the lengths of the U-curves corresponding to lines ℓ, ℓ_1, ℓ_2, supporting one edge of the polygon. The U-curve of minimum length among these is output. There are at most $3n$ such discrete angles (directions), and the length of a U-curve for one such angle can be computed in constant time from the the length of the U-curve for the previous angle. The algorithm is similar to the classic rotating calipers algorithm of Toussaint [34], and it takes $O(n)$ time by the previous observation.

To justify its correctness, it suffices to show that if each of the lines ℓ, ℓ_1, ℓ_2 is incident to only one vertex of P, then the corresponding U-curve is not minimal. Due to space limitations we omit the proof of Lemma 4.

Lemma 4. *Let P be a convex polygon tangent to a line ℓ at a vertex $v \in P$ only, and tangent to ℓ_1 and ℓ_2 at vertices p_1 and p_2 only. Then the corresponding U-curve $U(P, \ell)$ is not minimal.*

We thus conclude this section with the following result.

Theorem 2. *Given a convex polygon P with n vertices, the single-arc barrier (polygonal curve) $U_{\min}(P)$ can be computed in $O(n)$ time.*

Obviously, the single-arc barrier computed by **B1** is not longer than that computed by **A1**, so the approximation ratio of the algorithm **B1** is also bounded by $\frac{\pi+5}{\pi+2} = 1.5834\ldots$. One may ask again whether this single arc barrier computed by **B1** is optimal (in the class of single arc barriers). We can show again that this is not the case (details omitted here).

5 Arbitrary Barriers

Theorem 3. *Given a convex polygon P with n vertices, a (possibly disconnected) segment barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times longer than the optimal can be computed in $O(n)$ time.*

Proof. Consider the following algorithm **A2** which computes a (generally disconnected) barrier. First compute a minimum-perimeter rectangle R containing P; refer to Figure 3. Let a,b,c,d,e,f,g,h, i,j,k,l be the 12 segments on the boundary of R as shown in the figure; here b, e, h and k are (possibly degenerate) segments on the boundary of P contained in the left, bottom, right and top side of R. Let P_i, $i = 1, 2, 3, 4$ be the four polygonal paths on P's boundary, connecting these four segments as shown in the figure.

Consider four barriers for P, denoted B_i, for $i = 1, 2, 3, 4$. B_i consists of the polygonal path P_i extended at both ends on the corresponding rectangle sides,

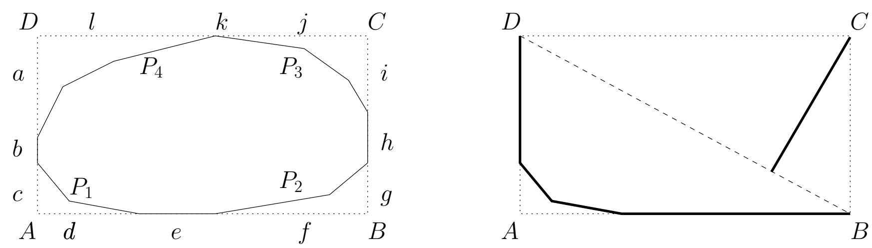

Fig. 3: The approximation algorithm **A2**.

and the height from the opposite rectangle vertex in the complementary right angled triangle; see Figure 3(right). The algorithm returns the shortest of the four barriers. Let h_A, h_B, h_C, h_D denote the four heights. We have $|h_A| = |h_B| = |h_C| = |h_D|$ and the following other length equalities:

$$\begin{aligned}
|B_1| &= |a| + |b| + |P_1| + |e| + |f| + |h_A|, \\
|B_2| &= |d| + |e| + |P_2| + |h| + |i| + |h_B|, \\
|B_3| &= |g| + |h| + |P_3| + |k| + |l| + |h_C|, \\
|B_4| &= |j| + |k| + |P_4| + |b| + |c| + |h_D|.
\end{aligned}$$

By adding them up we get

$$\begin{aligned}
\sum_{i=1}^{4} |B_i| = \Big(|b| + |e| + |h| + |k| + \sum_{i=1}^{4} |P_i|\Big) + \big(|a| + \ldots + |k|\big) \\
+ \big(|h_A| + |h_B| + |h_C| + |h_D|\big) = \mathrm{per}(P) + \mathrm{per}(R) + 4|h_A|. \qquad (6)
\end{aligned}$$

Expressing the rectangle area in two different ways yields $|h_A| = \frac{xy}{\sqrt{x^2+y^2}}$, where x and y are the lengths of the two sides of R. By Lemma 3 we have

$$\mathrm{per}(R) = 2(x+y) \leq \frac{4}{\pi}\mathrm{per}(P).$$

Under this constraint, $|h_A|$ is maximized for $x = y = \frac{\mathrm{per}(P)}{\pi}$, namely

$$|h_A| \leq \frac{\mathrm{per}(P)}{\pi\sqrt{2}} \quad \Rightarrow \quad 4|h_A| \leq \frac{2\sqrt{2}}{\pi}\mathrm{per}(P).$$

Hence from (6) we deduce that

$$\min_i |B_i| \leq \frac{1}{4}\left(1 + \frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}\right)\mathrm{per}(P).$$

Recall that $\mathrm{per}(P)/2$ is a lower bound on the weight of an optimal solution. The ratio between the length of the solution and the lower bound on the optimal solution is

$$\frac{\pi + 4 + 2\sqrt{2}}{2\pi} = \frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$$

Consequently, the approximation ratio of the algorithm **A2** is $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$. The algorithm takes $O(n)$ time, since computing the minimum-perimeter rectangle containing P takes $O(n)$ time with the standard technique of rotating calipers [29, 34]. This completes the proof of Theorem 3. □

6 Interior-restricted versus Unrestricted Barriers

In certain instances, it is infeasible to construct barriers guarding a specific domain outside the domain (which presumably belongs to someone else). We call such barriers constrained to the interior and the boundary of the domain, *interior-restricted*, or just *interior*, and all others *unrestricted*. For example, all four barriers for the unit square illustrated in Figure 1 are interior barriers.

In the late 1980s, Akman [1] soon followed by Dublish [8] had reported algorithms for computing a minimum interior-restricted barrier of a given convex polygon (they refer to such a barrier as an *opaque minimal forest* of the polygon). Both algorithms however have been shown to be incorrect by Shermer [33] in 1991. He also proposed (conjectured) a new exact algorithm instead, but apparently, so far no one succeeded to prove its correctness. To the best of our knowledge, the computational complexity of computing a shortest barrier (either interior-restricted or unrestricted) for a given convex polygon remains open.

Next we show that a minimum *connected interior* barrier for a convex polygon can be computed efficiently:

Theorem 4. *Given a convex polygon P, a minimum Steiner tree of the vertices of P forms a minimum connected interior barrier for P. Consequently, there is a fully polynomial-time approximation scheme for finding a minimum connected interior barrier for a convex polygon.*

Proof. Let B be an optimal barrier. For each vertex $v \in P$, consider a line ℓ_v tangent to P at v, such that $P \cap \ell_v = \{v\}$. Since B lies in P, ℓ_v can be only blocked by v, so $v \in B$. Now since B is connected and includes all vertices of P, its length is at least that of a minimum Steiner tree of P, as claimed. Recall that the minimum Steiner tree problem for n points in the plane in convex position admits a fully polynomial-time approximation scheme that achieves an approximation ratio of $1 + \varepsilon$ and runs in time $O(n^6/\epsilon^4)$ for any $\varepsilon > 0$ [30]. □

A minimum *single-arc interior* barrier for a convex polygon can be also computed efficiently. As it turns out, this problem is equivalent to that of finding a shortest traveling salesman path (i.e., Hamiltonian path) for the n vertices of the polygon.

Theorem 5. *Given a convex polygon P, a minimum Hamiltonian path of the vertices of P forms a minimum single-arc interior barrier for P. Consequently, there is an $O(n^2)$-time exact algorithm for finding a minimum single-arc interior barrier for a convex polygon with n vertices.*

Proof. The same argument as in the proof of Theorem 4 shows that any interior barrier for P must include all vertices of P. By the triangle inequality, the optimal single-arc barrier visits each vertex exactly once. Thus a minimum Hamiltonian path of the vertices forms a minimum single-arc interior barrier.

We now present a dynamic programming algorithm for finding a minimum Hamiltonian path of the vertices of a convex polygon. Let $\{v_0, \ldots, v_{n-1}\}$ be the n vertices of the convex polygon in counter-clockwise order; for convenience, the indices are modulo n, e.g., $v_n = v_0$. Denote by $\mathrm{dist}(i, j)$ the Euclidean distance between the two vertices v_i and v_j. For the subset of vertices from v_i to v_j counter-clockwise along the polygon, denote by $S(i, j)$ the minimum length of a Hamiltonian path starting at v_i, and denote by $T(i, j)$ the minimum length of a Hamiltonian path starting at v_j. Note that a minimum Hamiltonian path must not intersect itself. Thus the two tables S and T can be computed by dynamic programming with the base cases

$$S(i, i+1) = T(i, i+1) = \mathrm{dist}(i, i+1)$$

and with the recurrences

$$\begin{aligned} S(i, j) &= \min\{\mathrm{dist}(i, i+1) + S(i+1, j),\ \mathrm{dist}(i, j) + T(i+1, j)\}, \\ T(i, j) &= \min\{\mathrm{dist}(j, j-1) + T(i, j-1),\ \mathrm{dist}(j, i) + S(i, j-1)\}. \end{aligned}$$

Then the minimum length of a Hamiltonian path on the n vertices is

$$\min_i \min\{\mathrm{dist}(i, i+1) + S(i+1, i-1),\ \mathrm{dist}(i, i-1) + T(i+1, i-1)\}.$$

The running time of the algorithm is clearly $O(n^2)$. □

Remark. Observe that the unit square contains a disk of radius 1/2. According to the result of Eggleston mentioned earlier [10], the optimal (not necessarily interior-restricted) connected barrier for a disk of radius r has length $(\pi + 2)r$. This optimal barrier is a single curve consisting of half the disk perimeter and two segments of length equal to the disk radius. It follows that the optimal (not necessarily interior-restricted) connected barrier for the unit square has length at least $(\pi + 2)/2 = \pi/2 + 1 = 2.5707\ldots$. Compare this with the current best construction (illustrated in Figure 1, third from the left) of length $1 + \sqrt{3} = 2.7320\ldots$. Note that this third construction in Figure 1 gives the optimal connected interior barrier for the square because of Theorem 4. Further note that the first construction in Figure 1 gives the optimal single-arc interior barrier because of Theorem 5.

7 Concluding Remarks

Interesting questions remain open regarding the structure of optimal barriers and the computational complexity of computing such barriers. For instance:

(1) Does there exist an absolute constant $c \geq 0$ (perhaps zero) such that the following holds? The shortest barrier for any convex polygon with n vertices is a segment barrier consisting of at most $n + c$ segments.
(2) Is there a polynomial-time algorithm for computing a shortest barrier for a given convex polygon with n vertices?
(3) Can one give a characterization of the class of convex polygons whose optimal barriers are interior?

In connection with question (2) above, let us notice that the problem of deciding whether a given segment barrier B is an opaque set for a given convex polygon is solvable in polynomial time. Due to space limitations the proof of Theorem 6 is omitted.

Theorem 6. *Given a convex polygon P with n vertices, and a segment barrier B with k segments, there is a polynomial-time algorithm for deciding whether B is an opaque set for P.*

We have presented several approximation and exact algorithms for computing shortest barriers of various kinds, for a given convex polygon. The two approximation algorithms with ratios close to 1.58 probably cannot be improved substantially without either increasing their computational complexity or finding a better lower bound on the optimal solution than that given by Lemma 2. The question of finding a better lower bound is particularly intriguing, since even for the simplest polygons, such as a square, we don't possess any better tool. While much research up to date focused on upper or lower bounds for specific example shapes, obtaining a polynomial time approximation scheme (in the class of arbitrary barriers) for an arbitrary convex polygon is perhaps not out of reach.

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