Erdős-Hajnal-type results for ordered paths

János Pach [*†‡] István Tomon [§]

Abstract

An *ordered graph* is a graph with a linear ordering on its vertex set. We prove that for every positive integer k, there exists a constant $c_k > 0$ such that any ordered graph G on n vertices with the property that neither G nor its complement contains an induced monotone path of size k, has either a clique or an independent set of size at least n^{c_k}. This strengthens a result of Bousquet, Lagoutte, and Thomassé, who proved the analogous result for unordered graphs.

A key idea of the above paper was to show that any unordered graph on n vertices that does not contain an induced path of size k, and whose maximum degree is at most $c(k)n$ for some small $c(k) > 0$, contains two disjoint linear size subsets with no edge between them. This approach fails for ordered graphs, because the analogous statement is false for $k \geq 3$, by a construction of Fox. We provide further examples how this statement fails for ordered graphs avoiding other ordered trees as well.

1 Introduction

Erdős and Hajnal [10] proved that graphs avoiding some fixed induced subgraph or subgraphs have very favorable Ramsey-theoretic properties. In particular, they contain surprisingly large homogeneous (that is, complete or empty) subgraphs and bipartite subgraphs. According to the celebrated Erdős-Hajnal conjecture, every graph G on n vertices which does not contain some fixed graph H as an induced subgraph, has a clique or an independent set of size at least n^c, where $c = c(H) > 0$ is a constant that depends only on H. There is a rapidly growing body of literature studying this conjecture (see, e.g., [1, 2, 5, 6, 8, 11, 13, 15, 23]).

For any graph G and any disjoint subsets $A, B \subset V(G)$, we say that A *is complete to* B if $ab \in E(G)$ for every $a \in A, b \in B$. If $|A| = |B| = k$ and A is complete to B, then A and B are said to form a *bi-clique of size* k. Denote the maximum degree of the vertices in G by $\Delta(G)$. Following [13], a family of graphs $\mathcal{G}$ is said to have the *Erdős-Hajnal* property if there exists a constant $c = c(\mathcal{G}) > 0$ such that every $G \in \mathcal{G}$ has either a clique or an independent set of size at least $|V(G)|^c$. The family $\mathcal{G}$ has the *strong Erdős-Hajnal property* if there exists a constant $b = b(\mathcal{G}) > 0$ such that for every $G \in \mathcal{G}$, either G or its complement $\overline{G}$ has a bi-clique of size $b|V(G)|$. It was proved in [1]

[*]Rényi Institute, Budapest, *e-mail*: **pach@cims.nyu.edu**

[†]IST Austria, Vienna, partially supported by Austrian Science Fund (FWF), grant Z 342-N31.

[‡]MIPT, Moscow, partially supported by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926.

[§]ETH Zurich, Research partially supported by Swiss National Science Foundation grants no. 200020-175573. *e-mail*: **istvan.tomon@math.ethz.ch**

that if a *hereditary* family (that is, a family closed under taking induced subgraphs) has the strong Erdős-Hajnal property, then it also has the Erdős-Hajnal property.

The aim of this paper is to discuss Erdős-Hajnal type problems for ordered graphs. An *ordered graph* is a graph with a total ordering on its vertex set. With a slight abuse of notation, in every ordered graph, we denote this ordering by $\prec$. If the vertex set of G is a subset of the integers, then $\prec$ stands for the natural ordering. An ordered graph H is an *ordered subgraph* (or simply subgraph) of G if there exists an order preserving embedding from $V(H)$ to $V(G)$ that maps edges to edges. If, in addition, non-edges are mapped into non-edges, then H is called an *induced* ordered subgraph of G. If G does not have H as induced ordered subgraph, then we say that G *avoids* H. The ordered path with vertices $1,\dots,k$ and edges $\{i,i+1\}$, for $i=1,\dots,k-1$, is called a *monotone path of size* k.

Our main result is the following.

Theorem 1. *For any positive integer k, there exists $c=c(k)>0$ with the following property. If G is an ordered graph on n vertices such that neither G nor its complement contains an induced monotone path of size k, then G has either a clique or an independent set of size at least n^c.*

Our theorem obviously implies the analogous statement for unordered graphs, which was first established by Bousquet, Lagoutte, and Thomassé [5]. The idea of their proof was the following. We call a family of graphs, $\mathcal{H}$, *lopsided* if there exists a constant $c=c(\mathcal{H})>0$ with the following property: any graph G on n vertices which does not contain any element of $\mathcal{H}$ as an induced subgraph, and for which $\Delta(G)<cn$, the complement of G has a bi-clique of size at least cn. If $\mathcal{H}$ consists of a single graph H, then H is called lopsided. They proved that the (unordered) path of size k is lopsided. It follows from the arguments of Bousquet *et al.* that if $\mathcal{H}$ is lopsided, then the family of all graphs which avoid every element of $\mathcal{H}$ as an induced subgraph, and whose complements also avoid them, has the strong Erdős-Hajnal and, thus, the Erdős-Hajnal property.

Since then, this idea has been exploited to prove the Erdős-Hajnal conjecture for various other families of graphs: the family of graphs avoiding a tree T and its complement [8], the family of graphs avoiding all subdivisions of a graph H and the complements of these subdivisions [9], the family of graphs avoiding a graph H as a vertex minor [7], families of graphs avoiding a fixed cycle as a pivot minor [16], etc.

However, for ordered graphs, this method does not work even in the simplest case: for monotone paths. A construction of Fox [12] shows that, for every n and $\delta>0$, there exists an ordered graph G with $|V(G)|=n$ and $\Delta(G)<n^{\delta}$ which avoids the monotone path of size 3, and whose complement does not contain a bi-clique of size larger than $\frac{cn}{\log n}$, for a suitable constant $c=c(\delta)>0$. Hence, using the above terminology, the monotone path of size at least 3 is *not lopsided*.

Although monotone paths are not lopsided, they satisfy a somewhat weaker property, as is shown by the following theorem of the authors.

Theorem 2. *([19]) For any positive integer k, there exists a constant $c=c(k)>0$ with the following property. If G is an ordered graph on n vertices that does not contain an induced monotone path of size k, and $\Delta(G)<cn$, then the complement of G contains a bi-clique of size at least $\frac{cn}{\log n}$.*

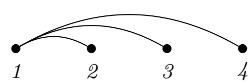
Unfortunately, Theorem 1 cannot be deduced from this weaker property. Our approach is based on a technique in [24], where it was shown that the family of string graphs has the Erdős-Hajnal property.

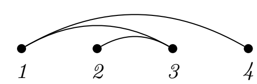
Recently, Seymour, Scott, and Spirkl [23] extended our Theorem 2 from monotone paths to all ordered forests T, albeit with a weaker bound $n^{1-o(1)}$ in place of $\frac{cn}{\log n}$. They proved that for any $0 < c < 1$, there exists $\epsilon = \epsilon(T, c) > 0$ with the following property. If G is an ordered graph on n vertices that does not contain T as an induced ordered subgraph and $\Delta(G) < \epsilon n$, then the complement of G contains a bi-clique of size at least ϵn^{1-c}. Therefore, if we want to guarantee a bi-clique of size $n^{1-o(1)}$ in $\overline{G}$, we need to assume that the maximum degree of G is $o(n)$. This is definitely a stronger condition than the one we had for monotone paths.

Our next construction shows that this stronger condition is indeed necessary. We also provide new examples of ordered trees T (that do not contain a monotone path of size 3), for which one cannot expect to find linear size bi-cliques.

Theorem 3. *For any $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ with the following property.*

For any positive integer $n \geq n_0$, there is an ordered graph G with n vertices and $\Delta(G) \leq \epsilon n$ such that the size of the largest bi-clique in $\overline{G}$ is at most $n^{1-\delta}$, and G does not contain either of the following ordered trees as an induced ordered subgraph:

S : 

P : 

The investigation of bipartite variants of the problems considered in this paper were initiated in [17]; see also [3, 22].

Our paper is organized as follows. In Section 2, we introduce the key concept needed for the proof of Theorem 1 and reduce Theorem 1 to another statement (Theorem 6). Sections 3 and 4 are devoted to the proof of this latter statement. The construction proving Theorem 3 will be presented in Sections 5.

Throughout this paper, we use the following notation, which is mostly conventional. For any graph G and any subset $U \subset V(G)$, we denote by $G[U]$ the subgraph of G induced by U. The *neighborhood* of U is defined as $N_G(U) = N(U) = \{v \in V(G) \setminus U : \exists u \in U, uv \in E(G)\}$. If $U = \{u\}$, instead of $N(U)$, we simply write $N(u)$. For a vertex $v \in V(G)$, let $G - v$ stand for the graph obtained from G by deleting the vertex v. Also, if G is an ordered graph, the *forward neighbourhood* of a vertex $v \in V(G)$, denoted by $N_G^+(y) = N^+(y)$ is the set of neighbours y such that $x \prec y$.

For easier readability, we omit the use of floors and ceilings, whenever they are not crucial.

2 The quasi-Erdős-Hajnal property

After introducing some notation and terminology, we outline our proof strategy for Theorem 1.

For any $k \geq 3$, let $\mathcal{P}_k$ denote the family of all ordered graphs G such that neither G nor its complement contains a monotone path of size k as an induced subgraph. Instead of proving that $\mathcal{P}_k$ has the Erdős-Hajnal property, we prove that it has the *quasi-Erdős-Hajnal property.* This concept was introduced by the second named author in [24], in order to show that the family of string graphs has the Erdős-Hajnal property.

Definition 4. *A family of graphs, $\mathcal{G}$, has the* quasi-Erdős-Hajnal property *if there is a constant $c = c(\mathcal{G}) > 0$ with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and t disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq (\frac{|V(G)|}{|X_i|})^c$ for $i = 1, \ldots, t$, and*

(i) *either there is no edge between X_i and X_j for $1 \leq i < j \leq t$,*

(ii) *or X_i is complete to X_j for $1 \leq i < j \leq t$.*

It was proved in [24] that in hereditary families, the quasi-Erdős-Hajnal property is equivalent to the Erdős-Hajnal property. We somewhat relax the definition of the quasi-Erdős-Hajnal property, and with a slight abuse of notation, we overwrite the previous definition as follows.

Definition 5. *A family of graphs, $\mathcal{G}$, has the* quasi-Erdős-Hajnal property *if there are two constants, $\alpha, \beta > 0$, with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and t disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq \alpha(\frac{|V(G)|}{|X_i|})^\beta$ for $i = 1, \ldots, t$, and*

(i) *either there is no edge between X_i and X_j for $1 \leq i < j \leq t$,*

(ii) *or X_i is complete to X_j for $1 \leq i < j \leq t$.*

It is easy to verify that the two definitions are in fact equivalent. If $\mathcal{G}$ satisfies Definition 4, then, obviously, it also satisfies Definition 5. In the reverse direction, setting $c = \frac{\beta}{1-\log_2 \alpha}$ if $\alpha \leq 1$, and $c = \beta$ if $\alpha > 1$, if the inequality $t \geq \alpha(\frac{|V(G)|}{|X_i|})^\beta$ holds for some $t \geq 2$, then we also have $t \geq (\frac{|V(G)|}{|X_i|})^c$.

Therefore, it is enough to show that $\mathcal{P}_k$ has the quasi-Erdős-Hajnal property. The advantage of the quasi-Erdős-Hajnal property compared to the Erdős-Hajnal property is that it allows us to establish the following lopsided statement, which will imply Theorem 1.

Theorem 6. *For every positive integer k, there exist two constants $\epsilon, \alpha > 0$ with the following property.*

Let G be an ordered graph on n vertices with maximum degree at most ϵn such that G does not contain a monotone path of size k as an induced subgraph. Then there exist $t \geq 2$ and t disjoint subsets $X_1, \ldots, X_t \subset V(G)$ such that $t \geq \alpha(\frac{n}{|X_i|})^{1/2}$ and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

In the inequality $t \geq \alpha(\frac{n}{|X_i|})^{1/2}$, the exponent $1/2$ has no significance: the statement remains true with any $0 < \beta < 1$ instead of $1/2$ (with the cost of changing ϵ and α). However, it is not true with $\beta = 1$, as it would contradict the aforementioned construction of Fox [12].

In the rest of this section, we show how Theorem 6 implies Theorem 1. Very similar ideas were used in [5, 8, 9]. The next two sections are devoted to the proof of Theorem 6.

By a classical result of Rödl [20], any graph G avoiding some fixed graph H contains a linear size subset that is either very dense or very sparse. A quantitatively stronger version of this result was proved by Fox and Sudakov [14].

Lemma 7. *[20] For every graph H and $\epsilon_0 > 0$, there exists $\delta_0 > 0$ with the following property.*

For any graph G with n vertices that does not contain H as an induced subgraph, there is a subset $U \subset V(G)$ such that $|U| \geq \delta_0 n$, and either $|E(G[U])| \leq \epsilon_0 \binom{|U|}{2}$ or $|E(G[U])| \geq (1-\epsilon_0)\binom{|U|}{2}$.

Lemma 7 applies to unordered graphs, but it can be easily extended to ordered graphs, using the following statement.

Lemma 8. *[21] For every ordered graph H, there exists an unordered graph H_0 with the property that introducing any total ordering on $V(H_0)$, the resulting ordered graph H_0' always contains H as an induced ordered subgraph.*

By the combination of these two lemmas, we obtain the following.

Lemma 9. *For every ordered graph H and $\epsilon > 0$, there exists $\delta > 0$ with the following property.*

For any ordered graph G with n vertices that does not contain H as an induced ordered subgraph, there exists a subset $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) \leq \epsilon|U|$ or $\Delta(\overline{G}[U]) \leq \epsilon|U|$.

Proof. By Lemma 8, there exists a graph H_0 such that introducing any total ordering on $V(H_0)$, the resulting ordered graph H_0' contains H as an induced ordered subgraph. Let $\epsilon_0 = \frac{\epsilon}{2}$, and let δ_0 be the constant given by Lemma 7 with respect to H_0 and ϵ_0.

Let G be an ordered graph with n vertices that does not contain H as an induced ordered subgraph. Then the underlying unordered graph of G does not contain H_0 as an induced subgraph. Hence, there exists $U' \subset V(G)$ such that $|U'| \geq \delta_0 n$, and either $|E(G[U'])| \leq \epsilon_0\binom{|U'|}{2}$ or $|E(G[U'])| \geq (1-\epsilon_0)\binom{|U'|}{2}$. Suppose that $|E(G[U'])| \leq \epsilon_0\binom{|U'|}{2}$, the other case can be handled similarly. Let W be the set of vertices in U' whose degree in $G[U]$ is larger than $2\epsilon_0|U|$. Then

$$\frac{1}{2}(2\epsilon_0|W|)|U'| \leq |E(G[U'])| \leq \epsilon_0\binom{|U'|}{2},$$

so that $|W| \leq \frac{|U'|}{2}$. Setting $U = U' \setminus W$, we have $\Delta(G[U]) \leq 2\epsilon_0|U'| \leq \epsilon|U|$ and

$$|U| \geq \frac{|U'|}{2} \geq \frac{\delta_0}{2}n.$$

Hence, $\delta = \frac{\delta_0}{2}$ will suffice. □

After this preparation, it is easy to deduce from Theorem 6 that $\mathcal{P}_k$ has the quasi-Erdős-Hajnal property and, therefore, the Erdős-Hajnal property.

Proof of Theorem 1. Let $\epsilon, \alpha > 0$ be the constants given by Theorem 6, and let $\delta > 0$ be the constant given by Lemma 9, where H is the monotone path of size k.

Let G be an ordered graph on n vertices such that neither G nor its complement contains a monotone path of length k as an induced subgraph. Then there exists $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) < \epsilon|U|$ or $\Delta(\overline{G}[U]) < \epsilon|U|$. Suppose that $\Delta(G[U]) < \epsilon|U|$, the other case can be handled similarly. Applying Theorem 6 to $G[U]$, we obtain that there exist $t \geq 2$ and t disjoint sets $X_1, \ldots, X_t \subset U$ such that

$$t \geq \alpha\left(\frac{|U|}{|X_i|}\right)^{1/2} \geq \alpha\delta^{1/2}\left(\frac{n}{|X_i|}\right)^{1/2}$$

for $i = 1, \ldots, t$, and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

Thus, the family $\mathcal{P}_k$ has the quasi-Erdős-Hajnal property with parameters $\alpha := \alpha\delta^{1/2}$ and $\beta := 1/2$. Therefore, $\mathcal{P}_k$ also has the Erdős-Hajnal property. □

In the next two sections, we present the proof of Theorem 6.

3 The embedding lemma

The backbone of the proof of Theorem 6 is the following technical lemma, whose proof is already contained in [24], within the proof Lemma 7. For convenience and to make this paper self-contained, it is also included here.

Lemma 10. *There exist two constants $\epsilon_1, \alpha_1 > 0$ with the following property. Let G be a bipartite graph with vertex classes A and B, $|A| = |B| = n$. Then at least one of the following three conditions is satisfied.*

(i) *There exist $t \geq 2$ and $2t$ disjoint sets $W_1, \ldots, W_t \subset A$ and $X_1, \ldots, X_t \subset B$ such that $t \geq \alpha_1(\frac{n}{|X_i|})^{1/2}$, and $X_i \subset N(W_i)$ for $i = 1, \ldots, t$, but $X_i \cap N(W_j) = \emptyset$ for $i \neq j$.*

(ii) *There exist $X_1 \subset A$ and $X_2 \subset B$ such that $2 > \alpha_1(\frac{n}{|X_i|})^{1/2}$ and there is no edge between X_1 and X_2.*

(iii) *There exists $v \in A$ such that $|N(v)| \geq \epsilon_1 n$.*

Proof. We show that $\epsilon_1 = \frac{1}{2000}$ and $\alpha_1 = \frac{1}{200}$ meet the above requirements.

Suppose that (iii) does not hold. Then the number of edges of H is at most $\epsilon_1 n^2$, so the number of vertices $w \in B$ such that $|N(w)| > \epsilon_1 n$ is at most $n/2$. Deleting all such vertices, and some more, we obtain a bipartite graph H' with vertex classes A' and B' of size $n' = n/2$ such that the maximum degree of H' is at most $2\epsilon_1 n = 4\epsilon_1 n'$.

Let $\epsilon = 4\epsilon_1 = \frac{1}{500}$ and $\alpha = \frac{1}{100}$. From now on, we shall only work with H', so with a slight abuse of notation, write $H := H'$, $A_0 := A'$, $B_0 := B'$, and $n := n'$. Therefore, we have $\Delta(H) \leq \epsilon n$.

In what follows, we describe an algorithm, which will be referred to as the *main algorithm*. It will output

(i)' either an integer $t \geq 2$ and $2t$ disjoint sets $W_1, \ldots, W_t \subset A$ and $X_1, \ldots, X_t \subset B$ such that $t \geq \alpha(\frac{n}{|X_i|})^{1/2}$, and $X_i \subset N(W_i)$ for $i = 1, \ldots, t$, but $X_i \cap N(W_j) = \emptyset$ for $i \neq j$;

(ii)' or two subsets $X_1 \subset A$ and $X_2 \subset B$ such that $2 > \alpha(\frac{n}{|X_i|})^{1/2}$ and there is no edge between X_1 and X_2.

We declare the following constants for the main algorithm. Let $J_0 = \lfloor \log_2 \epsilon n \rfloor + 1$, and for $j = 1, \ldots, J_0$, let $t_j = n^{1/2} 2^{j/2}$. Then

$$\sum_{i=1}^{J_0} t_i = \sum_{i=1}^{J_0} n^{1/2} 2^{i/2} \leq 2n\epsilon^{1/2} \frac{1}{1 - 2^{-1/2}} < \frac{n}{4}. \quad (1)$$

Also, declare the following variables. Let $J := J_0$, $A := A_0$, $B := B_0$, $A^* := \emptyset$ and $B^* := \emptyset$.

In each step of the main algorithm, we make the following changes: we move certain elements of A into A^*, move certain elements of B into B^*, and decrease J. We think of the elements of A^* and B^* as "leftovers". We make sure that at the end of each step of the algorithm, the following properties are satisfied:

1. $|A| + |A^*| = |B| + |B^*| = n$,
2. $|A^*|, |B^*| \leq 2 \sum_{i=J+1}^{J_0} t_i$,
3. for every $v \in B$, $|N(v) \cap A| < 2^J$.

Note that by (1) and conditions 1 and 2, we have $|A|, |B| \geq \frac{n}{2}$. These conditions are certainly satisfied at the beginning of the algorithm. Next, we describe a general step of our main algorithm.

Main algorithm. If $J = 0$, then stop the main algorithm, and output $X_1 = A, X_2 = B$. In this case, there is no edge between A and B, by condition 3 and $|A|, |B| \geq \frac{n}{2}$. By the choice of α, this output satisfies condition (ii)'.

Suppose next that $J \geq 1$. For $i = 1, \dots, J$, let V_i be the set of vertices $v \in B$ such that $2^{i-1} \leq |N(v) \cap A| < 2^i$, and let V_0 be the set of vertices $v \in B$ such that $N(v) \cap A = \emptyset$. Then, by condition 3, we have $B = \bigcup_{i=0}^{J} V_i$.

Let k, $1 \leq k \leq J$ be the largest integer for which $t_k < |V_k|$. First, consider the case where there is no such k. Then

$$n - \sum_{i=J+1}^{J_0} t_i - |V_0| \leq n - |B^*| - |V_0| = |B| - |V_0| = \sum_{i=1}^{J} |V_i| \leq \sum_{i=1}^{J} t_i,$$

where the first inequality follows from condition 2, and the first equality is the consequence of condition 1. Comparing the left-hand and right-hand sides, and using (1), we get $|V_0| \geq n/2$. In this case, stop the algorithm and output $X_1 = V_0$ and $X_2 = A$. Note that $\alpha(\frac{n}{|X_i|})^{1/2} < 2$ is satisfied for $i = 1, 2$, so this output satisfies condition (ii)'.

Suppose that there exists k with the desired property. Remove the elements of V_i for $i > k$ from B, and add them to B^*. Then we added at most $\sum_{i=k+1}^{J} t_i$ elements to B^*. Setting $J := k$, properties 1-3 are still satisfied.

Now we shall run a *sub-algorithm*. Let $Z_0 = V_k$. With help of the sub-algorithm, we construct a sequence $Z_0 \supset \cdots \supset Z_r$ satisfying the following properties. During each step of the sub-algorithm, we either find an output satisfying (i)', or we will move certain elements of A to A^*. At the end of the l-th step of this algorithm, Z_l will be the set of vertices in B that still have at least 2^{k-1} neighbours in A. We stop the algorithm if Z_l is too small.

Sub-algorithm. Suppose that Z_l has already been defined. If $|Z_l| < 2t_k$, then let $r = l$, stop the sub-algorithm, remove the elements of Z_l from B, and add them to B^*. Make the update $J := k - 1$, and move to the next step of the main algorithm. Note that B^* satisfies condition 2. Later, we will see that all the other properties are satisfied.

On the other hand, if $|Z_l| \geq 2t_k$, we define Z_{l+1} as follows. Let $x_l = \frac{|Z_l|}{t_k}$. Say that a vertex $v \in A$ is *heavy* if

$$|N(v) \cap Z_l| \geq \frac{x_l 2^k}{t_k}|Z_l| = \left(\frac{|Z_l|}{t_k}\right)^2 2^k = \frac{|Z_l|^2}{n} =: \Delta_l,$$

and let H_l be the set of heavy vertices. Counting the number of edges f between H_l and Z_l in two ways, we can write

$$|H_l|\Delta_l \leq f < |Z_l|2^k,$$

which gives $|H_l| < \frac{t_k}{x_l}$. Remove the elements of H_l from A and add them to A^*. Examine how the degrees of the vertices in Z_l changed, and consider the following two cases:

Case 1. At least $\frac{|Z_l|}{2}$ vertices in Z_l have at least 2^{k-1} neighbors in A.

Let T be the set of vertices in Z_l that have at least 2^{k-1} neighbors in A, so $|T| \geq \frac{|Z_l|}{2}$. Pick each element of A with probability $p = 2^{-k}$, and let S be the set of selected vertices. We say that $v \in T$ is *good* if $|N(v) \cap S| = 1$, and let Y be the set of good vertices. We have

$$\mathbb{P}(v \text{ is good}) = |N(v) \cap A|p(1-p)^{|N(v)\cap A|-1} \geq \frac{1}{2}(1-2^{-k})^{2^k} \geq \frac{1}{6},$$

so that $\mathbb{E}(|Y|) \geq \frac{|T|}{6} \geq \frac{|Z_l|}{12}$. Therefore, there exists a choice for S such that $|Y| \geq \frac{|Z_l|}{12}$. Let us fix such an S. For each $v \in S$, let Y_v be the set of elements $w \in Y$ such that $N(w) \cap S = \{v\}$. Also, note that

$$|Y_v| \leq |N(v) \cap Z_l| \leq \min\{\epsilon n, \Delta_l\} =: \Delta_l'.$$

In other words, the sets Y_v for $v \in S$ partition Y into sets of size at most Δ_l'. Here, we have

$$\frac{|Y|}{\Delta_l'} \geq \frac{|Z_l|}{12\Delta_l'} \geq \max\left\{\frac{n}{12|Z_l|}, \frac{|Z_l|}{\epsilon n}\right\}.$$

By the choice of ϵ, the right-hand side is always at least 6. But then we can partition S into $t \geq \frac{|Y|}{3\Delta_l'} \geq 2$ parts $W_1, \ldots, W_t$ such that the sets $X_i = \bigcup_{v \in W_i} Y_v$ have size at least Δ_l' for $i = 1, \ldots, t$. The resulting sets $X_1, \ldots, X_t$ satisfy that

$$t \geq \frac{|Y|}{3\Delta_l'} \geq \frac{n}{36|Z_l|} \geq \frac{1}{36}\left(\frac{n}{\Delta_l}\right)^{1/2} \geq \frac{1}{36}\left(\frac{n}{|X_i|}\right)^{1/2}.$$

Stop the main algorithm, and output t and the $2t$ disjoint sets $W_1, \ldots, W_t$ and $X_1, \ldots, X_t$. By the choice of α, this output satisfies (i)'.

Case 2. At most $\frac{|Z_l|}{2}$ vertices in Z_l have at least 2^{k-1} neighbours in A.

In this case, define Z_{l+1} as the set of elements of Z_l with at least 2^{k-1} neighbours in A (then Z_{l+1} is the set of all elements in B with at least 2^{k-1} neighbours in A as well). Also, move to the next step of the sub-algorithm.

We need to check that, if the main algorithm is not terminated, then after the sub-algorithm ends, conditions 1-3 are still satisfied. Conditions 1 and 3 are clearly true, and 2 holds for B^*.

It remains to show that 2 holds for A^* as well. Note that, as $|Z_{l+1}| \leq \frac{|Z_l|}{2}$ for $l = 0, \dots, r-1$, and $|Z_{r-1}| \geq 2t_k$, we have $|Z_l| \geq 2^{r-l} t_k$ and $x_l \geq 2^{r-l}$. Compared to the first step of the sub-algorithm, $|A^*|$ increased by

$$\sum_{l=0}^{r-1} |H_l| \leq \sum_{l=0}^{r-1} \frac{t_k}{x_l} \leq \sum_{l=0}^{r-1} \frac{t_k}{2^{r-l}} < t_k.$$

Therefore, condition 2 is also satisfied.

In every step of the main algorithm, J decreases by at least one, so the main algorithm will stop in a finite number of steps. When the algorithm stops, its output will satisfy either (i)' or (ii)'. □

4 The proof of Theorem 6

Now we are in a position to prove Theorem 6. Let G be an ordered graph. The *transitive closure* of G is the ordered graph G' on the vertex set $V(G)$ in which x and y are connected by an edge if and only if there exists a monotone path in G with endpoints x and y.

Proof of Theorem 6. Let ϵ_1, α_1 be the constants given by Lemma 10. Furthermore, define the following constants: $c_1 = \frac{\epsilon_1}{2}$, $c_{i+1} = \frac{\epsilon_1 c_i}{4}$ (for $i = 1, 2, \dots$), $\epsilon = \frac{c_k}{2}$, and $\alpha = \frac{\alpha_1 c_k^{1/2}}{2}$.

Let G be an ordered graph on n vertices such that

1. the maximum degree of G is at most ϵn,
2. there exist no t and t disjoint subsets $X_1, \dots, X_t \subset V(G)$ such that $t \geq \alpha(\frac{n}{|X_i|})^{1/2}$ and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

Then, we show that G contains a monotone path of size k as an induced subgraph. In particular, we find k vertices $x_1 \prec \dots \prec x_k$ with the following properties. For $s = 1, \dots, k$,

(a) $x_1, \dots, x_s$ is an induced monotone path.

(b) Let

$$U_s = V(G) \setminus \left(\bigcup_{i=1}^{s-1} N(x_i) \right),$$

let $G_s = G[U_s \cup \{x_s\}]$, and let G'_s be the transitive closure of G_s. Then the forward degree of x_s in G'_s is at least $c_s n$.

First, we find a vertex x_1 with the desired properties, that is, if G' is the transitive closure of G, then the forward degree of x_1 must be at least $c_1 n$. Let A be the set of the first $n/2$ elements of $V(G)$, and set $B = V(G) \setminus A$. Also, let H denote the bipartite subgraph of G' with parts A and B. By Lemma 10, at least one of the following three conditions is satisfied.

(i) There exist $t \geq 2$ and $2t$ disjoint sets $W_1, \dots, W_t \subset A$ and $X_1, \dots, X_t \subset B$ such that

$$t \geq \alpha_1 \left(\frac{|A|}{|X_i|}\right)^{1/2} = 2^{-1/2}\alpha_1 \left(\frac{n}{|X_i|}\right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2},$$

and $X_i \subset N_H(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N_H(W_j) = \emptyset$ for $i \neq j$.

(ii) There exist $X_1 \subset A$ and $X_2 \subset B$ such that

$$2 > \alpha_1 \left(\frac{|A|}{|X_i|}\right)^{1/2} = 2^{-1/2}\alpha_1 \left(\frac{n}{|X_i|}\right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2},$$

and there is no edge between X_1 and X_2.

(iii) There exists $v \in A$ such that $|N_H(v)| \geq \epsilon_1 |A| = c_1 n$.

As non-edges of G' are also non-edges of G, (ii) cannot hold, by property 2 of G (at the beginning of the proof). Suppose that (i) holds. Note that there is no edge between X_i and X_j in G, for $1 \leq i < j \leq t$. Suppose for contradiction that $x \in X_i$ and $y \in X_j$ are joined by an edge in G, for some $x \prec y$. Then there exists $w \in W_i$ such that $wx \in E(G')$, but $wy \notin E(G')$. This is a contradiction, as this means that there is a monotone path from w to x in G, so there is a monotone path from w to y as well. Hence, there is no edge between X_i and X_j for $1 \leq i < j \leq t$, which contradicts 2. Therefore, (iii) must hold: there exists a vertex $x_1 \in V(G)$ whose forward degree in $G' = G'_1$ is at least $c_1 n$.

Suppose that we have already found $x_1, \dots, x_s$ with the desired properties, for some $1 \leq s \leq k-1$. Then we define x_{s+1} as follows. Let X be the forward neighbourhood of x_s in G_s, let Y be the forward neighbourhood of x_s in G'_s, and let $Z = Y \setminus X$. As $|X| \leq \epsilon n$ and $|Y| \geq c_s n$, we have $|Z| \geq \frac{c_s}{2} n$. Let A be the set of the first $\frac{|Z|}{2}$ elements of Z with respect to $\prec$, and let $B = Z \setminus A$. A monotone path in G_s is said to be *good* if none of its vertices, with the possible exception of the first one, belongs to X. For every $v \in A$, there exists at least one element $x \in X$ such that $v \in N^+_{G'_s}(x)$; assign the largest (with respect to $\prec$) such element x to v. Then there is a good monotone path from x to v.

Define a bipartite graph H between A and B as follows. If $v \in A$ and $y \in B$, and $x \in X$ is the vertex assigned to v, then join v and y by an edge if there is a good monotone path from x to y. Applying Lemma 10 to H, we conclude that at least one of the following three statements is true.

(i) There exist $t \geq 2$ and $2t$ disjoint sets $W_1, \dots, W_t \subset A$ and $X_1, \dots, X_t \subset B$ such that

$$t \geq \alpha_1 \left(\frac{|A|}{|X_i|}\right)^{1/2} > \frac{\alpha_1 c_s^{1/2}}{2} \left(\frac{n}{|X_i|}\right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2},$$

and $X_i \subset N_H(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N_H(W_j) = \emptyset$ for $i \neq j$.

(ii) There exist $X_1 \subset A$ and $X_2 \subset B$ such that

$$2 > \alpha_1 \left(\frac{|A|}{|X_i|}\right)^{1/2} > \frac{\alpha_1 c_s^{1/2}}{2} \left(\frac{n}{|X_i|}\right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2},$$

and there is no edge between X_1 and X_2.

(iii) There exists $v \in A$ such that $|N_H(v)| \geq \epsilon_1 |A| = \frac{\epsilon_1 c_s}{4} n = c_{s+1} n$.

Suppose first that (i) holds. Then, as before, we show that there is no edge between X_i and X_j in G for $1 \leq i < j \leq t$. Suppose that $u \in X_i$ and $w \in X_j$ are joined by an edge in G, for some $u \prec w$. Then there exists $v \in W_i$ such that $vu \in E(H)$, but $vw \notin E(H)$. Let $x \in X$ be the vertex assigned to v. Then we can find a good monotone path from x to u. Since uw is an edge of G, there is a good monotone path from x to w, contradicting the assumption $vw \notin E(H)$. Therefore, there cannot be any edge between X_i and X_j in G, which means that (i) contradicts 2.

Suppose next that (ii) holds. Again, we can show that there is no edge between X_1 and X_2 in G, contradicting 2. Suppose that $v \in X_1$ and $y \in X_2$ are joined by an edge in G, and let $x \in X$ be the vertex assigned to v. There is a good monotone path from x to v in G_s, so there is a good monotone path from x to y, contradicting the assumption that vy is not an edge of H.

Therefore, we can assume that (iii) holds. Let $v \in A$ be a vertex of degree at least $c_{s+1}n$ in H, and let $x_{s+1} \in X$ be the vertex assigned to v. We show that x_{s+1} satisfies the desired properties. We have $U_{s+1} = U_s \setminus X$, and the forward degree of x_{s+1} in G'_{s+1} is exactly the number of vertices y such that there is a good monotone path from x_{s+1} to y. That is, the forward degree of x_{s+1} is at least $|N_H(v)| \geq c_{s+1}n$, as required. This completes the proof. □

5 The construction—Proof of Theorem 3

In this section, we present our construction for Theorem 3. The construction involves expander graphs, which are defined as follows.

Recall that for any graph H and any $U \subset V(H)$, we denote by $N(U) = N_H(U)$ the neighborhood of U in H. The *closed neighborhood* of U is defined as $U \cup N_H(U)$, and is denoted by $N[U] = N_H[U]$. The graph H is called an (n, d, λ)*-expander* if H is a d-regular graph on n vertices, and for every $U \subseteq V$ satisfying $|U| \leq |V|/2$, we have $|N_H[U]| \geq (1+\lambda)|U|$. By a well-known result of Bollobás [4], a random 3-regular graph on n vertices is a $(n, 3, \lambda_0)$-expander with high probability for some absolute constant $\lambda_0 > 0$. In the rest of this section, we fix such a constant λ_0. For explicit constructions of expander graphs see, e.g., [18].

For any positive integer r, let H^r denote the graph with vertex set $V(H)$ in which two vertices are joined by an edge if there exists a path of length at most r between them in H. Here we allow loops, so that in H^r every vertex is joined to itself. We need the following simple property of expander graphs.

Claim 11. *Let H be an (n, d, λ)-expander graph and let $r \geq 1$. For any subsets $X, Y \subseteq V(H)$ such that there is no edge between X and Y in H^r, we have $|X||Y| \leq n^2(1+\lambda)^{-r}$.*

Proof. Let $X_i = N_{H^i}[X]$ and $Y_i = N_{H^i}[Y]$ for $i = 0, 1, \ldots, r$. It follows from the definition of expanders that, if $|X_i| \leq \frac{n}{2}$, then

$$|X| \leq \frac{1}{2} n (1+\lambda)^{-i}.$$

Similarly, if $|Y_i| \leq \frac{n}{2}$, then $|Y| \leq \frac{1}{2} n (1+\lambda)^{-i}$. If X and Y are not connected by any edge in H^r, then X_i and Y_{r-i} must be disjoint for every i. Let ℓ be the largest number in $\{0, 1, \ldots, r\}$ such that $|X_\ell| \leq n/2$.

If $\ell = r$, then $|X| < n(1+\lambda)^{-r}$, and hence $|X||Y| \leq n^2(1+\lambda)^{-r}$.

If $\ell < r$, then $|X_{\ell+1}| > n/2$ and $|Y_{r-\ell-1}| \leq n/2$. Therefore, we have $|Y| \leq n(1+\lambda)^{-(r-\ell-1)}$. Using the inequality $1+\lambda \leq 2$, we obtain

$$|X||Y| \leq \frac{1}{4}n^2(1+\lambda)^{-r+1} \leq n^2(1+\lambda)^{-r}.$$

□

Claim 12. *For any d-regular graph H and $r \geq 1$, we have $\Delta(H^r) \leq (d+1)^r$.*

Proof. Trivial, by induction on r. □

Our construction is based on the following key lemma.

Lemma 13. *Let k, m, f be positive integers. Let $A_1, \ldots, A_k$ be disjoint sets of size m, and suppose that there exists an $(m, 3, \lambda_0)$-expander.*

Then there is a graph G on the vertex set $V = \bigcup_{i=1}^{k} A_i$ such that

1. $\Delta(G) \leq 4^{f2^k}$,
2. *there are no three vertices $x, y, z \in V$ such that $x \in A_a, y \in A_b, z \in A_c$ for some $a < b < c$, and $xy, xz \in E(G)$, but $yz \notin E(G)$,*
3. *for any $a \neq b$ and any pair of subsets $X \subset A_a$ and $Y \subset A_b$ not connected by any edge of G, we have $|X||Y| \leq m^2(1+\lambda_0)^{-f}$.*

Proof. Let H be an $(m, 3, \lambda_0)$-expander. Let $\phi : V \to V(H)$ be an arbitrary function such that ϕ is a bijection when restricted to the set A_i, for $i = 1, \ldots, k$. Define the graph G, as follows. Suppose that $x \in A_a$ and $y \in A_b$ for some $a < b$. Join x and y by an edge if there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(y)$ in H. By Claim 12, the maximum degree of G is at most $\sum_{i=1}^{k-1} 4^{f2^i} \leq 4^{f2^k}$, so that G has property 1.

To see that G also has property 2, consider $x \in A_a,\ y \in A_b,\ z \in A_c$ such that $a < b < c$ and $xy, xz \in E(G)$. We have to show that $yz \in E(G)$. By definition, there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(y)$ in H, and there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(z)$. But then there exists a path of length at most $f2^a \leq f2^{b-1}$ between $\phi(y)$ and $\phi(z)$, so yz is also an edge of G.

It remains to verify that G has property 3. If $1 \leq a < b \leq k$ and $X \subset A_a$ and $Y \subset A_b$ are not connected by any edge in G, then there is no edge between $\phi(X)$ and $\phi(Y)$ in $H^{f2^{a-1}}$. By Claim 11, we have $|X||Y| \leq m^2(1+\lambda_0)^{-f2^{a-1}} \leq m^2(1+\lambda_0)^{-f}$. □

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. Let $k = \frac{2}{\epsilon}$, $f = \frac{\log_2 n}{4 \cdot 2^k}$, and $m = \frac{n}{k}$. We show that the theorem holds with $\delta = \frac{\log_2(1+\lambda_0)}{2^k}$.

Let $A_1, \ldots, A_k$ be disjoint sets of size m. By Lemma 13, there exists a graph G_0 on $V = \bigcup_{i=1}^{m} A_i$ satisfying conditions 1-3 with the above parameters.

Define the ordered graph G on the vertex set V as follows. Let $\prec$ be any ordering on V satisfying $A_1 \prec \cdots \prec A_k$. For any $x \in A_a$ and $y \in A_b$, join x and y by an edge of G if either $a \neq b$ and $xy \in E(G_0)$, or $a = b$. Then the maximum degree of G is at most $\frac{n}{k} + \Delta(G_0) \leq \epsilon n$. Notice that the complement of G does not contain a bi-clique of size $n^{1-\delta}$. Indeed, if (X, Y) is a bi-clique in $\overline{G}$, then there exists $a \neq b$ such that $|X \cap A_a| \geq \frac{|X|}{k}$ and $|Y \cap A_b| \geq \frac{|Y|}{k} = \frac{|X|}{k}$. Thus, $\frac{|X|^2}{k^2} \leq \frac{m^2}{n^{2\delta}}$, which implies that $|X| \leq n^{1-\delta}$.

It remains to show that G contains neither S, nor P as an induced ordered subgraph. Let us start with S. Suppose that there are four vertices, $v_0 \prec v_1 \prec v_2 \prec v_3$, in G such that $v_0v_1, v_0v_2, v_0v_3 \in E(G)$, but $v_1v_2, v_2v_3, v_1v_3 \notin E(G)$. Let $v_0 \in A_a$, $v_1 \in A_b$, $v_2 \in A_c$, and $v_3 \in A_d$, then $a \leq b \leq c \leq d$. If $c = a$, then $b = a$, which implies $v_1v_2 \in E(G)$, contradiction. Therefore, $a < c \leq d$. As $v_2v_3 \notin E(G)$, we must have $c < d$ as well. But then the three vertices v_0, v_2, v_3 contradict property 2, so that G does not contain S an induced ordered subgraph.

To show that G does not contain P, we can proceed in a similar manner. Suppose for contradiction that there are four vertices, $v_0 \prec v_1 \prec v_2 \prec v_3$, in G such that $v_0v_2, v_0v_3, v_1v_2 \in E(G)$, but $v_0v_1, v_1v_3, v_2v_3 \notin E(G)$. Let $v_0 \in A_a$, $v_1 \in A_b$, $v_2 \in A_c$, and $v_3 \in A_d$, where $a \leq b \leq c \leq d$. We have $a < b$, otherwise $v_0v_1 \in E(G)$. In the same way, $c < d$, otherwise $v_2v_3 \in E(G)$. Therefore, $a < c < d$, and the vertices, v_0, v_2, and v_3, contradict condition 2 of Lemma 13. □

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, M. Sharir, *Crossing patterns of semi-algebraic sets,* J. Comb. Theory Ser. A **111** (2) (2005): 310–326.

[2] N. Alon, J. Pach, J. Solymosi, *Ramsey-type theorems with forbidden subgraphs,* Combinatorica **21** (2) (2001): 155–170.

[3] M. Axenovich, C. Tompkins, L. Weber, *Large homogeneous subgraphs in bipartite graphs with forbidden induced subgraphs,* arXiv:1903.09725, 2019.

[4] B. Bollobás, *The isoperimetric number of random regular graphs,* European J. Combin. **9** (1988): 241–244.

[5] N. Bousquet, A. Lagoutte, S. Thomassé, *The Erdős-Hajnal conjecture for paths and antipaths,* Journal of Combinatorial Theory, Ser. B **113** (2015): 261–264.

[6] M. Chudnovsky, *The Erdős-Hajnal conjecture–a survey,* J. Graph Theory **75** (2) (2014): 178–190.

[7] M. Chudnovsky, S. Oum, *Vertex-minors and the Erdős-Hajnal conjecture,* Discrete Math. **341** (2018): 3498–3499.

[8] M. Chudnovsky, A. Scott, P. Seymour, S. Spirkl, *Trees and linear anticomplete pairs,* arXiv:1809.00919 (2018).

[9] M. Chudnovsky, A. Scott, P. Seymour, S. Spirkl, *Sparse graphs without linear anticomplete pairs,* arXiv preprint, arXiv:1804.01060 (2018)

[10] P. Erdős, A. Hajnal, *Ramsey-type theorems,* Discrete Applied Mathematics **25** (1-2) (1989): 37–52.

[11] P. Erdős, A. Hajnal, J. Pach, *Ramsey-type theorem for bipartite graphs,* Geombinatorics **10** (2000): 64–68.

[12] J. Fox, *A bipartite analogue of Dilworth's theorem,* Order **23** (2006): 197–209.

[13] J. Fox, J. Pach, *Erdős-Hajnal-type results on intersection patterns of geometric objects,* In: Horizons of combinatorics, 79–103, Bolyai Soc. Math. Stud., 17, Springer, Berlin, 2008.

[14] J. Fox, B. Sudakov, *Induced Ramsey-type theorems,* Advances in Mathematics **219** (2008): 1771–1800.

[15] J. Fox, B. Sudakov, *Density theorems for bipartite graphs and related Ramsey-type results,* Combinatorica **29** (2009): 153–196.

[16] J. Kim, S. Oum, *The Erdős-Hajnal property for graphs with no fixed cycle as a pivot-minor,* arXiv preprint, arXiv:2003.12960 (2020)

[17] D. Korándi, J. Pach, I. Tomon, *Large homogeneous submatrices,* arXiv:1903.06608 (2019).

[18] M. Morgenstern, *Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power q,* J. Comb. Theory Ser. B **62** (1994): 44–62.

[19] J. Pach, I. Tomon, *Ordered graphs and large bi-cliques in intersection graphs of curves,* European Journal of Combinatorics 82 (2019): 102994.

[20] V. Rödl, *On universality of graphs with uniformly distributed edges,* Discrete Mathematics **59** (1986): 125–134.

[21] V. Rödl, P. Winkler, *A Ramsey-type theorem for orderings of a graph,* SIAM J. Discrete Math. **2** (1989): 402–406.

[22] A. Scott, P. Seymour, *Pure pairs. VI. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix,* manuscript (2020).

[23] A. Scott, P. Seymour, S. Spirkl, *Pure pairs. VI. Excluding an ordered tree,* manuscript (2020).

[24] I. Tomon, *String graphs have the Erdős-Hajnal property,* arXiv:2002.10350 (2020).