The number of edges in k-quasi-planar graphs*

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Abstract

A graph drawn in the plane is called *k-quasi-planar* if it does not contain k pairwise crossing edges. It has been conjectured for a long time that for every fixed k, the maximum number of edges of a k-quasi-planar graph with n vertices is $O(n)$. The best known upper bound is $n(\log n)^{O(\log k)}$. In the present note, we improve this bound to $(n \log n)2^{\alpha(n)^{c_k}}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for k-quasi-planar graphs in which every edge is drawn as an x-monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^{ck^6} n \log n$.

1 Introduction

A *topological graph* is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is *simple* if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called *geometric*.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see [2, 3, 4, 6, 10, 12, 16, 21, 23]). It follows from Euler's Polyhedral Formula that every topological graph on n vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called *k-quasi-planar* if it can be drawn as a topological graph with no k pairwise crossing edges. A graph is 2-quasi-planar if and only

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if it is planar. According to an old conjecture (see Problem 1 in Section 9.6 of [5]), for any fixed $k \geq 2$ there exists a constant c_k such that every k-quasi-planar graph on n vertices has at most $c_k n$ edges. Agarwal, Aronov, Pach, Pollack, and Sharir [4] were the first to prove this conjecture for *simple* 3-quasi-planar graphs. Later, Pach, Radoičić, and Tóth [17] generalized the result to *all* 3-quasi-planar graphs. Ackerman [1] proved the conjecture for $k = 4$.

For larger values of k, first Pach, Shahrokhi, and Szegedy [18] showed that every simple k-quasi-planar graph on n vertices has at most $c_k n(\log n)^{2k-4}$ edges. For $k \geq 3$ and for all (not necessarily simple) k-quasi-planar graphs, Pach, Radoičić, and Tóth [17] established the upper bound $c_k n(\log n)^{4k-12}$. Plugging into these proofs the above mentioned result of Ackerman [1], for $k \geq 4$, we obtain the slightly better bounds $c_k n(\log n)^{2k-8}$ and $c_k n(\log n)^{4k-16}$, respectively. For large values of k, the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a k-quasi-planar graph on n vertices is $n(\log n)^{O(\log k)}$.

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound $O(n \log n)$. He also extended this result to *simple* topological graphs whose edges are drawn as *x-monotone* curves [23].

The aim of this paper is to improve the best known bound, $n(\log n)^{O(\log k)}$, on the number of edges of a k-quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as x-monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$.

Theorem 1.1. *Let $G = (V, E)$ be a k-quasi-planar simple topological graph with n vertices. Then $|E(G)| \leq (n \log n) 2^{\alpha(n)^{c_k}}$, where $\alpha(n)$ denotes the inverse of the Ackermann function and c_k is a constant that depends only on k.*

Recall that the *Ackermann* (more precisely, the *Ackermann-Péter) function* $A(n)$ is defined as follows. Let $A_1(n) = 2n$, and $A_k(n) = A_{k-1}(A_k(n-1))$ for $k = 2, 3, \ldots$. In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of n *two*'s. Now let $A(n) = A_n(n)$, and let $\alpha(n)$ be defined as $\alpha(n) = \min\{k \geq 1 : A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

Theorem 1.2. *Let $G = (V, E)$ be a k-quasi-planar (not necessarily simple) topological graph with n vertices, whose edges are drawn as x-monotone curves. Then $|E(G)| \leq 2^{ck^6} n \log n$, where c is an absolute constant.*

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport-Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr's result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on k from double exponential to single exponential.

2 Generalized Davenport-Schinzel Sequences

The sequence $u = a_1, a_2, ..., a_m$ is called *l-regular* if any l consecutive terms are pairwise different. For integers $l, t \geq 2$, the sequence

$$S = s_1, s_2, ..., s_{lt}$$

of length lt is said to be of *type up*(l,t) if the first l terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

for every i, $1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c, a, b, c,$$

is a type *up*$(3,4)$ sequence or, in short, an *up*$(3,4)$ sequence. We need the following theorem of Klazar [13] on generalized Davenport-Schinzel sequences.

Theorem 2.1 (Klazar)**.** *For $l \geq 2$ and $t \geq 3$, the length of any l-regular sequence over an n-element alphabet that does not contain a subsequence of type up(l,t) has length at most*

$$n \cdot l2^{(lt-3)} \cdot (10l)^{10\alpha(n)^{lt}}.$$

For $l \geq 2$, the sequence

$$S = s_1, s_2, ..., s_{3l-2}$$

of length $3l-2$ is said to be of *type up-down-up*(l) if the first l terms are pairwise different and

$$s_i = s_{2l-i} = s_{(2l-2)+i}$$

for every $i, 1 \leq i \leq l$. For example,

$$a, b, c, d, c, b, a, b, c, d,$$

is an *up-down-up*(4) sequence. Valtr and Klazar [14] showed that any l-regular sequence over an n-element alphabet, which contains no subsequence of type up-down-up(l), has length at most $2^{l^c}n$ for some constant c. This has been improved by Pettie [20], who proved the following.

Lemma 2.2 (Pettie)**.** *For $l \geq 2$, the length of any l-regular sequence over an n-element alphabet, which contains no subsequence of type up-down-up(l), has length at most $2^{O(l^2)}n$.*

For more results on generalized Davenport-Schinzel sequences, see [15, 20, 19].

3 On intersection graphs of curves

In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection C of curves, no two of which intersect many times, contains a large subcollection C' such that in the partition of C' into its connected components $C_1, \ldots, C_t$ in the intersection graph of C, each component C_i has a vertex connected to all other $|C_i| - 1$ vertices.

For a graph $G = (V, E)$, a subset V_0 of the vertex set is said to be a *separator* if there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq \frac{2}{3}|V|$ such that no edge connects a vertex in V_1 to a vertex in V_2. We need the following separator lemma for intersection graphs of curves, established in [9].

Lemma 3.1 (Fox–Pach)**.** *There is an absolute constant c_1 such that every collection C of curves with x intersection points has a separator of size at most $c_1\sqrt{x}$.*

Call a collection C of curves in the plane *decomposable* if there is a partition $C = C_1 \cup \ldots \cup C_t$ such that each C_i contains a curve which intersects all other curves in C_i, and for $i \neq j$, the curves in C_i are disjoint from the curves in C_j. The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is include here, for completeness.

Lemma 3.2. *There is an absolute constant $c > 0$ such that every collection C of $m \geq 2$ curves such that each pair of them intersect in at most t points has a decomposable subcollection of size at least $\frac{cm}{t \log m}$.*

Proof of Lemma 3.2 We prove the following stronger statement. There is an absolute constant $c > 0$ such that every collection C of $m \geq 2$ curves whose intersection graph has at least x edges, and each pair of curves intersect in at most t points, has a decomposable subcollection of size at least $\frac{cm}{t \log m} + \frac{x}{m}$. Let $c = \frac{1}{576c_1^2}$, where $c_1 \geq 1$ is the constant in Lemma 3.1. The proof is by induction on m, noting that all collections of curves with at most three elements are decomposable. Define $d = d(m, x, t) := \frac{cm}{t \log m} + \frac{x}{m}$.

Let Δ denote the maximum degree of the intersection graph of C. We have $\Delta < d - 1$. Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in C that intersect it, is decomposable and its size is at least d, and we are done. Also, $\Delta \geq 2\frac{x}{m}$, since $2\frac{x}{m}$ is the average degree of the vertices in the intersection graph of C. Hence, if $\Delta \geq 2\frac{cm}{t \log m}$, then the desired inequality holds. Thus, we may assume $\Delta < 2\frac{cm}{t \log m}$.

Applying Lemma 3.1 to the intersection graph of C, we obtain that there is a separator $V_0 \subset C$ with $|V_0| \leq c_1\sqrt{tx}$, where c_1 is the absolute constant in Lemma 3.1. That is, there is a partition $C = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2|V|/3$ such that no curve in V_1 intersects any curve in V_2. For $i = 1, 2$, let $m_i = |V_i|$ and x_i denote the number of pairs of curves in V_i that intersect, so that

$$x_1 + x_2 \geq x - \Delta|V_0| \geq x - 2\frac{cm}{t \log m} c_1 \sqrt{tx}. \tag{1}$$

As no curve in V_1 intersects any curve in V_2, the union of a decomposable subcollection of V_1 and a decomposable subcollection of V_2 is decomposable. Thus, by the induction hypothesis, C contains decomposable subcollection of size at least

$$\begin{aligned} d(m_1, x_1, t) + d(m_2, x_2, t) &= \frac{cm_1}{t \log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{t \log m_2} + \frac{x_2}{m_2} \\ &\geq \frac{c(m_1 + m_2)}{t \log(2m/3)} + \frac{(x_1 + x_2)}{2m/3}. \end{aligned}$$

We split the rest of the proof into two cases.

Case 1. $x \geq t^{-1}\left(12c_1c\frac{m}{\log m}\right)^2$. In this case, by (1), we have $x_1 + x_2 \geq \frac{5}{6}x$ and hence there is a decomposable subcollection of size at least

$$\begin{aligned} d(m_1, x_1, t) + d(m_2, x_2, t) &\geq \frac{c(m_1 + m_2)}{t \log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{t \log m} \\ &\geq d + \frac{x}{4m} - \frac{c_1 c\sqrt{tx}}{t \log m} > d, \end{aligned}$$

completing the analysis.

Case 2. $x < t^{-1}\left(12c_1c\frac{m}{\log m}\right)^2$. There is a decomposable subcollection of size at least

$$\begin{aligned} d(m_1,x_1,t)+d(m_2,x_2,t) &\geq \frac{c(m_1+m_2)}{t\log(2m/3)} \geq \frac{c}{t}\left(m-c_1\sqrt{tx}\right)\left(\frac{1}{\log m}+\frac{1}{2\log^2 m}\right) \\ &\geq \frac{c}{t}\left(\frac{m}{\log m}+\frac{m}{2\log^2 m}-\frac{2c_1\sqrt{tx}}{\log m}\right) \geq \frac{c}{t}\left(\frac{m}{\log m}+\frac{m}{4\log^2 m}\right) \\ &\geq \frac{c}{t}\left(\frac{m}{\log m}+\frac{m}{4\log^2 m}\right) \geq \frac{cm}{t\log m}+\frac{x}{m}=d, \end{aligned}$$

where we used $c=\frac{1}{4(12c_1)^2}=\frac{1}{576c_1^2}$. □

4 Simple Topological Graphs

In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

Lemma 4.1. *Let $G=(V,E)$ be a k-quasi-planar simple topological graph with n vertices. Suppose that G has an edge that crosses every other edge. Then we have $|E| \leq n\cdot 2^{\alpha(n)^{c'_k}}$, where $\alpha(n)$ denotes the inverse Ackermann function and c'_k is a constant that depends only on k.*

Proof of Lemma 4.1. Let $k\geq 5$ and $c'_k = 40\cdot 2^{k^2+2k}$. To simplify the presentation, we do not make any attempt to optimize the value of c'_k. Label the vertices of G from 1 to n, i.e., let $V=\{1,2,\ldots,n\}$. Let $e=uv$ be the edge that crosses every other edge in G. Note that $d(u)=d(v)=1$.

Let E' denote the set of edges that cross e. Suppose without loss of generality that no two of elements of E' cross e at the same point. Let $e_1,e_2,...,e_{|E'|}$ denote the edges in E' listed in the order of their intersection points with e from u to v. We create two sequences of vertices $S_1=p_1,p_2,...,p_{|E'|}$ and $S_2=q_1,q_2,...,q_{|E'|}\subset V$, as follows. For each $e_i\in E'$, as we move along edge e from u to v and arrive at the intersection point with e_i, we turn left and move along edge e_i until we reach its endpoint u_i. Then we set $p_i=u_i$. Likewise, as we move along edge e from u to v and arrive at edge e_i, we turn right and move along edge e_i until we reach its other endpoint w_i. Then we set $q_i=w_i$. Thus, S_1 and S_2 are sequences of length $|E'|$ over the alphabet $\{1,2,...,n\}$. See Figure 1 for a small example.

We need two lemmas. The first one is due to Valtr [23].

Lemma 4.2 (Valtr)**.** *For $l\geq 1$, at least one of the sequences S_1,S_2 defined above contains an l-regular subsequence of length at least $|E'|/(4l)$.* □

Since each edge in E' crosses e exactly once, the proof of Lemma 4.2 can be copied almost verbatim from the proof of Lemma 4 in [23]. Indeed, the only fact about the sequences S_1 and S_2 it uses is that the edges $e_{j_1},e_{j_1+1},...,e_{j_2}$ are spanned by the vertices $p_{j_1},...,p_{j_2}$ and $q_{j_1},...,q_{j_2}$, for each pair $j_1<j_2$.

For the rest of this section, we set $l=2^{k^2+k}$ and $t=2^k$.

Lemma 4.3. *Neither of the sequences S_1 and S_2 has a subsequence of type $up(l,t)$.*

Proof. By symmetry, it suffices to show that S_1 does not contain a subsequence of type $up(l,t)$. The argument is by contradiction. We will prove by induction on k that the existence of such a

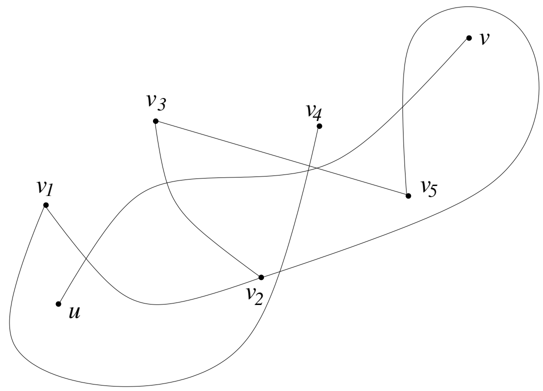


Figure 1: In this example, $S_1 = v_1, v_3, v_4, v_3, v_2$ and $S_2 = v_2, v_2, v_1, v_5, v_5$.

sequence would imply that G has k pairwise crossing edges. The base cases $k = 1, 2$ are trivial. Now assume the statement holds up to $k-1$. Let

$$S = s_1, s_2, ..., s_{lt}$$

be our $up(l,t)$ sequence of length lt such that the first l terms are pairwise distinct and for $i = 1, 2, ..., l$ we have

$$s_i = s_{i+l} = s_{i+2l} = s_{i+3l} = \cdots = s_{i+(t-1)l}.$$

For each $i = 1, 2, ..., l$, let $v_i \in V$ denote the vertex s_i. Moreover, let $a_{i,j}$ be the arc emanating from vertex v_i to the edge e corresponding to s_{i+jl} for $j = 0, 1, 2, ..., t-1$. We will think of s_{i+jl} as a point on $a_{i,j}$ very close but not on edge e. For simplicity, we will let $s_{lt+q} = s_q$ for all $q \in \mathbb{N}$ and $a_{i,j} = a_{i,j'}$ for all $j \in \mathbb{Z}$, where $j' \in \{0, 1, 2, \ldots, t-1\}$ is such that $j \equiv j' \pmod{t}$. Hence there are l distinct vertices $v_1, ..., v_l$, each vertex of which has t arcs emanating from it to the edge e.

Consider the arrangement formed by the t arcs emanating from v_1 and the edge e. Since G is simple, these arcs partition the plane into t regions. By the pigeonhole principle, there is a subset $V' \subset \{v_1, ..., v_l\}$ of size

$$\frac{l-1}{t} = \frac{2^{k^2+k} - 1}{2^k}$$

such that all of the vertices of V' lie in the same region. Let $j_0 \in \{0, 1, 2, ..., t-1\}$ be an integer such that V' lies in the region bounded by a_{1,j_0}, a_{1,j_0+1}, and e. See Figure 2. In the case $j_0 = t-1$, the set V' lies in the unbounded region.

Let $v_i \in V'$ and a_{i,j_0+j_1} be an arc emanating from v_i for $j_1 \geq 1$. Notice that a_{i,j_0+j_1} cannot cross both a_{1,j_0} and a_{1,j_0+1}. Indeed, as a_{i,j_0+j_1} can cross each of a_{1,j_0} and a_{1,j_0+1} at most once, had it crossed both of them, its endpoint s_{1,j_0+j_1} would be in the shaded region on Figure 2. Suppose that a_{i,j_0+j_1} crosses a_{1,j_0+1}. Then all arcs emanating from v_i,

$$A = \{a_{i,j_0+1}, a_{i,j_0+2}, ..., a_{i,j_0+j_1-1}\}$$

must also cross a_{1,j_0+1}. Indeed, let γ be the simple closed curve created by the arrangement

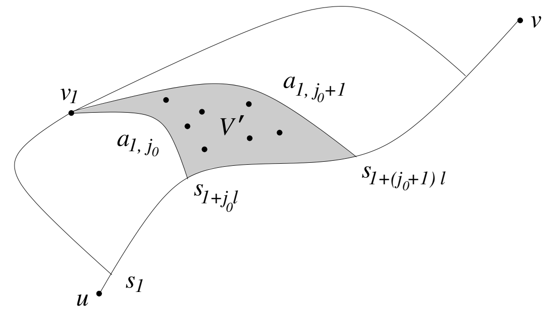


Figure 2: Vertices of V' lie in the region enclosed by $a_{1,j_0}, a_{1,j_0+1}, e$.

$$a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.$$

Since $a_{i,j_0+j_1}, a_{1,j_0+1}$, and e pairwise intersect at precisely one point, γ is well defined. We define points $x = a_{i,j_0+j_1} \cap a_{1,j_0+1}$ and $y = a_{1,j_0+1} \cap e$, and orient γ in the direction from x to y along γ.

In view of the fact that a_{i,j_0+j_1} intersects a_{1,j_0+1}, the vertex v_i must lie to the right of γ. Moreover, since the arc from x to y along a_{1,j_0+1} is a subset of γ, the points corresponding to the subsequence

$$S' = \{s_q \in S \mid 2 + (j_0+1)l \leq q \leq (i-1) + (j_0+j_1)l\}$$

must lie to the left of γ. Hence, γ separates vertex v_i and the points of S'. Therefore, using again that G is simple, each arc from A must cross a_{1,j_0+1} (these arcs cannot cross a_{i,j_0+j_1}). See Figure 3.

By the same argument, if the arc a_{i,j_0-j_1} crosses a_{1,j_0} for $j_1 \geq 1$, then the arcs emanating from v_i,

$$a_{i,j_0-1}, a_{i,j_0-2}, \ldots, a_{i,j_0-j_1+1}$$

must also cross a_{1,j_0}. Since $a_{i,j_0+t/2} = a_{i,j_0-t/2}$, we have the following observation.

Observation 4.4. *For half of the vertices $v_i \in V'$, the arcs emanating from v_i satisfy*

1. $a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+t/2}$ *all cross* a_{1,j_0+1}*, or*

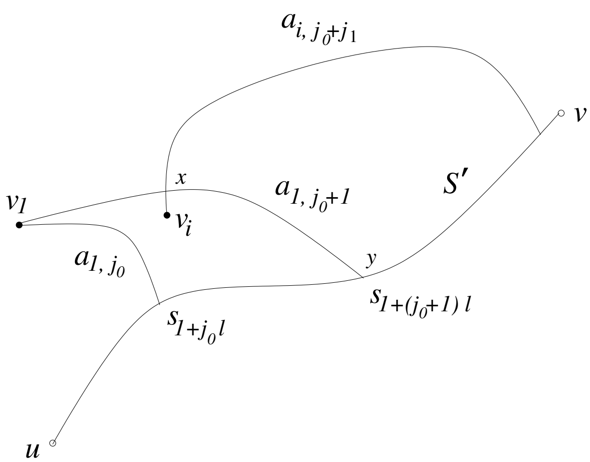
2. $a_{i,j_0-1}, a_{i,j_0-2}, \ldots, a_{i,j_0-t/2}$ *all cross* a_{1,j_0}*.*

Since $t/2 = 2^{k-1}$ and

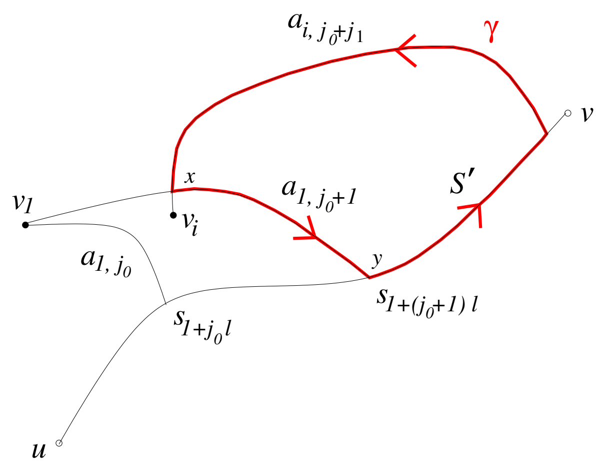
$$\frac{|V'|}{2} \geq \frac{l-1}{2t} = \frac{2^{k^2+k}-1}{2 \cdot 2^k} \geq 2^{(k-1)^2+(k-1)},$$

by Observation 4.4, we obtain a *up*$(2^{(k-1)^2+(k-1)}, 2^{k-1})$ sequence such that the corresponding arcs all cross either a_{1,j_0} or a_{1,j_0+1}. By the induction hypothesis, it follows that there exist k pairwise crossing edges. □

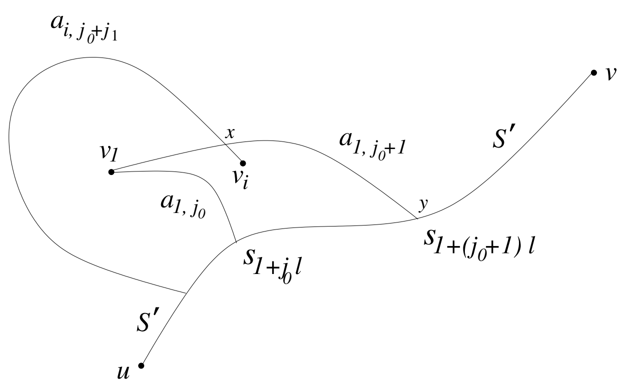
Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, S_1 contains an l-regular subsequence of length $|E'|/(4l)$. By Theorem 2.1 and Lemma 4.3, this subsequence has length at most



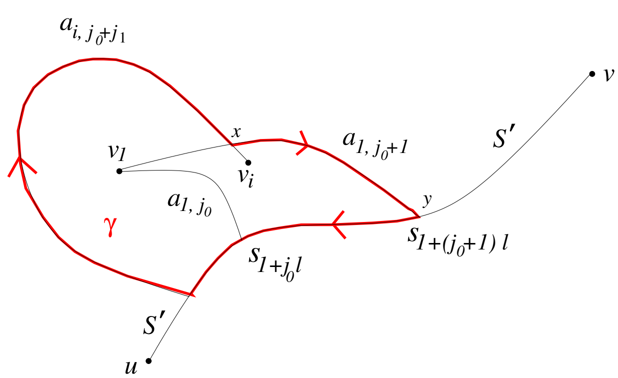
(a) The case when $j_0 + j_1 \mod t \leq t-1$.



(b) γ defined from Figure 3(a).



(c) The case when $j_0 + j_1 \mod t < j_0$. Recall $a_{i,j_0+j_1} = a_{i,j_0+j_1 \mod 2^k}$.



(d) γ defined from Figure 3(c).

Figure 3: Defining γ and its orientation.

$$n \cdot l2^{(lt-3)} \cdot (10l)^{10\alpha(n)^{lt}}.$$

Therefore, we have

$$\frac{|E'|}{4 \cdot l} \leq n \cdot l2^{(lt-3)} \cdot (10l)^{10\alpha(n)^{lt}},$$

which implies

$$|E'| \leq 4 \cdot n \cdot l^2 2^{(lt-3)} \cdot (10l)^{10\alpha(n)^{lt}}.$$

Since $c'_k = 40 \cdot lt = 40 \cdot 2^{k^2+2k}$, $\alpha(n) \geq 2$ and $k \geq 5$, we have

$$|E| = |E'| + 1 \leq n \cdot 2^{\alpha(n)^{c'_k}},$$

which completes the proof of Lemma 4.1. $\square$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $G = (V, E)$ be a k-quasi-planar simple topological graph on n vertices. By Lemma 3.2, there is a subset $E' \subset E$ such that $|E'| \geq c|E|/\log|E|$, where c is an absolute constant and E' is decomposable. Hence, there is a partition

$$E' = E_1 \cup E_2 \cup \cdots \cup E_t$$

such that each E_i has an edge e_i that intersects every other edge in E_i, and for $i \neq j$, the edges in E_i are disjoint from the edges in E_j. Let V_i denote the set of vertices that are the endpoints of the edges in E_i, and let $n_i = |V_i|$. By Lemma 4.1, we have

$$|E_i| \leq n_i 2^{\alpha(n_i)^{c'_k}} + 2n_i,$$

where the $2n_i$ term accounts for the edges that share a vertex with e_i. Hence,

$$\frac{c|E|}{\log|E|} \leq \sum_{i=1}^{t} n_i 2^{\alpha(n_i)^{c'_k}} + 2n_i \leq n 2^{\alpha(n)^{c'_k}} + 2n,$$

Therefore, we obtain

$$|E| \leq (n \log n) 2^{\alpha(n)^{c_k}},$$

for a sufficiently large constant c_k. $\square$

5 x-Monotone Topological Graphs

The aim of this section is to prove Theorem 1.2.

Proof of Theorem 1.2. For $k \geq 2$, let $g_k(n)$ be the maximum number of edges in a k-quasi-planar topological graph whose edges are drawn as x-monotone curves. We will prove by induction on n that

$$g_k(n) \leq 2^{ck^6} n \log n$$

where c is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let $G = (V, E)$ be a k-quasi-planar topological graph whose edges are drawn as x-monotone curves, and let the vertices be labeled $1, 2, ..., n$. Let L be a vertical line that partitions the vertices into two parts, V_1 and V_2, such that $|V_1| = \lfloor n/2 \rfloor$ vertices lie to the left of L, and $|V_2| = \lceil n/2 \rceil$ vertices lie to the right of L. Furthermore, let E_1 denote the set of edges induced by V_1, let E_2 denote the set of edges induced by V_2, and let E' be the set of edges that intersect L. Clearly, we have

$$|E_1| \leq g_k(\lfloor n/2 \rfloor) \quad \text{and} \quad |E_2| \leq g_k(\lceil n/2 \rceil).$$

It suffices that show that

$$|E'| \leq 2^{ck^6/2} n, \tag{2}$$

since this would imply

$$g_k(n) \leq g_k(\lfloor n/2 \rfloor) + g_k(\lceil n/2 \rceil) + 2^{ck^6/2} n \leq 2^{ck^6} n \log n.$$

In the rest of the proof, we only consider the edges belonging to E'. For each vertex $v_i \in V_1$, consider the graph G_i whose vertices are the edges with v_i as a left endpoint, and two vertices in G_i are adjacent if the corresponding edges cross at some point to the left of L. Since G_i is an *incomparability graph* (see [7], [11]) and does not contain a clique of size k, G_i contains an independent set of size $|E(G_i)|/(k-1)$. We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to v_i. After repeating this process for all vertices in V_1, we are left with at least $|E'|/(k-1)$ edges.

Now we continue this process on the other side. For each vertex $v_j \in V_2$, consider the graph G_j whose vertices are the edges with v_j as a right endpoint, and two vertices in G_j are adjacent if the corresponding edges cross at some point to the right of L. Since G_j is an incomparability graph and does not contain a clique of size k, G_j contains an independent set of size $|E(G_j)|/(k-1)$. We keep all edges that corresponds to this independent set, and discard all other edges incident to v_j. After repeating this process for all vertices in V_2, we are left with at least $|E'|/(k-1)^2$ edges.

We order the remaining edges $e_1, e_2, ..., e_m$ in the order in which they intersect L from bottom to top. (We assume without loss of generality that any two intersection points are distinct.) Define two sequences, $S_1 = p_1, p_2, ..., p_m$ and $S_2 = q_1, q_2, ..., q_m$, such that p_i denotes the left endpoint of edge e_i and q_i denotes the right endpoint of e_i. We need the following lemma.

Lemma 5.1. *Neither of the sequences S_1 and S_2 has subsequence of type up-down-up(k^3+2).*

Proof. By symmetry, it suffices to show that S_1 does not have a subsequence of type *up-down-up*(k^3+2). Suppose for contradiction that S_1 does contain such a subsequence. Then there is a sequence

$$S = s_1, s_2, ..., s_{3(k^3+2)-2}$$

such that the integers $s_1, ..., s_{k^3+2}$ are pairwise distinct and

$$s_i = s_{2(k^3+2)-i} = s_{2(k^3+2)-2+i}$$

for $i = 1, 2, ..., k^3+2$.

For each $i \in \{1, 2, ..., k^3+2\}$, let $v_i \in V_1$ denote the label (vertex) of s_i and let x_i denote the x-coordinate of the vertex v_i. Moreover, let a_i be the arc emanating from vertex v_i to the point on L that corresponds to $s_{2(k^3+2)-i}$. Let $A = \{a_2, a_3, ..., a_{k^3+1}\}$. Note that the arcs in A are enumerated downwards with respect to their intersection points with L, and they correspond to the elements of the "middle" section of the up-down-up sequence. We define two partial orders on A as follows.

$$a_i \prec_1 a_j \quad \text{if} \quad i < j, \quad x_i < x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect,}$$

$$a_i \prec_2 a_j \quad \text{if} \quad i < j, \quad x_i > x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect.}$$

Clearly, $\prec_1$ and $\prec_2$ are partial orders. If two arcs are not comparable by either $\prec_1$ or $\prec_2$, then they cross. Since G does not contain k pairwise crossing edges, by Dilworth's theorem [7], there exist k arcs $\{a_{i_1}, a_{i_2}, ..., a_{i_k}\}$ such that they are pairwise comparable by either $\prec_1$ or $\prec_2$. Now the proof falls into two cases.

Case 1. Suppose that $a_{i_1} \prec_1 a_{i_2} \prec_1 \cdots \prec_1 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, ..., v_{i_k}$ to the points corresponding to $s_{2(k^3+2)-2+i_1}, s_{2(k^3+2)-2+i_2}, ..., s_{2(k^3+2)-2+i_k}$ are pairwise crossing. See Figure 4.

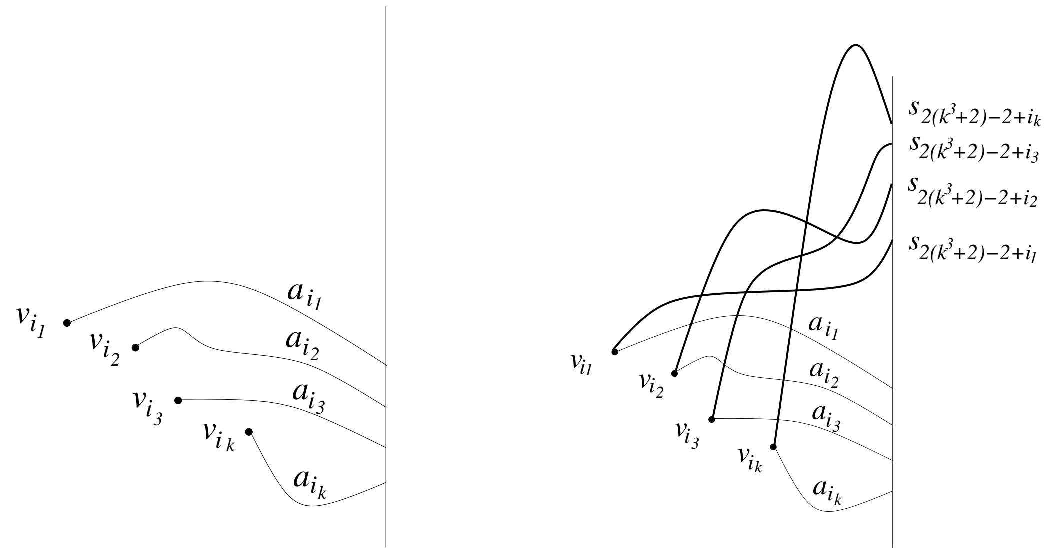


Figure 4: Case 1 of Lemma 5.1.

Case 2. Suppose that $a_{i_1} \prec_2 a_{i_2} \prec_2 \cdots \prec_2 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, ..., v_{i_k}$ to the points corresponding to $s_{i_1}, s_{i_2}, ..., s_{i_k}$ are pairwise crossing. See Figure 5.

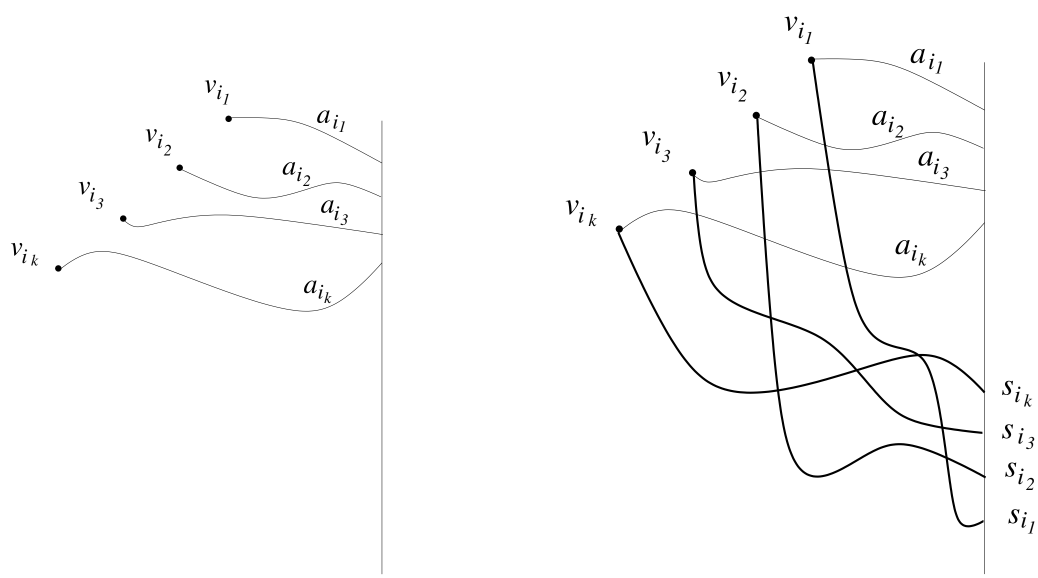


Figure 5: Case 2.

$\square$

We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, S_1, say, contains a (k^3+2)-regular subsequence of length

$$\frac{|E'|}{4(k^3+2)(k-1)^2}.$$

By Lemmas 2.2 and 5.1, this subsequence has length at most $2^{c'k^6}n$, where c' is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3+2)(k-1)^2} \leq 2^{c'k^6}n,$$

which implies that

$$|E'| \leq 4k^5 2^{c'k^6}n \leq 2^{ck^6/2}n$$

for a sufficiently large absolute constant c. □

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