

Ramsey-Turán numbers for semi-algebraic graphs

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Abstract

A *semi-algebraic graph* $G = (V, E)$ is a graph where the vertices are points in \mathbb{R}^d , and the edge set E is defined by a semi-algebraic relation of constant complexity on V . In this note, we establish the following Ramsey-Turán theorem: for every integer $p \geq 3$, every K_p -free semi-algebraic graph on n vertices with independence number $o(n)$ has at most $\frac{1}{2} \left(1 - \frac{1}{\lfloor p/2 \rfloor - 1} + o(1)\right) n^2$ edges. Here, the dependence on the complexity of the semi-algebraic relation is hidden in the $o(1)$ term. Moreover, we show that this bound is tight.

1 Introduction

Over the past decade, several authors have shown that many classical theorems in extremal graph theory can be significantly improved if we restrict our attention to semi-algebraic graphs, that is, graphs whose vertices are points in Euclidean space, and edges are defined by a semi-algebraic relation of constant complexity [1, 5, 8, 11, 9, 4]. In this note, we continue this sequence of works by studying Ramsey-Turán numbers for semi-algebraic graphs.

More formally, a graph $G = (V, E)$ is a *semi-algebraic graph* with *complexity* at most t , if its vertex set V is an ordered set of points in \mathbb{R}^d , where $d \leq t$, and if there are at most t polynomials $g_1, \dots, g_s \in \mathbb{R}[x_1, \dots, x_{2d}]$, $s \leq t$, of degree at most t and a Boolean formula Φ such that for vertices $u, v \in V$ such that u comes before v in the ordering,

$$(u, v) \in E \quad \Leftrightarrow \quad \Phi(g_1(u, v) \geq 0; \dots; g_s(u, v) \geq 0) = 1.$$

At the evaluation of $g_\ell(u, v)$, we substitute the variables x_1, \dots, x_d with the coordinates of u , and the variables x_{d+1}, \dots, x_{2d} with the coordinates of v . Here, we assume that the complexity t is a fixed parameter, and $n = |V|$ tends to infinity.

The classical theorem of Turán gives the maximum number of edges in a K_p -free graph on n vertices.

Theorem 1.1 (Turán, [13]). *Let $G = (V, E)$ be a K_p -free graph with n vertices. Then*

$$|E| \leq \frac{1}{2} \left(1 - \frac{1}{p-1} + o(1)\right) n^2.$$

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The only graph for which this bound is tight is the complete $(p-1)$ -partite graph whose parts are of size as equal as possible. This graph can easily be realized as an intersection graph of segments in the plane, which is a semi-algebraic graph with complexity at most four. Therefore, Turán's theorem cannot be improved by restricting it to semi-algebraic graphs.

Let H be a fixed graph. The *Ramsey-Turán number* $\mathbf{RT}(n, H, \alpha)$ is defined as the maximum number of edges that an n -vertex graph of independence number at most α can have without containing H as a (not necessarily induced) subgraph. Ramsey-Turán numbers were introduced by Andrásfai [2] and were motivated by the classical theorems of Ramsey and Turán and their connections to geometry, analysis, and number theory. According to one of the earliest results in Ramsey-Turán theory, which appeared in [7], for every $p \geq 2$, we have

$$\mathbf{RT}(n, K_{2p-1}, o(n)) = \frac{1}{2} \left(1 - \frac{1}{p-1} \right) n^2 + o(n^2). \quad (1)$$

For excluded K_4 , a celebrated result of Szemerédi [12] and Bollobás-Erdős [3] states that

$$\mathbf{RT}(n, K_4, o(n)) = \frac{1}{8} n^2 + o(n^2).$$

This was generalized by Erdős, Hajnal, Sós, and Szemerédi [6] to all cliques of even size. For every $p \geq 2$, we have

$$\mathbf{RT}(n, K_{2p}, o(n)) = \frac{1}{2} \cdot \frac{3p-5}{3p-2} n^2 + o(n^2). \quad (2)$$

For more results in Ramsey-Turán theory, consult the survey of Simonovits and Sós [10].

In the present note, we establish asymptotically tight bounds on Ramsey-Turán numbers for semi-algebraic graphs. We define $\mathbf{RT}_t(n, K_p, o(n))$ as the maximum number of edges that n -vertex K_p -free semi-algebraic graphs with complexity at most t can have, if their independence number is $o(n)$. Strictly speaking, this definition and all above results apply to *sequences* of graphs with n vertices, as n tends to infinity.

It turns out that if the size of the excluded clique is *even*, then the answer to the Ramsey-Turán question significantly changes when the graphs are required to be semi-algebraic. However, in the odd case, we obtain the same asymptotics for the Ramsey-Turán function as in (1). More precisely, we have

Theorem 1.2. *For any fixed integers $t \geq 10$ and $p \geq 2$, we have*

$$\mathbf{RT}_t(n, K_{2p-1}, o(n)) = \mathbf{RT}_t(n, K_{2p}, o(n)) = \frac{1}{2} \left(1 - \frac{1}{p-1} \right) n^2 + o(n^2).$$

2 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. One of the main tools used in the proof is the following regularity lemma for semi-algebraic graphs. Given a graph $G = (V, E)$, a vertex partition is called *equitable* if any two parts differ in size by at most one. Given two disjoint subsets $V_i, V_j \subset V$, we say that the pair (V_i, V_j) is *homogeneous* if $V_i \times V_j \subset E$ or $(V_i \times V_j) \cap E = \emptyset$.

Lemma 2.1 ([9]). *For any positive integer t , there exists a constant $c = c(t) > 0$ with the following property. Let $0 < \varepsilon < 1/2$ and let $G = (V, E)$ be a semi-algebraic graph with complexity at most t .*

Then V has an equitable partition $V = V_1 \cup \dots \cup V_K$ into K parts, where $1/\varepsilon < K < (1/\varepsilon)^c$, such that all but an ε -fraction of the pairs of parts are homogeneous.

The upper bound in Theorem 1.2 follows from

Theorem 2.2. *Let $\varepsilon > 0$ and let $G = (V, E)$ be an n -vertex semi-algebraic graph with complexity at most t . If G is K_{2p} -free and $|E| > \frac{1}{2} \left(1 - \frac{1}{p-1} + \varepsilon\right) n^2$, then G has an independent set of size γn , where $\gamma = \gamma(t, p, \varepsilon)$.*

Proof. We apply Lemma 2.1 with parameter $\varepsilon/4$ to obtain an equitable partition $\mathcal{P} : V = V_1 \cup \dots \cup V_K$ such that $\frac{4}{\varepsilon} \leq K \leq \left(\frac{4}{\varepsilon}\right)^c$, where $c = c(t)$ and all but an at most $\frac{\varepsilon}{4}$ -fraction of all pairs of parts in \mathcal{P} are homogeneous (complete or empty with respect to E). If $n \leq 10K$, then G has an independent set of size one, and the theorem holds trivially. So, we may assume $n > 10K$.

By deleting all edges inside each part, we have deleted at most

$$K \binom{\lceil n/K \rceil}{2} \leq \frac{4n^2}{5K} \leq \varepsilon \frac{n^2}{5}$$

edges. Deleting all edges between non-homogeneous pairs of parts, we lose an additional at most

$$\left\lceil \frac{n}{K} \right\rceil^2 \frac{\varepsilon K^2}{4 \cdot 2} \leq \varepsilon \frac{n^2}{5}$$

edges. In total, we have deleted at most $2\varepsilon n^2/5$ edges of G . The only edges that remain in G are edges between homogeneous pairs of parts, and we have at least $\frac{1}{2} \left(1 - \frac{1}{p-1} + \varepsilon/5\right) n^2$ edges. By Turán's theorem (Theorem 1.1), there is at least one remaining copy of K_p , and its vertices lie in p distinct parts $V_{i_1}, \dots, V_{i_p} \in \mathcal{P}$ that form a complete p -partite subgraph. If any of the parts V_{i_j} forms an independent set in G , then there is an independent set of order $|V_{i_j}| \geq \lfloor n/K \rfloor \geq \gamma n$, where $\gamma = \gamma(t, \varepsilon, p)$, and we are done. Otherwise, there is an edge in each of the p parts, and the endpoints of these p edges form a K_{2p} in G , a contradiction. \square

The lower bound on $\mathbf{RT}(n, K_{2p-1}, o(n))$ and $\mathbf{RT}(n, K_{2p}, o(n))$ in Theorem 1.2 is constructive and is based on the following result of Walczak.

Lemma 2.3 ([14]). *For any pair of positive integers n and p , where n is a multiple of $p-1$, there is a collection S of $n/(p-1)$ segments in the plane whose intersection graph G_S is triangle-free and has no independent set of size $c_p n / \log \log n$. Here c_p is a suitable constant.*

The construction. Take $p-1$ dilated copies of a set S meeting the requirements in Lemma 2.3, and label them as S_1, \dots, S_{p-1} , so that S_i lies inside a ball with center $(i, 0)$ and radius $1/10$. Set $V = S_1 \cup \dots \cup S_{p-1}$. Note that $|S_i| = n/(p-1)$ so that $|V| = n$. Let $G = (V, E)$ be the graph whose vertices are the elements of V , and two vertices (that is, two segments) are connected by an edge if and only if they cross or their left endpoints are at least $1/2$ apart. The graph G consists of a complete $(p-1)$ -partite graph, where each part induces a copy of the triangle-free graph G_S . Clearly, G is K_{2p-1} -free and does not contain any independent set of size $c_p n / \log \log n$. Moreover,

$$|E(G)| \geq \frac{1}{2} \left(1 - \frac{1}{p-1}\right) n^2.$$

Since every segment can be represented by a point in \mathbb{R}^4 and the intersection and distance relations have bounded description complexity (see [1]), E is a semi-algebraic relation with complexity at most c , where c is an absolute constant.

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