# Unsplittable multiple coverings with balls

A system $\underline{B}=\{B_i : i\in I\}$ of balls in $\mathbb{R}^d$ is said to form a k-<u>fold covering</u> if every point $x\in\mathbb{R}^d$ is covered by at least k elements of $\underline{B}$. A 1-fold covering is simply called a <u>covering</u>. Let $r(B_i)$ and $c(B_i)$ denote the radius and the centre of $B_i$, respectively. For any two points $x,y\in\mathbb{R}^d$, let $d(x,y)$ stand for their (euclidean) distance.

<u>Theorem 1.</u> For any natural numbers k and $d\geq 3$, there exists a k-fold covering of $\mathbb{R}^d$ with open unit balls $\{B_i : i\in I\}$, which cannot be decomposed into two coverings. Moreover, we may assume that $\inf_{i\neq j\in I} d(c(B_i),c(B_j)) > 0$.

<u>Theorem 2.</u> For any natural numbers k and $d\geq 3$, there exists a constant $c_d>0$ such that every k-fold covering of $\mathbb{R}^d$ with unit balls $\underline{B}=\{B_i : i\in I\}$, having the property that no point of $\mathbb{R}^d$ is covered by more than $c_d 2^{k/d}$ members of $\underline{B}$, can be decomposed into two coverings.

## 2. Proof of Theorem 1

The following construction will be basic for our purposes.

Theorem 2.1. For every natural number k there exist a finite point system $P_k$ and a finite system of (not necessarily equal) closed discs $\underline{D}_k$ in the plane with the property that for any 2-colouring of the elements of $P_k$ we can find a disc $D \in \underline{D}_k$ such that

(i) $|D \cap P_k| \geq k$,

(ii) $D \cap P_k$ is monochromatic.

Proof. Given any disc D in the plane, let c(D) and r(D) denote the center and the radius of D, respectively.

If k=1 then the statement is trivial. Let $k \geq 2$ be fixed. We are going to construct $\underline{D}_i = \underline{D}_i' \cup \underline{D}_i''$ and $P_i$ $(i \leq k)$ by recursion.

Let $D^*$ be a disc of radius $\delta_2 = 1/10$, and let $\underline{D}_2' = (D^1, D^2, \ldots, D^k)$ be a system of unit discs such that $d(c(D^s), c(D^t))$ (i.e., the distance between the centers of $D^s$ and $D^t$) is at most $\delta_2$ and each $D^t$ has exactly one point (say, $p^t$) in common with $D^*$. Set $\underline{D}_2'' = (D^*)$, $\underline{D}_2 = \underline{D}_2' \cup \underline{D}_2''$ and $P_2 = \{c(D), p^1, p^2, \ldots, p^k\}$.

Assume now that $\underline{D}_i = \underline{D}_i' \cup \underline{D}_i''$ and $P_i$ have already been defined for some $2 \leq i < k$, and

(1) every $D \in \underline{D}_i'$ has a boundary point p(D) not contained in any other element of .

An easy continuity argument shows that there exists a sufficiently small positive number

(2) $$\delta_{i+1} \leq \frac{1}{10} \min\{\delta_i, \min_{D \in \underline{D}_i'} d(p(D), \bigcup_{D \neq \tilde{D} \in \underline{D}_i} \tilde{D})\}$$

such that for any $D \in \underline{D}_i'$ one can find k distinct unit discs $D^1, D^2, \ldots, D^k$ and a disc $D^*$ of radius $\delta_{i+1}$ with $c(D^*) = p(D)$ satisfying

(3) $$d(c(D^t), c(D)) \le \delta_{i+1}\ ,$$

(4) $$D^t \cap P_i = (\mathrm{int} D^t) \cap P_i = D \cap P_i\ ,$$

(5) $D^t$ and $D^*$ have exactly one point (say, $p^t(D)$) in common,

for all $1 \le t \le k$. Set

(6) $$\underline{D}'_{i+1} = \{D^1, D^2, \ldots, D^k : D \in \underline{D}'_i\},\ \underline{D}''_{i+1} = \underline{D}''_i \cup \{D^* : D \in \underline{D}'_i\},\ \underline{D}_{i+1} = \underline{D}'_{i+1} \cup \underline{D}''_{i+1},$$
$$P_{i+1} = P_i \cup \{p^1(D), p^2(D), \ldots, p^k(D) : D \in \underline{D}'_i\}.$$

Let $D^t$ be any element of $\underline{D}'_{i+1}$ ($D \in \underline{D}'_i$, $1 \le t \le k$) and let $E$ be any element of $\underline{D}_{i+1}$ distinct from $D^t$ and $D^*$. If $E \in \underline{D}''_i$ then, by (2),

$$d(p^t(D), E) \ge d(p(D), E) - d(p(D), p^t(D)) \ge 10\delta_{i+1} - \delta_{i+1} > 0.$$

Similarly, if $E = \tilde{D}^s$ or $E = \tilde{D}^*$ for some $\tilde{D} \ne D$ ($\tilde{D} \in \underline{D}'_i$, $1 \le s \le k$), then

$$d(p^t(D), E) \ge d(p(D), \tilde{D}) - d(p(D), p^t(D)) - d(c(E), c(\tilde{D})) \ge 8\delta_{i+1} > 0$$

and

$$d(p^t(D), E) \ge d(p(D), \tilde{D}) - d(p(D), p^t(D)) - r(E) \ge 8\delta_{i+1} > 0,$$

respectively. If $E = D^s$ for some $s \ne t$, then $d(p^t(D), E)$ is clearly positive. This shows that $p^t(D)$ is not covered by any element of $\underline{D}_{i+1}$ different from $D^t$ and $D^*$, so we can find a boundary point of $D^t$ sufficiently close to $p^t(D)$ which does not belong to any other element of $\underline{D}_{i+1}$. Thus, (1) remains valid for $i+1$, and the algorithm can be repeated.

Let $D$ be any element of $\underline{D}'_i$. By (2)-(6) it follows that

(7) $$D^t \cap P_{i+1} = (D \cap P_i) \cup \{p^t(D)\} \qquad (1 \le t \le k),$$
$$D^* \cap P_{i+1} = D^* \cap P_k = \{p^1(D), p^2(D), \ldots, p^k(D)\}.$$

We will prove by induction on $i$ that $\underline{D}_i$ and $P_i$ ($2 \le i \le k$) meet the requirements of the theorem. For $i=2$ this is obviously true. Let $2 \le i < k$, and let $f: P_{i+1} \to \{B, W\}$ be any colouring of the elements of $P_{i+1}$ with 2 colours (Black and White). Applying the induction hypothesis to $f|_{P_i}$ (the restriction of $f$ to $P_i$), we obtain that there exists a $D \in \underline{D}_i$ such that $D\cap P_i$ is monochromatic, i.e., $f(x)=f(y)$ for any $x,y\in D\cap P_i$.

Assume first that $D\in\underline{D}_i'$ and all elements of $D\cap P_i$ are, say, black. If $f(p^t(D))=B$ for some $1\le t\le k$, then, by (7), $D^t\cap P_{i+1}=(D\cap P_i)\cup(p^t(D))$ is monochromatic. If $f(p^t(D))=W$ for all $1\le t\le k$, then $D^*\cap P_{i+1}$ is coloured completely white. Since $D^t, D^*\in\underline{D}_{i+1}$, $|D^t|=i+1$ and $|D^*|=k\ge i+1$, in both cases there exists a disc in $\underline{D}_{i+1}$ containing at least $i+1$ points of $P_{i+1}$, which are coloured the same.

Suppose next that $D\in\underline{D}_i''$. Then, by (7), $D\cap P_{i+1}=D\cap P_i$ and $|D\cap P_{i+1}|=k\ge i+1$. Thus, in this case $D\in\underline{D}_i''\subseteq\underline{D}_{i+1}$ satisfies conditions (i) and (ii) of the theorem with $i+1$ instead of $k$. This completes the induction, and hence the proof. □

We will make use of the following special feature of our construction.

<u>Lemma 2.2</u>. Let $\underline{D}_k$ and $P_k$ denote the same as above. For any $D_1\in\underline{D}_k''$, set

$$X(D_1) = (\mathrm{bd}\ D_1)\cap\Big(\bigcup_{D_1\neq E\in\underline{D}_k} E\Big).$$

Then $X(D_1)$ can be covered by an angular region of size at most $2\pi/3$, whose apex is at the centre of $D_1$.

<u>Proof</u>. Let $i+1$ be the smallest integer such that $D_1\in\underline{D}_{i+1}''$. Then $D_1=D^*$ for some $D\in\underline{D}_i'$ $(1\le i<k)$, $r(D^*)=\delta_{i+1}$ and, by (2), $D^*$ is obviously disjoint from all other elements of $\underline{D}_{i+1}''$. Similarly, no element of $\underline{D}_j''$ $(j>i+1)$ intersects $D^*$.

By the first part of (6), for any $E\in\underline{D}_k'$ there exist a sequence of unit discs $(E_i,E_{i+1},\dots,E_k=E)$ and a sequence of integers $(t_i,t_{i+1},\dots,t_{k-1})$ such that $E_j\in\underline{D}_j'$, $1\le t_j\le k$ and

$$E_{j+1}=(E_j)^{t_j} \qquad (i\le j<k). \tag{8}$$

If $E_i\neq D$ then, by (2), (3) and (8),

$$d(c(D^*),E)=d(p(D),E_k)\ge d(p(D),E_i)-\sum_{i\le j<k} d(c(E_j),c(E_{j+1}))$$

$$\ge 10\delta_{i+1}-\sum_{i\le j<k}\delta_{j+1}>\delta_{i+1}=r(D^*),$$

so $E$ is disjoint from $D^*$.

Assume next that $E_i=D$, and let $x$ denote the intersection point of bd $D^*$ and the segment $[c(D),c(D^*)]=[c(D),p(D)]$. By (3),

$$d(x,p^t(D)) \le 2\delta_{i+1}\frac{\delta_{i+1}}{\delta_{i+1}+1} \qquad (1\le t\le k)$$

On the other hand,

$$d(c(E_{i+1}),c(E)) \le \sum_{i<j<k} d(c(E_j),c(E_{j+1})) \le \sum_{i<j<k} \delta_{j+1} < \delta_{i+1}/8.$$

Easy trigonometric calculations show that for any $y\in(\text{bd } D^*)\cap E$

$$d(x,y) \le d(x,p^{t_i}(D)) + d(p^{t_i}(D),y) \le \frac{\delta_{i+1}}{5} + (2d(c(E_{i+1}),c(E))\delta_{i+1})^{1/2}$$
$$\le \frac{3}{4}\delta_{i+1},$$

and the result follows. □

In what follows, we will turn the above planar construction into a 3-dimensional arrangement of balls.

We will think of $\mathbb{R}^2$ as of a horizontal plane in $\mathbb{R}^3$. The terms 'above' and 'below' will be used in this sense.

<u>Lemma 2.3</u>. Let $\underline{D}_k$ and $P_k\subset\mathbb{R}^2$ be the same as above, $R>1$. For every $D\in\underline{D}_k$, let $B(D)\subset\mathbb{R}^3$ be a closed ball of radius $R$, whose centre is above $\mathbb{R}^2$ and $B(D)\cap\mathbb{R}^2=D$. If $B$ is any closed ball of radius $R$ such that

$$B \subseteq \bigcup_{D\in\underline{D}_k} B(D),$$

then $B = B(D)$ for some $D\in\underline{D}_k$.

<u>Proof</u>. Let $B$ be a closed ball of radius $R$, which is completely covered by $B(D)$'s, and set $D_o=B\cap\mathbb{R}^2$. Obviously, the centre of $B$ must be above $\mathbb{R}^2$, and $r(D_o) \le \max_{D\in\underline{D}_k} r(D) = 1$, otherwise the south pole of $B$ cannot be covered by any $B(D)$.

Pick a disc $D_1\in\underline{D}_k$ such that $B(D_1)$ covers the north pole of $B$. Then $r(D_o)\ge r(D_1)$ with equality if and only if $D_o=D_1$ and hence $B=B(D_1)$.

Thus, we may assume that $r(D_1) < r(D_o) \leq 1$, which implies that $D_1 \in \underline{D}_k''$. From the fact that $B(D_1)$ covers the north pole of $B$ it follows that $B$ covers the south pole of $B(D_1)$. Consequently, $D_o = B \cap \mathbb{R}^2$ contains $c(D_1)$ in its interior. This implies that $D_o$ covers a circular arc of bd $D_1$ whose angle is larger than $2\pi/3$, which contradicts Lemma 2.2, because

$$(\mathrm{bd}\ D_1) \cap D_o \subseteq (\mathrm{bd}\ D_1) \cap \Big(\bigcup_{D_1 \neq E \in \underline{D}_k} E\Big) = X(D_1). \qquad \square$$

Remark 2.4. Enlarging some circles in $\underline{D}_k$ a little bit, we can obviously attain that $D \cap P_k = (\mathrm{int}\ D) \cap P_k$ and all other properties remain valid. It is also clear that in this case Lemma 2.3 can be stated in the following 'quantitative' form. There exists a small constant $\varepsilon_k > 0$ with the property that, for any ball $B$ of radius $R$, we can either find a disc $D \in \underline{D}_k$ such that

$$B \cap P_k = D \cap P_k = (\mathrm{int}\ D) \cap P_k ,$$

or $B \setminus \big(\bigcup_{D \in \underline{D}_k} B(D)\big)$ contains a ball of radius $\varepsilon_k$.

Theorem 2.5. Let $k \geq 2$ be a natural number, $R > 1$. Then there exists a subset $Q_k \subset \mathbb{R}^3$ with the property that any closed ball of radius $R$ contains at least $k$ elements of $Q_k$, and for any 2-colouring of $Q_k$ there exists a ball $B$ of radius $R$ such that $B \cap Q_k$ is monochromatic.

Proof. Let $\underline{D}_k$ and $P_k$ be the same as above, and set

$$Q_k = P_k \cup \Big(\mathbb{R}^3 \setminus \bigcup_{D \in \underline{D}_k} B(D)\Big).$$

If $B$ is any closed ball of radius $R$ such that $B \notin \{B(D) : D \in \underline{D}_k\}$, then, by Lemma 2.3, $B$ contains at least one and hence infinitely many points of $\mathbb{R}^3 \setminus \bigcup_{D \in \underline{D}_k} B(D) \subset Q_k$. If $B = B(D)$ for some $D \in \underline{D}_k$, then $|B \cap Q_k| = |B \cap P_k| = |D \cap P_k| = k$.

Let $f$ be any coloring of $Q_k$ by 2 colours. This induces a 2-colouring of $P_k$; so, according to the definition of $\underline{D}_k$ (part (ii) of Theorem 2.1), there exists a $D\in\underline{D}_k$ such that $B(D)\cap Q_k = D\cap P_k$ is monochromatic. □

A point set $S\subseteq \mathbb{R}^n$ is called $\varepsilon$-discrete if $d(x,y)\geq\varepsilon$ for any two distinct elements $x,y\in S$.

Theorem 2.5'. For any natural number $k$, there exist $\varepsilon_k''>0$ and an $\varepsilon_k''$-discrete subset $Q_k'\subseteq\mathbb{R}^3$ with the property that any open unit ball contains at least $k$ elements of $Q_k'$, and for any 2-colouring of $Q_k'$ one can find a closed unit ball $B$ such that $B\cap Q_k'$ is monochromatic.

Proof. Let $\underline{D}_k$ and $P_k$ satisfy the slightly stronger properties stated in Remark 2.4, and let $S_k$ be an $\varepsilon_k'$-discrete subset of $\mathbb{R}^3\setminus\bigcup_{D\in\underline{D}_k} B(D) = T_k$ such that every ball $B'\subseteq T$ whose radius is at least $\varepsilon_k$ contains at least $k$ elements of $S_k$. Set $Q_k' = P_k\cup S_k$. Then any open ball of radius $R$ covers at least $k$ elements of $Q_k'$, and for any 2-colouring of $Q_k'$ there exist a $D\in\underline{D}_k$ such that all elements of $B(D)\cap Q_k'=D\cap P_k$ have the same colour. Changing the scale so that $R$ becomes the unit distance, we obtain the result. □

Theorem 2.5". For any natural numbers $k$ and $d\geq 3$, there exist an $\varepsilon_{k,d}>0$ and an $\varepsilon_{k,d}$-discrete subset $Q_{k,d}\subseteq\mathbb{R}^d$ with the property that any open unit ball contains at least $k$ elements of $Q_{k,d}$, and for any 2-colouring of $Q_{k,d}$ one can find a closed unit ball $B$ such that $B\cap Q_{k,d}$ is monochromatic.

Proof. We will prove by induction on $d$ the following little stronger statement. There exist a $Q_{k,d}\subseteq\mathbb{R}^d$ with the required properties and a finite system $\underline{B}_{k,d}$ of closed unit balls such that for any 2-colouring of $Q_{k,d}$ one can find a $B\in\underline{B}_{k,d}$ for which all elements of $B\cap Q_{k,d}$ have the same colour .

For $d=3$, this follows from the proof of Theorem 2.5'.

Assume now that $Q_{k;d}$ and $\underline{B}_{k;d}$ have already been defined for some $d\geq 3$. Given any $B\in\underline{B}_{k;d}$, let $B'$ denote a unit ball in $\mathbb{R}^{d+1}$ whose intersection with the hyperplane $\mathbb{R}^d$ is $B$. Let $\varepsilon>0$, and let $S_{k;d+1}$ denote a maximal $\varepsilon$-discrete subset of $\mathbb{R}^{d+1}\setminus \bigcup_{B\in\underline{B}_{k;d}} B'$. It is easy to see that, if $\varepsilon$ is sufficiently small, then

$$Q_{k;d+1} = Q_{k;d} \cup S_{k;d+1} ;$$

$$\underline{B}_{k;d+1} = \{ B' : B\in\underline{B}_{k;d} \}$$

satisfy all the conditions. $\square$

Replacing each element $q\in Q_{k;d}$ by a unit ball centered at $q$, we obtain a k-fold covering of $\mathbb{R}^d$, which meets the requirements of Theorem 1. In this sense, Theorems 1 and 2.5" are 'dual' statements.

## 3. Proof of Theorem 2

Let $\underline{B}=\{B_i : i\in I\}$ be a k-fold covering of $\mathbb{R}^d$ with unit balls such that no point of $\mathbb{R}^d$ is contained in more than t members of $\underline{B}$. Assume without loss of generality that the balls are in general position. For any $B_i$, let bd $B_i$ denote the surface of $B_i$. Let $C_j$ ($j\in J$) be the connected components of $\mathbb{R}^d \setminus \bigcup_{i\in I} \mathrm{bd}\, B_i$, and define a hypergraph $H(\underline{B})=H$ in the following way. Set

$$V(H) = \{B_i : i\in I\},$$

$$E(H) = \{E_j : j\in J\} \quad \text{where} \quad E_j=\{B_i : C_j\subset B_i\in \underline{B}\}.$$

($V(H)$ and $E(H)$ denote the vertex set and the edge set of $H$, as usual.) The fact that $\underline{B}$ is a k-fold covering implies that

(9) $$|E_j| \geq k \qquad \text{for every } j\in J.$$

Fix now a $j_o\in J$, and let $E_j$ be any edge of H such that $E_j\cap E_{j_o}\neq\emptyset$. Then all elements (balls) belonging to $E_j$ are contained in a ball B of radius 4 (around any point of $C_{j_o}$). Since no point of B is covered by more than t members of $\underline{B}$, we obtain

$$|\bigcup\{E_j : E_j\cap E_{j_o}\neq\emptyset\}| \leq \frac{t \text{ Vol } B}{\text{Vol } B_j} = t4^d.$$

On the other hand, it is easy to see that $t4^d$ balls cut the space into at most $(t4^d)^d$ different pieces, hence

(10) $$|\{E_j : E_j\cap E_{j_o}\neq\emptyset\}| \leq t^d 4^{d^2} \quad \text{for any } j_o\in J.$$

The following result is an easy consequence of the Lovász Local Lemma (cf. [1],[6]).

<u>Theorem 3.1.</u> ([2],[3]) Let H be a hypergraph whose every edge has at least k elements. If every edge of H meets at most $2^{k-3}$ other edges, then there

exists a 2-colouring of V(H) such that no edge is monochromatic. □

By (9) and (10), we can apply this result to our hypergraph H, provided that $t \leq 2^{(k/d)-2d-1}$. Thus, we obtain a colouring $f : V(H) \to \{R,G\}$ with 2 colours (say, red and green) such that every $C_j$ is covered by both red and green balls. Consequently, $\underline{B} = \underline{B}_R \cup \underline{B}_G$,

$$\underline{B}_R = \{B_i \in \underline{B} : f(B_i)=R\},$$

$$\underline{B}_G = \{B_i \in \underline{B} : f(B_i)=G\}$$

is a decomposition of $\underline{B}$ into two coverings. □

Remark 3.2. The same proof shows that Theorem 2 remains valid (apart from the value of the constant) for every k-fold covering $\underline{B}=\{B_i : i \in I\}$ with balls satisfying $\frac{\inf_{i\in I} r(B_i)}{\sup_{i\in I} r(B_i)} > \varepsilon$ for some $\varepsilon>0$.

## References


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