Game saturation of intersecting families

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Abstract

We consider the following combinatorial game: two players, Fast and Slow, claim k-element subsets of $[n] = \{1, 2, ..., n\}$ alternately, one at each turn, such that both players are allowed to pick sets that intersect all previously claimed subsets. The game ends when there does not exist any unclaimed k-subset that meets all already claimed sets. The score of the game is the number of sets claimed by the two players, the aim of Fast is to keep the score as low as possible, while the aim of Slow is to postpone the game's end as long as possible. The game saturation number $gsat(\mathbb{I}_{n,k})$ is the score of the game when both players play according to an optimal strategy. We prove that $\Omega_k(n^{k/3-5}) = gsat(\mathbb{I}_{n,k}) = O_k(n^{k-k^{1/2}/2})$ holds.

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1 Introduction

A typical problem in extremal combinatorics is to determine the most number of hyperedges that a hypergraph on n vertices may contain provided it satisfies some property $\mathcal{P}$. Let us denote the maximum value by $ex(\mathcal{P})$ which is often called the *extremal number* of $\mathcal{P}$. Another often considered problem is to find the minimum number of edges that a hypergraph may contain provided it is maximal (i.e. unextendable) with respect to property $\mathcal{P}$. Let us denote the minimum value by $sat(\mathcal{P})$ which is said to be the *saturation number*

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of $\mathcal{P}$. In this paper we study a game played by two players, Fast and Slow such that Fast's aim is to create a maximal hypergraph the size of which is as close to the saturation number as possible while Slow's aim is to create a maximal hypergraph the size of which is as close to the extremal number as possible.

More formally, the *saturation game* $(X, \mathcal{D})$ is played on the *board* X according to the *rule* $\mathcal{D} \subseteq 2^X$, where $\mathcal{D}$ is downward closed (or decreasing) family of subsets of X, that is $E \subset D \in \mathcal{D}$ implies $E \in \mathcal{D}$. Two players Fast and Slow pick one unclaimed element of the board at each turn alternately such that at any time i during the game, the set C_i of all elements claimed thus far belongs to $\mathcal{D}$. The elements $x \in X \setminus C_i$ for which $\{x\} \cup C_i \in \mathcal{D}$ holds will be called the *legal moves* at time $i+1$ as these are the elements of the board that can be claimed by the player on turn. The game ends when there is no more legal moves, that is when C_i is a maximal set in $\mathcal{D}$ and the score of the game is the size of C_i. The aim of Fast is to finish the game as fast as possible and thus obtain a score as low as possible while the aim of Slow is to keep the game going as long as possible. The *game saturation number* $gsat(\mathcal{D})$ is the score of the game when both players play according to an optimal strategy. To be precise we have to distinguish two cases depending on which player takes the first move. Let $gsat_F(\mathcal{D})$ and $gsat_S(\mathcal{D})$ denote the score of the saturation game $(X, \mathcal{D})$ when both players play according to an optimal strategy and the game starts with Fast's or Slow's move, respectively. In most cases, the board X is either $\binom{[n]}{k}$ for some $1 \leq k \leq n$ or $2^{[n]}$. Clearly, the inequalities $sat(\mathcal{D}) \leq gsat_F(\mathcal{D}), gsat_S(\mathcal{D}) \leq ex(\mathcal{D})$ hold.

The first result concerning saturation games is due to Füredi, Reimer and Seress [6]. They considered the case when the board X is the edge set of the complete graph on n vertices and $\mathcal{D} = \mathcal{D}_{n,K_3}$ is the family of all triangle-free subgraphs of K_n. They established the lower bound $\frac{1}{2}n \log n \leq gsat_F(\mathcal{D}_{n,K_3}), gsat_S(\mathcal{D}_{n,K_3})$ and claimed without proof an upper bound $\frac{n^2}{5}$ via personal communication with Paul Erdős. Their paper mentions that the first step of Fast's strategy is to build a C_5-factor. However, as it was recently pointed out by Hefetz, Krivelevich and Stojakovic [7], Slow can prevent this to happen. Indeed, in his first $\lfloor \frac{n-1}{2} \rfloor$ moves, Slow can create a vertex x with degree $\lfloor \frac{n-1}{2} \rfloor$, and because of the triangle-free property, the neighborhood of x must remain an independent set throughout the game. But clearly, a graph that contains a C_5-factor cannot have an independent set larger than $2n/5$.

Recently, Cranston, Kinnersley, O and West [2] considered the saturation game when the board $X = X_G$ is the edge set of a graph G and $\mathcal{D} = \mathcal{D}_G$ consists of all (partial) matchings of G.

In this paper, we will be interested in intersecting families. That is the board $X = X_{n,k}$ will be the edge-set of the complete k-graph on n vertices and $\mathcal{D} = \mathbb{I}_{n,k}$ is the set of intersecting families that is $\mathbb{I}_{n,k} = \{\mathcal{F} \subseteq X_{n,k} : F \cap G \neq \emptyset \ \forall F, G \in \mathcal{F}\}$. Note that by the celebrated theorem of Erdős, Ko and Rado [4] we have $ex(\mathbb{I}_{n,k}) = \binom{n-1}{k-1}$ provided $2k \leq n$. The saturation number $sat(\mathbb{I}_{n,k})$ is not known. J-C Meyer [9] conjectured this to be $k^2 - k + 1$ whenever a projective plane of order $k-1$ exists. This was disproved by Füredi

[5] by constructing a maximal intersecting family of size $3k^2/4$ and later improved by Boros, Füredi, and Kahn to $k^2/2$ [1]. The best known lower bound on $sat(\mathbb{I}_{n,k})$ is $3k$ due to Dow, Drake, Füredi, and Larson [3].

We mentioned earlier that the game saturation number might depend on which player starts the game. This is indeed the case for intersecting families. If k equals 2, then after the first two moves the already claimed edges are two sides of a triangle. Thus if Fast is the next to move, he can claim the last edge of this triangle and the game is finished, thus $gsat_F(\mathbb{I}_{n,2}) = 3$. On the other hand, if Slow can claim the third edge, then he can pick an edge containing the intersection point of the first two edges and then all such edges will be claimed one by one and we obtain $gsat_S(\mathbb{I}_{n,2}) = n - 1$.

The main result of the present paper is the following theorem that bounds away $gsat(\mathbb{I}_{n,k})$ both from $ex(\mathbb{I}_{n,k})$ and $sat(\mathbb{I}_{n,k})$ if n is large enough compared to k.

Theorem 1.1. *For all $k \geq 2$ the following holds:*

$$\Omega_k\left(n^{\lfloor k/3 \rfloor - 5}\right) \leq gsat_F(\mathbb{I}_{n,k}), gsat_S(\mathbb{I}_{n,k}) \leq O_k\left(n^{k-k^{1/2}/2}\right).$$

2 Proof of Theorem 1.1

We start this section by defining an auxiliary game that will enable us to prove Theorem 1.1.

We say that a set S *covers* a family $\mathcal{F}$ of sets if $S \cap F \neq \emptyset$ holds for every set $F \in \mathcal{F}$. The *covering number* $\tau(\mathcal{F})$ is the minimum size of a set S that covers $\mathcal{F}$. Note that if $\mathcal{F}$ is an intersecting family of k-sets, then $\tau(\mathcal{F}) \leq k$ holds as by the intersecting property any set $F \in \mathcal{F}$ covers $\mathcal{F}$. The following proposition is folklore, but for sake of self-containedness we present its proof.

Proposition 2.1. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a maximal intersecting family with covering number τ, then the following inequalities hold:*

$$\binom{n-\tau}{k-\tau} \leq |\mathcal{F}| \leq k^{\tau}\binom{n-\tau}{k-\tau}.$$

Proof. The first inequality follows from the following observation: if S covers $\mathcal{F}$, then all k-subset of $[n]$ that contain S must belong to $\mathcal{F}$ by maximality.

To obtain the second inequality note that denoting the *maximal degree* of $\mathcal{F}$ by $\Delta(\mathcal{F})$ the inequality $|\mathcal{F}| \leq k\Delta(\mathcal{F})$ holds. Indeed, by the intersecting property we have $|\mathcal{F}| \leq \sum_{x \in F} d(x)$ for any set $F \in \mathcal{F}$ and the right hand side is clearly not more than $k\Delta(\mathcal{F})$. Let $d_j(\mathcal{F})$ denote the maximum number of sets in $\mathcal{F}$ that contain the same j-subset, thus $d_1(\mathcal{F}) = \Delta(\mathcal{F})$ holds by definition. For any $j < \tau$ and j-subset J that is contained in some $F \in \mathcal{F}$ there exists an $F' \in \mathcal{F}$ with $J \cap F' = \emptyset$. Thus $d_j \leq kd_{j+1}$ is true. Since $d_\tau \leq \binom{n-\tau}{k-\tau}$ holds, we obtain

$d_1(\mathcal{F}) \leq k^{\tau-1}\binom{n-\tau}{k-\tau}$ and thus the second inequality follows by the first observation of this paragraph. $\square$

The main message of Proposition 2.1 is that if k is fixed and n tends to infinity, then the order of magnitude of the size of a maximal intersecting family $\mathcal{G}$ is determined by its covering number. As $|\mathcal{G}| = \Theta_k(n^{k-\tau(\mathcal{G})})$ holds, a strategy in the saturation game that maximizes $\tau(\mathcal{F})$ is optimal for Slow up to a constant factor, and a strategy that minimizes $\tau(\mathcal{F})$ is optimal for Fast up to a constant factor.

Therefore from now on we will consider the τ-*game* in which two players: *minimizer* and *Maximizer* take unclaimed elements of $X = X_{n,k} = \binom{[n]}{k}$ alternately such that at any time during the game the set of all claimed elements should form an intersecting family. The game stops when the claimed elements form a maximal intersecting family $\mathcal{F}$. The score of the game is the covering number $\tau(\mathcal{F})$ and the aim of *minimizer* is to keep the score as low as possible while *Maximizer*'s aim is to reach a score as high as possible. Let $\tau_m(n,k)$ (resp. $\tau_M(n,k)$) denote the score of the game when both players play according to their optimal strategy and the first move is taken by *minimizer* (resp. *Maximizer*). The following simple observation will be used to define strategies.

Proposition 2.2. *Let $\mathcal{G} = \{G_1, ..., G_{k+1}\}$ be an intersecting family of k-sets. Assume that there exists a set C such that $G_1 \setminus C, ..., G_{k+1} \setminus C$ are non-empty and pairwise disjoint. Then we have $\tau(\mathcal{F}) \leq |C|$ for any intersecting family $\mathcal{G} \subseteq \mathcal{F}$ of k-sets.*

Proof. To see the statement, observe that if a k-set F is disjoint from C, then it cannot meet all $G_i \setminus C$'s. $\square$

Theorem 1.1 will follow from the following two lemmas and Proposition 2.1.

Lemma 2.3. *For any positive integer k, there exists $n_0 = n_0(k)$ such that if $n \geq n_0$, then*

$$\tau_m(n,k), \tau_M(n,k) \leq \lceil 2k/3 \rceil + 2$$

holds.

Proof. We have to provide a strategy for *minimizer* that ensures the covering number of the resulting family is small. Let us first assume that *minimizer* starts the game and let $m_0, M_1, m_1, M_2, m_2, ...$ denote the k-sets claimed during the game. *minimizer*'s strategy will involve sets A_i, C_i for $2 \leq i \leq k$ with the properties:

a) $A_i \subseteq m_0$, $C_{i-1} \subseteq C_i$, $|C_i| \leq |C_{i-1}| + 1$,
b) the sets $m_0 \setminus (A_i \cup C_i)$ and $m_1 \setminus C_i, ..., m_i \setminus C_i$ are non-empty and pairwise disjoint,
c) $A_i \cup C_i$ meets all sets m_j, M_j for $j \leq i$,
d) C_i meets all m_j's and all but at most one of the M_j's for $j \leq i$.

First we make sure that *minimizer* is able to claim k-sets $m_0, m_1, ..., m_k$ such that sets A_i, C_i with the above properties exist. *minimizer* can claim an arbitrary m_0, and after *Maximizer*'s first move M_1, he can pick $a_1 \in m_0 \cap M_1$ and claim $m_1 = \{a_1\} \cup N_1$ where N_1 is a $(k-1)$-set disjoint from $m_0 \cup M_1$. *minimizer*'s strategy distinguishes two cases for claiming m_2 depending on *Maximizer*'s second move M_2. If $a_1 \in M_2$, then *minimizer* claims $m_2 = \{a_1\} \cup N_2$ with N_2 being a $(k-1)$-set disjoint from $m_0 \cup m_1 \cup M_1 \cup M_2$ and we define $A_2 = \emptyset, C_2 = \{a_1\}$. If $a_1 \notin M_2$, then by the intersecting property there exists $c_2 \in M_1 \cap M_2$. Let *minimizer* claim $m_2 = \{a_1, c_2\} \cup N_2$ with N_2 being a $(k-2)$-set disjoint from $m_0 \cup m_1 \cup M_1 \cup M_2$ and put $A_2 = \emptyset, C_2 = \{a_1, c_2\}$. In both cases, the properties a)-d) hold.

Let us assume that *minimizer* is able to claim k-sets $m_0, m_1, ..., m_{i-1}$ and define sets A_i, C_i. The strategy of *minimizer* will distinguish several cases depending on *Maximizer*'s move M_i.

Case I: C_{i-1} meets all previously claimed k-sets.

• If $M_i \cap C_{i-1} \neq \emptyset$, then let $m_i = C_{i-1} \cup N_i$ with $|N_i| = k - |C_{i-1}|$ and $N_i \cap (\cup_{j=0}^{i-1} m_j \cup \cup_{j=1}^{i} M_j) = \emptyset$. This is a legal move for *minimizer* and $C_i = C_{i-1}, A_i = A_{i-1}$ satisfy properties a)-d).

• If $M_i \cap C_{i-1} = \emptyset$ and there exists $a_i \in (M_i \cap m_0) \setminus A_{i-1}$, then let $m_i = C_{i-1} \cup \{a_i\} \cup N_i$ with $|N_i| = k - |C_{i-1}| - 1$ and $N_i \cap (\cup_{j=0}^{i-1} m_j \cup \cup_{j=1}^{i} M_j) = \emptyset$. As m_i meets all previously claimed k-sets, this is a legal move for *minimizer*, and the sets $C_i = C_{i-1}, A_i = A_{i-1} \cup \{a_i\}$ satisfy properties a)-d).

• If none of the above subcases happen, then we must have $M_i \cap C_{i-1} = \emptyset$ and $\emptyset \neq M_i \cap m_0 \subseteq A_{i-1}$, as *Maximizer* must pick M_i such that it intersects all previously claimed k-sets, in particular it should intersect m_0. Let $a \in M_i \cap m_0$ and thus $a \in A_{i-1}$ and let *minimizer* claim $m_i = C_{i-1} \cup \{a\} \cup N_i$ with $|N_i| = k - |C_{i-1}| - 1$ and $N_i \cap (\cup_{j=0}^{i-1} m_j \cup \cup_{j=1}^{i} M_j) = \emptyset$. The sets $A_i = A_{i-1} \setminus \{a\}, C_i = C_{i-1} \cup \{a\}$ satisfy a)-d).

Case II: There exists an M_j $(j \leq i-1)$ with $M_j \cap C_{i-1} = \emptyset$.

• As *Maximizer* picks M_i such that it meets all previously claimed k-sets, there must exist an element $c \in M_i \cap M_j$. By c), we have that $C_{i-1} \cup \{c\}$ meets all previously claimed k-sets, thus $m_i = C_{i-1} \cup \{c\} \cup N_i$ with $|N_i| = k - |C_{i-1}| - 1$ and $N_i \cap (\cup_{j=0}^{i-1} m_j \cup \cup_{j=1}^{i} M_j) = \emptyset$ is a legal move for *minimizer*. The sets $A_i = A_{i-1} \setminus \{c\}$ and $C_i = C_{i-1} \cup \{c\}$ satisfy properties a)-d).

We have just seen that *minimizer* is able to claim k-sets $m_1, m_2, ..., m_k$ such that there exist sets A_i, C_i $(2 \leq i \leq k)$ satisfying the properties a)-d).

Claim 2.4. For any $2 \leq i \leq k$, the inequality $|C_i| \leq 3 + \lfloor \frac{2(i-2)}{3} \rfloor$ holds.

Proof of Claim. Let $\alpha_i = |\{j : 2 < j \leq i, |C_{j-1}| = |C_j|\}|$, i.e. the number of steps when we are in the first two subcases of Case 1. Let $\beta_i = |\{j : 2 < j \leq i, |C_i| = |C_{i-1}| + 1, j\text{th turn is in Case 1}\}|$ and $\gamma_i = |\{j : 2 < j \leq i, j\text{th turn is in Case 2}\}|$. Clearly, we have $\alpha_i + \beta_i + \gamma_i = i - 2$. If the jth turn is in the last subcase of Case 1 or in Case 2, then C_j meets all previously claimed k-subsets and m_j, as well. Thus we obtain $\gamma_i + \beta_i \leq \alpha_i + \beta_i + 1$ and therefore $\gamma_i \leq \alpha_i + 1$. Also, as in the last subcase of Case 1 the size of A_j decreases, and this size only increases if we are in the first two subcases of Case 1, we obtain $\beta_i \leq \alpha_i$. From these three inequalities it follows that $1 + (i-2)/3 \leq \alpha_i$ holds and thus statement of Claim 2.4. □

Let M_{k+1} the next move of *Maximizer*. By property d), there can be at most one set M_j that is disjoint from C_k. If *minimizer* picks an element m of $M_{k+1} \cap M_j$ and claims the k-set $m_{k+1} = C_k \cup \{m\} \cup N_{k+1}$ with $|N_{k+1}| = k - |C_k| - 1$ and $N_k \cap (\cup_{j=0}^{k} m_j \cup \cup_{j=1}^{k+1} M_j) = \emptyset$, then the set $C = C_k \cup \{m\}$ and $m_1, ..., m_{k+1}$ satisfy the conditions of Proposition 2.2. This proves the lemma if *minimizer* takes the first move of the game.

If *Maximizer* starts the game, then *minimizer* can imitate his previous strategy to obtain a sequence of moves $M_1, m_1, M_2, m_2, ..., M_k, m_k$ with the slight modification that m_1 takes the role of m_0 in property a). In this way, *minimizer* obtains a C_k with the same size as before and in his *k+1*st and *k+2*nd moves, he can add two more element to obtain a set C that is just one larger and satisfies the conditions of Proposition 2.2 together with $m_2, ..., m_{k+1}, m_{k+2}$. Details are left to the reader. □

In the following proof we will use the notation $deg_{\mathcal{F}}(x) = |\{F : x \in F \in \mathcal{F}\}|$ for any set X and a family $\mathcal{F}$ of subsets of X.

Lemma 2.5. *For any positive integer k, if $k^{3/2} \leq n$, then*

$$\frac{1}{2}k^{1/2} \leq \tau_m(n,k), \tau_M(n,k)$$

holds.

Proof. Note that

$$\frac{|\mathcal{F}|}{max_{x \in X} deg(\mathcal{F})(x)} \leq \tau(\mathcal{F})$$

holds for any set X and a family $\mathcal{F}$ of subsets of X. Therefore a possible strategy for *Maximizer* is to keep

$$\max_{s \in [n]} deg_{\{M_l : l \leq j\}}(s)$$

as small as possible. If he is able to do so long enough, then even the sets claimed by him will ensure that the covering number of the resulting family is large. We claim that *Maximizer* can choose legal steps M_1, ..., $M_{\lfloor k^{1/2} \rfloor}$ such that

$$\max_{s \in [n]} deg_{\{M_l : l \leq \lfloor k^{1/2} \rfloor\}}(s) \leq 2$$

holds.

In order to establish the aim above, *Maximizer* chooses his set M_j with $M_j = M_j^1 \cup M_j^2$, $M_j^1 \cap M_j^2 = \emptyset$, $|M_j^1| = \lfloor k^{1/2} \rfloor - 1 =: l$ and $|M_j^2| = k - l + 1$. The M_j^1's are independent of how *minimizer* picks his sets, they are chosen to ensure that $M_i \cap M_j \neq \emptyset$ holds for any pair $1 \leq i < j \leq l+1$. The other part M_j^2 is supposed to ensure that M_j meets all j or $j-1$ sets that *minimizer* has claimed by that point of the game (depending on who started the game).

We define the M_j^1's inductively: let $M_1^1 = [l]$ and assume the elements of $M_1^1, ..., M_{j-1}^1$ are enumerated increasingly as $v_1^1, ...v_l^1, v_1^2, ..., v_l^2, ..., v_1^{j-1}, ..., v_l^{j-1}$, then let

$$M_j^1 = \{v_{j-1}^1, v_{j-1}^2, ..., v_{j-1}^{j-1}\} \cup \{i_j, i_j+1, ..., i_j+l-j\}$$

where $i_j = 1 + \sum_{h=0}^{j-1} l - h$. By definition, M_j^1 meets all previous M_i^1's in exactly one point and all intersection points are different, thus we obtain that the maximum degree is 2. Also, since the M_j^1 introduces $l-j+1$ new points, we have $U := \bigcup_{j=1}^{l+1} M_j^1 = \left[\frac{l(l+1)}{2}\right]$ and $\frac{l(l+1)}{2} \leq k/2$.

We still have to show that *Maximizer* can define the M_j^2's such that M_j^2 intersect all previously claimed sets of *minimizer* and the maximum degree is kept at most 2. *Maximizer* tries to pick the M_j^2's with $U \cap M_j^2 = \emptyset$ and such that the following two properties hold for $j \leq \lfloor k^{1/2} \rfloor$:

(1) $|\{x : deg_{\{M_l^2 : l \leq j\}}(x) = 2\}| \leq \frac{j^2}{2}$, and

(2) $\{x : deg_{\{M_l^2 : l \leq j\}}(x) \geq 3\} = \emptyset$.

We prove by induction on j that he can choose M_j^2 satisfying (1) and (2). M_1^2 can be chosen arbitrarily with the restriction that it is disjoint from U and if *minimizer* starts the game, then it should meet m_1. Note that the latter is possible as $|U| \leq k/2$ and thus $|m_1 \setminus U| \geq k/2$ holds.

Assume *Maximizer* was able to pick $M_1^2, ..., M_{j-1}^2$ for some $1 < j \leq \lfloor k^{1/2} \rfloor$ satisfying (1) and (2) and now he has to pick M_j^2. Observe that by the inductive hypothesis for all $1 \leq h < j$ we have

$$|m_h \setminus (\{x : deg_{\{M_i^2 : i < j\}}(x) = 2\} \cup U)| > k/2 - (j-1)^2 \geq 2k^{1/2},$$

and the sets $M_h^2 \setminus (\{x : deg_{\{M_i^2 : i < j\}}(x) = 2\} \cup U)$ with $1 \leq h < j$ are pairwise disjoint. Thus if *Maximizer* picks M_j^2 such that it is disjoint from $\{x : deg_{\{M_i^2 : i < j\}}(x) \geq 2\} \cup U$, then (2) is clearly satisfied. Let us fix $x_h \in m_h \setminus (\{x : deg_{\{M_i^2 : i < j\}}(x) \geq 2\} \cup U)$ for all $1 \leq h < j$ and let $M_j^2 = \{x_h : 1 \leq h < j\} \cup M$ where $|M| = k - l + 1 - |\{x_h : 1 \leq h < j\}|$ and $M \cap \bigcup_{h=1}^{j-1}(m_h \cup M_h) = \emptyset$. It is possible to satisfy this latter condition as $n \geq k^{3/2}$. We see that the new degree-2 elements are the x_h's and thus their number is at most $j-1$ As $(j-1)^2 + j - 1 \leq j^2$ we know that $\{M_h : h \leq j\}$ satisfies (1).

This inductive construction showed that the maximum degree of $\{M_h : 1 \leq h \leq l+1\}$ is 2 and thus its covering number is at least $(l+1)/2 = \lfloor \frac{k^{1/2}}{2} \rfloor$. $\square$

3 Concluding remarks and open problems

The main result of the present paper, Theorem 1.1 states that the exponent of $gsat(\mathbb{I}_{n,k})$ grows linearly in k. We proved that the constant of the linear term is at least 1/3 and at most 1. We deduced this result by obtaining lower and upper bounds on the covering number of the family of sets claimed during the game and using a well-known relation between the covering number and the size of an intersecting family. However we were not able to determine the order of magnitude of the covering number. We conjecture this to be linear in k. If it is true, this would have the following consequence.

Conjecture 3.1. *There exists a constant $c > 0$ such that for any $k \geq 2$ and $n \geq n_0(k)$ the inequality $gsat_F(\mathbb{I}_{n,k}), gsat_S(\mathbb{I}_{n,k}) \leq O(n^{(1-c)k})$ holds.*

In the proof of Lemma 2.5, there are two reasons for which *Maximizer* cannot continue his strategy for more than $\sqrt{k}$ steps. First of all the union of the M^1_j's becomes too large, and second of all the intersection points of the M^2_j's and the sets of *minimizer* should be disjoint. The first problem can probably be overcome by a result of Kahn [8] who showed the existence of an r-uniform intersecting family $\mathcal{F}_r$ with $|\mathcal{F}| = O(r)$, $|\bigcup_{F \in \mathcal{F}_r} F| = O(r)$ and $\tau(\mathcal{F}_r) = r$.

As we mentioned in the introduction, the answers to game saturation problems considered so far, did not depend on which player makes the first move, while this is the case for intersecting families if $k = 2$. However, if we consider the τ-game, both our lower and upper bounds differ by at most one depending on whether it is *Maximizer* or *minimizer* to make the first move. Thus we formulate the following conjecture.

Conjecture 3.2. *There exists a constant c such that $|\tau_M(n,k) - \tau_m(n,k)| \leq c$ holds independently of n and k.*

Intersecting families are most probably one of the two most studied classes of families in extremal set system theory. The other class is that of Sperner families. The downset $\mathbb{S}_n = \{\mathcal{F} \subseteq 2^{[n]} : \mathcal{F} \text{ is Sperner}\}$ is another example for which the two game saturation numbers differ a lot. Clearly, if Fast starts the game, then he can claim either the empty set or $[n]$ to finish the game immediately, thus we have $gsat_F(\mathbb{S}_n) = 1$. It is not very hard to see that if Slow starts with claiming a set $F \subset [n]$ of size $\lfloor n/2 \rfloor$, then the game will last at least a linear number of turns. Indeed, consider the family $\mathcal{N}_F = \{F \setminus \{x\} \cup \{y\} : x \in F, y \in [n] \setminus F\}$. $\mathcal{N}_F$ has size about $n^2/4$, while

$$\max_{G:\, G \not\subset F, F \not\subset G} |\{F' \in \mathcal{N}_F : F' \subseteq G \text{ or } G \subseteq F'\}| = \lceil n/2 \rceil.$$

This shows that $gsat_S(\mathbb{S}_n) \geq n/2$. One can improve this bound, but we were not able to obtain a superpolynomial lower bound nor an upper bound $o(\binom{[n]}{\lfloor n/2 \rfloor}) = o(sat(\mathbb{S}_n))$.

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