Profile vectors in the lattice of subspaces

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Abstract

The profile vector $f(\mathcal{U}) \in \mathbb{R}^{n+1}$ of a family $\mathcal{U}$ of subspaces of an n-dimensional vector space V over $GF(q)$ is a vector of which the ith coordinate is the number of subspaces of dimension i in the family $\mathcal{U}$ $(i = 0, 1, ..., n)$. In this paper, we determine the profile polytope of intersecting families (the convex hull of the profile vectors of all intersecting families of subspaces).

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1 Introduction

Many problems in extremal set theory consider a set $\mathbb{A}$ of families of subsets of an n-element set all having some fixed property. All subsets F possess a *weight* $w(F)$ depending only on $|F|$, the size of F and we ask for the family $\mathcal{F}$ with the largest weight $w(\mathcal{F}) = \sum_{F\in\mathcal{F}} w(F)$. (Note, that asking for the family with largest size is equivalent to asking for the family with largest weight for the constant weight 1.)

The basic tool for dealing with all kinds of weight functions *simultaneously* is the *profile vector* $f(\mathcal{F})$ of a family $\mathcal{F}$ which is defined by

$$f(\mathcal{F})_i = |\{F \in \mathcal{F} : |F| = i\}| \quad (i = 0, 1, ..., n).$$

With this notation the weight of a family for a given weight function w is simply the inner product of the weight vector and the profile vector. Therefore, as we know from linear programming, for any weight function the maximum weight is taken at one of the extreme points of the convex hull of the profile vectors (the *profile polytope*) of all families in $\mathbb{A}$. We denote the set of profile vectors by $\mu(\mathbb{A})$, its convex hull by $\langle\mu(\mathbb{A})\rangle$, the set of extreme points by $E(\mathbb{A})$ and the families having a profile in $E(\mathbb{A})$, the extremal families by $\mathcal{E}(\mathbb{A})$. If the weights are non-negative, then increasing any coordinate of the profile vector increases the weight of the family, so the maximum for these weights is taken at an extreme point which is maximal with respect to the coordinate-wise ordering. We call these vectors *essential* extreme points and denote them by $E^*(\mathbb{A})$ and the corresponding families by $\mathcal{E}^*(\mathbb{A})$. Note that to prove that a set of profiles are the extreme points of the profile polytope one has to express all profiles as a convex combination of these vectors, while to prove that a set of profiles are the essential extreme points of the polytope it is enough to dominate (a vector f dominates g if it is larger in the coordinate-wise ordering) any other profiles.

The systematic investigation of profile vectors and profile polytopes was started by P.L. Erdős, P. Frankl and G.O.H. Katona in [4] and [5], an overview of the topic can be found in the book of K. Engel [3].

The notion of profile vector can be introduced for any ranked poset P (a poset P is said to be ranked if there exist a non-negative integer l and a mapping $r : P \to \{0, 1, ..., l\}$ such that for any $p_1, p_2 \in P$ if p_2 covers p_1, we have $r(p_1) + 1 = r(p_2)$ and $r(p) = 0$ for some $p \in P$). In this case the profile of a family $\mathcal{F} \subseteq P$ is defined by

$$f(\mathcal{F})_i = |\{p \in \mathcal{F} : \text{rank } (p) = i\}| \quad (i = 0, 1, ..., n),$$

where rank (p) denotes the rank of an element p and n is the largest rank in P. Several results are known about profile vectors in the generalized context as well (see e.g. [3], [6], [11]).

One of the most studied ranked poset is the poset $L_n(q)$ of subspaces of an n-dimensional vector space V over the finite field $GF(q)$ with q elements (the ordering is just set-theoretic inclusion). In this case the rank of a subspace is just its dimension, so the profile vector $f(\mathcal{U})$ of a family $\mathcal{U}$ of subspaces is a vector of length $n+1$ (indexed from 0 to n) with $f(\mathcal{U})_i = |\{U \in \mathcal{U} : \dim U = i\}|$, $i = 0, 1, ..., n$. In this paper we determine the profile polytope of intersecting families in the poset $L_n(q)$. A family $\mathcal{U}$ of subspaces is called intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq 1$ (and t-intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq t$). Two subspaces U, U' are said to be disjoint if $\dim(U \cap U') = 0$ i.e. $U \cap U' = \{\underline{0}\}$.

We will use the symbol $\left[{n \atop k}\right]_q = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}$ for the Gaussian (q-nomial) coefficient denoting the number of k-dimensional subspaces of an n-dimensional space over $GF(q)$ (and q will be omitted, when it is clear from the context). The set of all k-dimensional subspaces of a vector space V will be denoted by $\left[{V \atop k}\right]$.

With the above notations the main result of the present paper is the following theorem.

Theorem 1 *The essential extreme points of the profile polytope of the set of intersecting families of subspaces are the vectors v_i $(1 \leq i \leq n/2)$ for even n and there is an additional essential extreme point v^+ for odd n, where*

$$(v_i)_j = \begin{cases} 0 & \text{if } \ 0 \leq j < i \\ \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} & \text{if } \ i \leq j \leq n-i \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } \ j > n-i. \end{cases} \tag{1}$$

and

$$(v^+)_j = \begin{cases} 0 & \text{if } \ 0 \leq j < n/2 \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } \ j > n/2. \end{cases} \tag{2}$$

2 Intersecting families of subspaces

In this section we determine the essential extreme points of the profile polytope of the set of intersecting families of subspaces. (Since the intersecting property is hereditary -i.e. after removing any of its members an intersecting family stays intersecting-, we know (cf. [5]) that any extreme point can be obtained from one of the essential extreme points by changing some of the non-zero coordinates to zero.) This was implicitly done in [2] by Bey, but he only stated that his results concerning the Boolean lattice stay valid in the context of $L_n(q)$. What is more important, our approach is different from his: our main tool in determining some inequalities concerning the profile vectors of intersecting families of subspaces is Theorem 2. This is a generalization of a theorem of Hsieh [10] which might be of independent interest.

To simplify our counting arguments we introduce the following

Notation. If $k + d \leq n$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q^{*(d)}$ denotes the number of k-dimensional subspaces of an n-dimensional vector space V over $GF(q)$ that are disjoint from a fixed d-dimensional subspace W of V.

Here are some basic facts about these numbers:

Facts.

$$\textbf{I.} \qquad \begin{bmatrix} n \\ k \end{bmatrix}^{*(d)} = \begin{bmatrix} n-d \\ k \end{bmatrix} q^{dk},$$

$$\textbf{II.} \qquad \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}} \leq \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(n-k)}} = \frac{1}{q^{n-k}} \leq \frac{1}{q^{k+1}} \qquad (\text{if } 2k+1 \leq n),$$

and so inductively for any $1 \leq p \leq k-1$

$$\textbf{III.} \qquad \frac{{n-p \brack k-p}^{*(d)}}{{n \brack k}^{*(d)}} \leq \frac{1}{q^{p(k+1)}} \qquad \text{(if } 2k+1 \leq n).$$

To determine the profile polytope of intersecting families we follow the so-called *method of inequalities*. Briefly it consists of the following steps:

- establish as many linear inequalities valid for the profile of any intersecting family as possible (each inequality determines a halfspace, therefore the profiles must lie in the intersection of all halfspaces determined by the inequalities),
- determine the extreme points of the polytope determined by the above halfspaces,
- for all of the above extreme points find an intersecting family having this extreme point as its profile vector.

The last step gives that the extreme points of the polytope determined by the halfspaces are the extreme points of the profile polytope that we are looking for.

The following theorem on intersecting families was first proved by Hsieh [10] (only for $n \geq 2k+1$) in 1977, then by Greene and Kleitman [9] (for the cases $k|n$ so especially if $n = 2k$) in 1978.

Theorem 2 *(Erdős - Ko - Rado for vector spaces, Hsieh's theorem) If $\mathcal{F} \subseteq {V \brack k}$ is an intersecting family of subspaces and $n \geq 2k$, then*

$$|\mathcal{F}| \leq {n-1 \brack k-1}.$$

The above theorem yields to the following inequalities concerning the profile vector of any intersecting family:

- $0 \leq f_i \leq {n-1 \brack i-1}, \quad 0 \leq i \leq n/2$
- $0 \leq f_i \leq {n \brack i}, \quad n/2 < i \leq n$

To establish more inequalities we will need the following statement:

Theorem 3 *The following generalization of Hsieh's theorem holds:*
(a) *if $2k \leq n$ and $d = 0$ or $d = n-k$*
or
(b) *if $n \geq 2k+1$ and $k+d \leq n$*

then for any intersecting family $\mathcal{F}$ of k-dimensional subspaces of an n-dimensional vector space V with all members disjoint from a fixed d-dimensional subspace U of V

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Note that the $d = 0$ case is just Hsieh's theorem.

Proof: If $k|d|n$ or $k|n$ and $d = 0$ then the argument of Greene and Kleitman [9] works. One can partition $V \setminus U$ into isomorphic copies of $V_k \setminus \{\underline{0}\}$, where V_k is a k-dimensional vector space over $GF(q)$. Since $\mathcal{F}$ cay contain at most one of the k-dimensional spaces of each partitioning set, the statement of the theorem follows.

So now we can assume $2k + 1 \leq n$. We follow the argument in [10]. First we verify the validity of the lemmas from [10] in our context. For $x \in V$ ($A \leqslant V$) let $\mathcal{F}_x$ ($\mathcal{F}_A$) denote the set of subspaces in $\mathcal{F}$ containing x (A).

Lemma A (the analogue of Lemma 4.2. in [10]) *Suppose $n \geq 2k + 1$ and let $\mathcal{F}$ be an intersecting family of k-subspaces of an n-dimensional space V such that all k-subspaces belonging to $\mathcal{F}$ are disjoint from a fixed d-dimensional subspace W of V (where $d \leq n - k$). If for all $x \in V$ we have $|\mathcal{F}_x| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}$, then*

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} \text{ or } |\mathcal{F}_A| \leq \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p-1}$$

for all 2-dimensional subspaces A, where $1 \leq p \leq k - 1$.

Proof: First we check the validity of the following consequence of the "facts":

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} > q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} \geq \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}, \tag{3}$$

for $1 \leq s \leq p$. Indeed,

$$\frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}}{\begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}} \geq \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}}{\begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(n-k)}} = q^{p(n-k)} > q^p \left(\frac{q^k - 1}{q - 1}\right)^p = q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^p,$$

where the first inequality is Fact III and the second one uses the assumption $n \geq 2k+1$.

Let us take an arbitrary 2-dimensional subspace $\langle x, y \rangle \subset V$. If $U \in \mathcal{F}$ implies $U \cap \langle x, y \rangle \neq \{\underline{0}\}$, then by (3) (and the assumption of the lemma) we have

$$|\mathcal{F}| \leq \sum_{Z \subset \langle x,y \rangle, Z 1-\dim} |\mathcal{F}_Z| \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}.$$

Thus we can suppose there is some $U_1 \in \mathcal{F}$ such that $U_1 \cap \langle x, y \rangle = \{\underline{0}\}$. Take $\underline{0} \neq z_1 \in U_1$. If $U \in \mathcal{F}$ implies $U \cap \langle x, y, z_1 \rangle \neq \{\underline{0}\}$, then (again using (3))

$$|\mathcal{F}| \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is some $U_2 \in \mathcal{F}$ such that $U_2 \cap \langle x, y, z_1 \rangle = \{\underline{0}\}$. Hence $|\mathcal{F}_{x,y,,z_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-4 \\ k-4 \end{bmatrix}^{*(d)}$, and so $|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-4 \\ k-4 \end{bmatrix}^{*(d)}$.

Suppose that for $1 \leq j \leq i, \underline{0} \neq z_j \in U_j$ and $\langle x, y, z_1, ..., z_j \rangle \cap U_{j+1} = \{\underline{0}\}$. Take $\underline{0} \neq z_{i+1} \in U_{i+1}$. If $U \in \mathcal{F}$ implies $U \cap \langle x, y, z_1, ..., z_{i+1} \rangle \neq \{\underline{0}\}$, then by (3)

$$|\mathcal{F}| \leq \begin{bmatrix} i+3 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is some $U_{i+2} \in \mathcal{F}$ such that $U_{i+2} \cap \langle x, y, z_1, ..., z_{i+1} \rangle = \{\underline{0}\}$. Hence we have

$$|\mathcal{F}_{x,y,Z_1,...,z_{i+1}}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-i-4 \\ k-i-4 \end{bmatrix}^{*(d)},$$

and by induction we obtain

$$|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ i \end{bmatrix}^{*(d)}.$$

Thus for $1 \leq i \leq p$, either we have $|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$ or $|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{i-1} \begin{bmatrix} n-1-i \\ k-1-i \end{bmatrix}^{*(d)}$, as a special case with $i = p$ we have $|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p-1} \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)}$. $\square$

We will need one more lemma from Hsieh's paper (actualized to our context):

Lemma B (the analogue of Lemma 4.3. in [10]) *Let $\mathcal{F}$ be a family of intersecting k-subspaces of an n-dimensional space V of which all subspaces are disjoint from a fixed d-dimensional subspace W of V. Furthermore if*
(a) *$q \geq 3$ and $n \geq 2k+1$ and for all x we have $|\mathcal{F}_x| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1}$,*
or if
(b) *$q = 2$ and*
- $n \geq 2k+1$
- and for all x we have $|\mathcal{F}_x| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{\min\{k-1, n-k-d\}} \prod_{i=1}^{k-1-(n-k-d)} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right)$ (if $k-1 \leq n-k-d$, then the product is empty and equals 1),
then

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Proof: In all cases $|\mathcal{F}|$ is at most $\begin{bmatrix} k \\ 1 \end{bmatrix}$ times the bound on $|\mathcal{F}_x|$.

Now if $q \geq 3$, then

$$|\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^k = \left(\frac{q^k-1}{q-1}\right)^k \leq q^{k^2-1} \leq q^{(k-1)(n-k)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)} \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

If $q = 2$, then for any $n \geq 2k+1$ and $d = n-k$ we have

$$|\mathcal{F}| \leq \prod_{i=1}^{k-1} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) < \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \right) < (q^k)^{k-1} q^{k-1} =$$

$$q^{k^2-1} \leq q^{(k-1)(n-k)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}.$$

Since $n \geq 2k+1$, we have $n-2k+1 \geq 2$ holds. This gives

$$|\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^k = \left(\frac{q^k-1}{q-1}\right)^k < q^{2(k-1)} \frac{(q^{2k-2}-1)(q^{2k-3}-1)...(q^k-1)}{(q^{k-1}-1)(q^{k-2}-1)...(q-1)} \leq$$

$$\leq q^{(k-1)(n-2k+1)} \begin{bmatrix} 2k-2 \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-2k+1)}.$$

This establishes the lemma for $0 \leq d \leq n-2k+1$. For the remaining cases ($n-2k+1 < d < n-k$) put $a_d = \begin{bmatrix} k \\ 1 \end{bmatrix}^{n-k-d+1} \prod_{i=1}^{k-1-(n-k-d)} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right)$, $b_d = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$. We have to prove that $\frac{a_d}{b_d} \leq 1$ holds for all $n-2k+1 < d < n-k$. To see this observe that

$$\frac{\frac{a_d}{b_d}}{\frac{a_{d+1}}{b_{d+1}}} = \frac{\begin{bmatrix} k \\ 1 \end{bmatrix}}{\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} d-(n-2k) \\ 1 \end{bmatrix}} \cdot \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d+1)}}{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}} = \frac{\begin{bmatrix} k \\ 1 \end{bmatrix}}{\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} d-(n-2k) \\ 1 \end{bmatrix}} \cdot \frac{\begin{bmatrix} n-2-d \\ k-1 \end{bmatrix} q^{(d+1)(k-1)}}{\begin{bmatrix} n-1-d \\ k-1 \end{bmatrix} q^{d(k-1)}} =$$

$$\frac{q^k-1}{q^k-q^{d-(n-2k)}} \cdot \frac{q^{n-k-d}-1}{q^{n-d-1}-1} q^{k-1} = \frac{q^{n+k-d-1}-q^{2k-1}-q^{n-d-1}+q^{k-1}}{q^{n+k-d-1}-q^{2k-1}-q^k+q^{d-(n-2k)}} \leq 1.$$

Thus the sequence $\frac{a_d}{b_d}$ is monotone increasing, and since $\frac{a_{n-k}}{b_{n-k}} \leq 1$ holds, so does $\frac{a_d}{b_d} \leq 1$ for all $n-2k+1 < d < n-k$.

This finishes the proof of the lemma. □

Before we get into the details of the proof of Theorem 3, we just collect its main ideas:

the heart of the proof is the concept of *covering number*. For a family of *subsets* $\mathcal{F} \subseteq 2^{[n]}$ this is the size of the smallest set $S \subseteq [n]$ that intersect all sets in $\mathcal{F}$ (S need not be in $\mathcal{F}$). For a family of subspaces $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ its covering number is the smallest number τ such that there is a τ-dimensional subspace U of V that intersects all subspaces that belong to $\mathcal{F}$. Observe that the proof of Lemma A was done by an induction on the covering number. The proof of Theorem 2 is again based on an induction on the covering number of $\mathcal{F}$. (During the proof, almost all computations will use the "facts" about Gaussian coefficients, all inequalities without any further remarks follow from them.)

If $x \in \cap\mathcal{F}$ for some $\underline{0} \neq x \in V$ then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$. Thus we can suppose that $\cap\mathcal{F} = \{\underline{0}\}$.

Let $x_1 \neq \underline{0}$ be such that $|\mathcal{F}_{x_1}| = \max_{x \in V} |\mathcal{F}_x|$.

By our assumption, there is some $A_1 \in \mathcal{F}$ not containing x_1. Thus $|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)}$.

Suppose that there are two independent vectors $z_1, z_2 \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z_i \rangle \neq \{\underline{0}\}$ for $i = 1, 2$. If $u_i \in \langle x_1, z_i \rangle \setminus \langle x_1 \rangle$, then the u_i's are independent. Thus

$$|\mathcal{F}| \leq |\mathcal{F}_{x_1}| + \sum_{U_i \subset (\langle x_1, z_i \rangle \setminus \langle x_1 \rangle) \cup \{\underline{0}\},\ dim(U_i)=1} |\mathcal{F}_{U_1, U_2}|$$

$$\leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \right)^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(*(d))}.$$

Thus we can suppose that there is at most one $z \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z \rangle \neq \{\underline{0}\}$. Suppose that $z \in A_1$ is such. Take $x \in A_1 \setminus \langle z \rangle$, then there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$ and hence $|\mathcal{F}_{x_1, x}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$. Thus

$$|\mathcal{F}_{x_1}| \leq |\mathcal{F}_{x_1, z}| + \sum_{X \subset (A_1 \setminus \langle z \rangle) \cup \{\underline{0}\},\ \ dim(X)=1} |\mathcal{F}_{x_1, X}| \leq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}.$$

But then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, z \rangle, dim(X)=1} |\mathcal{F}_X| \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*d} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)} \right) \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $x \in A_1$ there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$, and hence $|\mathcal{F}_{x_1, x}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$. Thus $|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$.

In general, suppose that for $1 \leq p \leq k-3$ we have non-zero vectors $y_1, y_2, ..., y_p \in V$ and $A_1, A_2, ..., A_{p+1} \in \mathcal{F}$ such that $y_i \in A$ and $A_{i+1} \cap \langle x_1, y_1, .., y_p \rangle = \{\underline{0}\}$ for $1 \leq i \leq p$. (We have just proved that either for any $y_1 \in A_1$ there exists such an $A_2 \in \mathcal{F}$ or the statement of the theorem holds.) Thus

$$|\mathcal{F}_{x_1, y_1, ..., y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so inductively we obtain that

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

By Lemma A, we have

$$|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

for all 2-dimensional $\langle x, y\rangle \subset V$.

Suppose that there are $p+2$ linearly independent vectors $z_1, z_2, ..., z_{p+2}$ in A_{p+2} such that $\langle x_1, y_1, ..., y_p, z_i\rangle \cap A \neq \{\underline{0}\}$ for all $A \in \mathcal{F}$ and $i = 1, 2, ..., p+2$. Let $u_i \in \langle x_1, y_1, ..., y_p, z_i\rangle \setminus \langle x_1, y_1, ..., y_p\rangle$, $i = 1, 2, ..., p+2$, then $u_1, u_2, ..., u_{p+2}$ are independent. Thus

$$|\mathcal{F}| \leq \sum_{X\subset\langle x_1,y_1,...,y_p\rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i\subset(\langle x_1,y_1,...,y_p,z_i\rangle\setminus\langle x_1,y_1,...,y_p\rangle)\cup\{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1,U_2,...,U_{p+2}}|$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^{p+2} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + q^{(p+1)(k-1)} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\begin{bmatrix} p+1 \\ 1 \end{bmatrix} + 1 \right) \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*d} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there are at most $p+1$ such z_i's. Hence

$$|\mathcal{F}_{x_1,y_1,...,y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that we do have independent vectors $z_1, z_2 \in A_{p+2}$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, y_1, ..., y_p, z_i\rangle \neq \{\underline{0}\}$ for $i = 1, 2$. Then

$$|\mathcal{F}| \leq \sum_{X\subset\langle x_1,y_1,...,y_p\rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i\subset(\langle x_1,y_1,...,y_p,z_i\rangle\setminus\langle x_1,y_1,...,y_p\rangle)\cup\{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1,U_2}|$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right) +$$

$$+ \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$= \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix}^2 + q^{2(p+1)} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \right) \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+1 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is at most one such z. Hence

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that $z_1 \in A_{p+1}$ is such a z, then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, ..., y_p, z_1 \rangle, dim(x)=1} |\mathcal{F}_X| \leq \begin{bmatrix} p+2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right)$$

$$< \begin{bmatrix} p+2 \\ 1 \end{bmatrix} (\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \frac{1}{q} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+2 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q^{p+2}} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $z \in A_{p+1}$, there is some $A \in \mathcal{F}$ such $A \cap \langle x_1, y_1, ..., y_p, z \rangle = \{\underline{0}\}$. Take $y_{p+1} \in A_{p+1}$, and let A_{p+2} be such that $A \cap \langle x_1, y_1, ..., y_p, y_{p+1} \rangle = \{\underline{0}\}$.

We obtained, that either the statement of the theorem holds, or there are linearly independent vectors $x_1, y_1, ..., y_{k-1}$ and $A_i \in \mathcal{F}$ $i = 1, ...k-1$ such that $y_i \in A_i$ and $\langle x_1, y_1, ...y_{i-1} \rangle \cap A_i = \{\underline{0}\}$. Furthermore we can suppose that y_i maximizes $|\mathcal{F}_{x_1, y_1, ..., y_{i-1}, z}|$ for $z \in A_i$.

If $q \geq 3$, this means that either $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$ or $|\mathcal{F}_x| \leq |\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1}$ and then we are done by Lemma B.

If $q = 2$, we have to sharpen our estimates on $|\mathcal{F}_{x_1}|$. We know that for j independent vectors $x_1, y_1, ..., y_{j-1}$ with $U \cap \langle x_1, y_1, ..., y_{j-1} \rangle = \{\underline{0}\}$ there exists a subspace $A_j \in \mathcal{F}$ such that $A_j \cap \langle x_1, y_1 ..., x_{j-1} \rangle = \{\underline{0}\}$. Then we would have $|\mathcal{F}_{x_1, y_1, ..., y_{j-1}}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}^{*(d)}$. (Note that $U \cap \langle x_1, y_1, ..., y_{j-1} \rangle = \{\underline{0}\}$ must hold, as otherwise any subspace containing $x_1, y_1, ..., y_{j-1}$ would intersect U nontrivially, therefore $\mathcal{F}_{x_1, y_1, ..., y_{j-1}}$ would be empty, and thus, by the maximality assumption on the choice of y_{i-1}, $\mathcal{F}$ would be empty.) Suppose further that for some positive l we have $j+k+d = n+l$. Then $\dim(\langle x_1, y_1 ..., y_{j-1}, A_j \rangle \cap U) \geq l$ and so (denoting $\langle x_1, y_1 ..., y_{j-1}, A_j \rangle \cap U$ by U_j) $\dim(\langle x_1, y_1 ..., y_{j-1}, U_j \rangle \cap A_j) \geq l$ as well, therefore when choosing among the vectors of A_j a subspace of dimension at least l is forbidden. Therefore we have the following better estimate on the number of subspaces in $\mathcal{F}$ containing $x_1, y_1, ..., y_{j-1}$:

$$\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l \\ 1 \end{bmatrix} \right) \begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}^{*(d)}.$$

Hence we have that either the statement of the theorem holds or the degree of any vector x is bounded by the expression given in the conditions of Lemma B. So Lemma B establishes our theorem in this case, too. $\Box$

Corollary. *For the profile vector f of any family $\mathcal{F}$ of intersecting subspaces of an n-dimensional vector space V, and for any $k < n/2$ and $n/2 < d \leq n-k$, the following holds*

$$c_{k,d}f_k + f_d \leq \begin{bmatrix} n \\ d \end{bmatrix},$$

where $c_{k,d} = q^d \frac{\begin{bmatrix} n-k \\ d \end{bmatrix}}{\begin{bmatrix} n-d-1 \\ k-1 \end{bmatrix}}$, and equality holds in case of $f_k = 0, f_d = \begin{bmatrix} n \\ d \end{bmatrix}$ or $f_k = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, f_d = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}$.

Proof: Let us doublecount the disjoint pairs formed by the elements of $\mathcal{F}_k = \{U \in \mathcal{F} : \dim U = k\}$ and $\mathcal{F}'_d = \begin{bmatrix} V \\ d \end{bmatrix} \setminus \mathcal{F}_d = \{U \leqslant V, U \notin \mathcal{F} : \dim U = d\}$. On the one hand, for each $U \in \mathcal{F}_k$ there are exactly $q^{dk}\begin{bmatrix} n-k \\ d \end{bmatrix}$ such pairs (this uses the first *fact* about q-nomial coefficients), while on the other hand by Theorem 3 we know, that for any $W \in \mathcal{F}'_d$ there are at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} = q^{d(k-1)}\begin{bmatrix} n-d-1 \\ k-1 \end{bmatrix}$ such pairs. This proves the required inequality and it is easy to see that equality holds in the cases stated in the Corollary. $\Box$

Having established these inequalities, we are able to prove our main theorem.

Proof of Theorem 1: First of all, for any $x \in V$, for the families $\mathcal{G}_i = \{U : x \in U, i \leq \dim U \leq n-i\} \cup \{U : \dim U > n-i\}$ $(1 \leq i \leq n/2)$ $f(\mathcal{G}_i) = v_i$ holds, and if n is odd then the profile of the family $\mathcal{G}^+ = \{U : \dim U > n/2\}$ is v^+, and clearly none of these vectors can be dominated by any convex combination of the others.

We want to dominate the profile vector f of any fixed intersecting family $\mathcal{F}$ with a convex combination of the vectors v_j (and possibly v^+ if n is odd). We define the coefficients of the v_js recursively. Let i denote the index of the smallest non-zero coordinate of f. For all $j < i$ let $\alpha_j = 0$. Now if for all $j' < j$ $\alpha_{j'}$ has already been defined, let

$$\alpha_j = \max\left\{\frac{f_j}{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}} - \sum_{j'=i}^{j-1}\alpha_{j'}, 0\right\}.$$

Note, that for all j $(i \leq j \leq n/2)$ the jth coordinate of $\sum_{j'=i}^{j}\alpha_{j'}v_{j'}$ is at least f_j (and equality holds if when choosing α_j, the first expression is taken as maximum), so these vectors already dominates the “first part” of f.

When all α_js $(i \leq j \leq n/2)$ are defined, then let $\alpha^+ = 1 - \sum_{j'=i}^{n/2}\alpha_{j'}$ and let α^+ be the coefficient of v^+ if n is odd or add α^+ to the coefficient of $v_{n/2}$ if n is even. Note also that α^+ is non-negative since for all $i \leq j \leq k \leq n/2$ $(v_j)_k = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ and by Hsieh's theorem $0 \leq f_k \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Therefore this is really a convex combination of the v_js.

The easy observation that this convex combination dominates f in the coordinates larger then $n-i$ follows from the fact that all v_js (and v^+ as well) have $\begin{bmatrix} n \\ d \end{bmatrix}$ in the dth coordinate, therefore so does the convex combination which is clearly an upper bound for f_d.

All what remains is to prove the domination in the dth coordinates for all $n/2 < d \leq n-i$, that is to prove the inequality

$$f_d \leq \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j + \begin{bmatrix} n \\ d \end{bmatrix} (1 - \sum_{j=i}^{n-d} \alpha_j).$$

Let $k \leq n-d$ be the largest index with $\alpha_k > 0$. Then we have

$$f_d \leq \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} f_k = \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j = (1 - \sum_{j=i}^{k} \alpha_j) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j$$

$$= (1 - \sum_{j=i}^{n-d} \alpha_j) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j$$

where the inequality is just the Corollary, the first equality follows from the fact that $\alpha_k > 0$, the second equality uses again the Corollary (the statement about when equality holds) and the last equality uses the defining property of k (for all $k < j \leq n-d$ $\alpha_j = 0$).

This proves the theorem. □

Note that, the (essential) extreme points are 'the same' as in the Boolean case, one just has to change the binomial coefficients to the corresponding q-nomial coefficients and the structure of the extremal families are really the same.

3 Concluding remarks

The authors of this paper in [8] introduced a generalization of the notion of profile vector, the socalled *l-chain profile vector*, where the coordinates are indexed by $j_1, j_2, ..., j_l$ $(0 \leq j_1 < j_2 < ... < j_l \leq \text{rank } (P))$ and count the number of chains of length l in the family where the ith element of the chain should have rank j_i for all $1 \leq i \leq l$ (so the l-chain profile vector of a family has $\binom{n+1}{l}$ coordinates in the Boolean poset and in $L_n(q)$ as well).As the set of intersecting family is upward closed (i.e. if $\mathcal{F}$ is an intersecting family of subspaces of V, then so is $U(\mathcal{F}) = \{W \leqslant V : \exists U \in \mathcal{F} \ \ (U \leqslant V)\}$), one can obtain the essential extreme points of the l-chain profile polytope for any l as described in [8].

The profile polytope of t-intersecting families has not yet been determined neither in the Boolean case nor in the poset of subspaces, but in both cases we know how large can be the ith coordinate of the profile for all $0 \leq i \leq n$.

Theorem 4 (Frankl - Wilson [7]) *If $\mathcal{U} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is a t-intersecting family and $n \geq 2k-t$, then*

$$|\mathcal{U}| \leq \max\{\begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix}\}.$$

The corresponding extremal families are
i, $\mathcal{U}_0 = \{U \in \begin{bmatrix} V \\ k \end{bmatrix} : T \subseteq U\}$ where T is a fixed t-dimensional subspace of V,
ii, $\mathcal{U}_1 = \begin{bmatrix} W \\ k \end{bmatrix}$ where W is a fixed $2k-t$-dimensional subspace of V.

Theorem 5 (Ahlswede - Khachatrian [1]) *If $1 \leq t \leq k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a t-intersecting family, then*

$$|\mathcal{F}| \leq \max_{0 \leq r \leq \frac{n-t}{2}} |\mathcal{F}_r|,$$

where $\mathcal{F}_r = \{F \in \binom{[n]}{k} : |F \cap [1, t+2r]| \geq t+i\}$ for $0 \leq r \leq \frac{n-t}{2}$.

These two theorems show that in the case of subspaces the extremal family is always one of two candidates, while in the Boolean case (as n goes to infinity) there are arbitrary many candidates (in fact for all r Theorem 5 in its full strength gives the range of k where $\mathcal{F}_r$ is the extremal family). Therefore one may suspect that it can be much easier to determine the profile polytope in the lattice of subspaces, than determining it in the Boolean case.

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