Induced and non-induced forbidden subposet problems

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Abstract

A poset Q contains another poset P if there is an injection $i: P \to Q$ such that for every $p_1, p_2 \in P$ the fact $p_1 \leq p_2$ implies $i(p_1) \leq i(p_2)$. A P-free poset is one that does not contain P. We say that Q contains an induced copy of P if for the injection above $p_1 \leq p_2$ holds if and only if $i(p_1) \leq i(p_2)$. Q is induced P-free if it does not contain an induced copy of P. The problem of determining the maximum size $La(n, P)$ that a P-free subposet of the Boolean lattice B_n can have, attracted the attention of many researchers, but little is known about the induced version of these problems. In this paper we determine the asymptotic behavior of $La^*(n, P)$, the maximum size that an induced P-free subposet of the Boolean lattice B_n can have for the case when P is the the complete two-level poset $K_{r,s}$ or the complete multi-level poset $K_{r,s_1,\dots,s_j,t}$ when all s_i's either equal 4 or are large enough and satisfy an extra condition. We also show lower and upper bounds for the non-induced problem in the case when P is the complete three-level poset $K_{r,s,t}$. These bounds determine the asymptotics of $La(n, K_{r,s,t})$ for some values of s independently of the values of r and t.

1 Introduction

We use standard notation: 2^X denotes the power set of X, $\binom{X}{k}$ denotes the set of k-element subsets of X and for two sets $A \subset B$ the interval $\{G : A \subseteq G \subseteq B\}$ is denoted by $[A, B]$.

The very first theorem in extremal finite set theory is due to Sperner [14] and it states that if $\mathcal{F} \subseteq 2^{[n]}$ is a family of sets that does not contain two sets F_1, F_2 with $F_1 \subsetneq F_2$, then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ holds. Such families are called *antichains* or *Sperner families*. A first generalization is due to Erdős [6], who proved that if $\mathcal{F}$ does not contain any $(k+1)$-chains, i.e.

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$k+1$ sets $F_1, F_2, \dots, F_{k+1}$ with $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k+1}$, then $|\mathcal{F}| \le \Sigma(n,k) := \sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}$ holds. Such families are called *k-Sperner families*.

These two theorems have many applications and generalizations. One such generalization is the topic of forbidden subposet problems first introduced by Katona and Tarján [11]. We say that a poset Q *contains* another poset P if there is an injection $i: P \to Q$ such that for every $p_1, p_2 \in P$ the fact $p_1 \le p_2$ implies $i(p_1) \le i(p_2)$. If Q does not contain P, then it is said to be *P-free*. If $\mathcal{P}$ is a set of posets, then Q is $\mathcal{P}$-free if it is P-free for all $P \in \mathcal{P}$. The parameter introduced by Katona and Tarján is the quantity $La(n,P)$ that denotes the maximum size of a P-free subposet of B_n, the Boolean poset of all subsets of $[n]$ ordered by inclusion. With this notation Erdős's theorem states that $La(n, P_{k+1}) = \Sigma(n,k)$, where P_{k+1} denotes the path on $k+1$ elements, i.e. a total ordering on $k+1$ elements.

In the same paper, Katona and Tarján introduced the induced version of the problem. We say that Q *contains an induced copy of* P if there is an injection $i: P \to Q$ such that for any $p_1, p_2 \in P$ we have $p_1 \le p_2$ if and only if $i(p_1) \le i(p_2)$. If Q does not contain an induced copy of P, then Q is said to be *induced P-free.* The analogous extremal number is denoted by $La^*(n,P)$ and obviously the inequality $La(n,P) \le La^*(n,P)$ holds for any poset P. The notation for multiple forbidden subposets is $La(n,\mathcal{P})$ and $La^*(n,\mathcal{P})$.

As any poset P is contained by $P_{|P|}$, we clearly have $La(n,P) \le La(n, P_{|P|}) = \Sigma(n, |P|-1)$. Strengthenings of this general bound were obtained by Burcsi and Nagy [2], Chen and Li [4] and recently by Grósz, Methuku and Tompkins [10]. Therefore it is natural to compare $La(n,P)$ to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Unfortunately, it is not known whether $\pi(P) = \lim_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists. The following conjecture was first stated in [9].

Conjecture 1.1. *For any poset P let $e(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^{k} \binom{[n]}{j+i}$ is P-free. Then $\pi(P)$ exists and is equal to $e(P)$.*

This conjecture has been verified for many classes of posets. The most remarkable result is due to Bukh.

Theorem 1.2. *Let T be a tree poset. Then $\Sigma(n, h(T)-1) \le La(n,T) \le (h(T)-1+O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ holds.*

Much less is known about the induced version of the problem. It has only been proved recently by Methuku and Pálvölgyi [13] that for every poset P there exists a constant c_P such that $La^*(n,P) \le c_P \binom{n}{\lfloor \frac{n}{2} \rfloor}$ holds. (For a special class of posets this has already been established by Lu and Milans [12].) As the list of known results on forbidden induced subposet problems is very short here we enumerate all such theorems.

Theorem 1.3 (Katona, Tarján [11])**.** *For $n \ge 3$ we have $La(n, \{\wedge, \vee\}) = La^*(n, \{\wedge, \vee\}) = 2\binom{n-1}{\lfloor n/2 \rfloor}$.*

Theorem 1.4 (Katona, Tarján [11] and Carroll, Katona [3]). $(1+\frac{1}{n}+O(\frac{1}{n^2}))\binom{n}{\lfloor n/2\rfloor} \leq La(n,\vee) = La(n,\wedge) \leq La^*(n,\vee) = La^*(n,\wedge) \leq (1+\frac{2}{n}+O(\frac{1}{n^2}))\binom{n}{\lfloor n/2\rfloor}$.

Finally, the induced version of Theorem 1.2 has been proved, but only with an $o(1)$ error term instead of $O(\frac{1}{n})$.

Theorem 1.5 (Boehnlein, Jiang [1]). *Let T be a tree poset. Then $\Sigma(n,h(T)-1) \leq La^*(n,T) \leq (h(T)-1+o(1))\binom{n}{\lfloor\frac{n}{2}\rfloor}$ holds.*

Before we state our results, let us formulate the induced analogue of Conjecture 1.1.

Conjecture 1.6. *Let P be a poset and let $e^*(P)$ denote the largest integer k such that for any j and n the family $\cup_{i=1}^k \binom{[n]}{j+i}$ is induced P-free. Then $\pi^*(P) = \lim_{n\to\infty}\frac{La^*(n,P)}{\binom{n}{\lfloor\frac{n}{2}\rfloor}}$ exists and is equal to $e^*(P)$.*

In the present paper, we address both the induced and the non-induced problem for complete multi-level posets. Let $K_{r_1,r_2,\dots,r_s}$ denote the poset on $\sum_{i=1}^s r_i$ elements $a_1^1, a_2^1, \dots, a_{r_1}^1$, $a_1^2, a_2^2, \dots, a_{r_2}^2, \dots, a_1^s, a_2^s, \dots, a_{r_s}^s$ with $a_\alpha^i < a_\beta^j$ if and only if $i<j$. Our first result gives not only the asymptotics of $La^*(n,K_{r,s})$, but also the order of magnitude of the second order term of the extremal value.

Theorem 1.7. *For any positive integers $2 \leq r,s$ we have $\Sigma(n,2)+(\frac{r+s-2}{n}-O_{r,t}(\frac{1}{n^2}))\binom{n}{\lfloor n/2\rfloor} \leq La^*(n,K_{r,s}) \leq (2+\frac{2(r+s-2)}{n}+O_{r,s}(\frac{1}{n^2}))\binom{n}{\lfloor n/2\rfloor}$.*

Note that the same upper bound for $La(n,K_{r,s})$ follows from Theorem 1.2 as $K_{r,s}$ is an (induced) subposet of $K_{r,1,s}$ and $K_{r,1,s}$ is a tree poset. By the same argument, Theorem 1.5 implies the asymptotics of $La^*(n,K_{r,s})$ but its error term is worse than that of Theorem 1.7. Let us remark that $La(n,K_{2,2}) = \Sigma(n,2)$ was shown by De Bonis, Katona, Swanepoel, [5]. As they also showed the uniqueness of the extremal family, it was known that the strict inequality $La(n,K_{2,2}) < La^*(n,K_{2,2})$ holds. Theorem 1.7 tells us the order of magnitude of the gap between these two parameters.

Then we turn our attention to the three level case of $K_{r,s,t}$. To do so we need to introduce the following notation: for positive integers r,t let

$$f(r,t) = \begin{cases} 0 & \text{if } r=t=1, \\ 1 & \text{if } r=1, t>1 \text{ or } r>1, t=1, \\ 2 & \text{if } r,t>2. \end{cases}$$

Also, for any integer $s \geq 2$ let us define $m = m_s = \lceil \log_2(s - f(r,t) + 2) \rceil$ and $m' = m'_s = \min\{m : s \leq \binom{m}{\lceil m/2 \rceil}\}$ and for any real number z, let z^+ denote $\max\{0, z\}$.

Theorem 1.8. *Let $s - f(r,t) \geq 2$.*

(1) If $s - f(r,t) \in [2^{m-1} - 1, 2^m - \binom{m}{\lceil \frac{m}{2} \rceil} - 1]$, then $\Sigma(n, m + f(r,t)) + \left(\frac{(r-2)^+ + (t-2)^+}{n} - O_{r,s,t}(\frac{1}{n^2})\right)\binom{n}{\lceil \frac{n}{2} \rceil} \leq La(n, K_{r,s,t}) \leq (m + f(r,t) + \frac{2(r+s-2)}{n} + O_{r,s,t}(\frac{1}{n^2}))\binom{n}{\lceil \frac{n}{2} \rceil}$. Hence, $\pi(K_{r,s,t}) = e(K_{r,s,t}) = m + f(r,t)$.

(2) If $s - f(r,t) \in [2^m - \binom{m}{\lceil \frac{m}{2} \rceil}, 2^m - 2]$, then $\Sigma(n, m + f(r,t)) + (\frac{(r-2)^+ + (t-2)^+}{n} - O_{r,s,t}(\frac{1}{n^2}))\binom{n}{\lceil \frac{n}{2} \rceil} \leq La(n, K_{r,s,t}) \leq (m + f(r,t) + 1 - \frac{2^m - s + f(r,t) - 1}{\binom{m}{\lceil \frac{m}{2} \rceil}})\binom{n}{\lceil \frac{n}{2} \rceil}$ holds.

Note that the special case $r = t = 1$ of Theorem 1.8 was already obtained by Griggs, Li and Lu [8]. Let us state a result that covers the case $s = 2$, $f(r,t) > 0$.

Theorem 1.9. *For any pair of integers r, t with $f(r,t) > 0$ we have $\Sigma(n,3) + (\frac{(r-2)^+ + (t-2)^+}{n} - O_{r,t}(\frac{1}{n^2}))\binom{n}{\lceil \frac{n}{2} \rceil} \leq La(n, K_{r,2,t}) \leq (3 + \frac{2(r+s-2)}{n} + O_{r,s,t}(\frac{1}{n^2}))\binom{n}{\lceil \frac{n}{2} \rceil}$. In particular, $\pi(K_{r,2,t}) = 3$ holds.*

It is easy to verify that three consecutive levels in B_n form an unextendable family of $K_{1,2,2}$-free and $K_{2,2,2}$-free family of sets, but from our proofs it does not follow that they are of largest possible size. However we formulate the following conjecture.

Conjecture 1.10. *If n is large enough, then $La(n, K_{1,2,2}) = La(n, K_{2,2,1}) = La(n, K_{2,2,2}) = \Sigma(n,3)$ holds.*

Then we turn our attention to the general case of $K_{r,s_1,s_2,\ldots,s_j,t}$. As there are more technical details in calculating $e(K_{r,s_1,s_2,\ldots,s_j,t})$ than in calculating $e^*(K_{r,s_1,s_2,\ldots,s_j,t})$ we will only consider the induced problem in its full generality.

Proposition 1.11. *(1) If $s_i \geq 2$ holds for all $1 \leq i \leq j$, then we have $e^*(K_{r,s_1,s_2,\ldots,s_j,t}) = f(r,t) + \sum_{i=1}^{j} m'_{s_i}$.*

(2) Let us write $w = |\{i : s_{i-1} = s_i = 1\}|$. Then $e^*(K_{r,s_1,s_2,\ldots,s_j,t}) = w + e^*(K_{r,\sigma_1,\sigma_2,\ldots,\sigma_{j'},t})$, where $\sigma_1, \sigma_2, \ldots, \sigma_{j'}$ is the sequence obtained from $s_1, s_2, \ldots, s_j$ by removing all its ones.

Proof. To see (i) let $\mathcal{F}$ consist of $f(r,t) + \sum_{i=1}^{j} m'_{s_i}$ consecutive levels of $2^{[n]}$ and suppose we find an induced copy of $K_{r,s_1,s_2,\ldots,s_j,t}$. If $F_1, \ldots, F_r$ and $F'_1, \ldots, F'_t$ play the role of the bottom

r and the top t sets, then $|\cap_{i=1}^t F'_i| - |\cup_{k=1}^r F_j| < \sum_{l=1}^j m'_{s_i}$ holds. If $F_1^{j'}, \dots, F_{s_{j'}}^{j'}$ play the role of the sets of the j'th middle level of $K_{r,s_1,s_2,\dots,s_j,t}$, then their union has size at least $s_{j'}$ more than the union of the sets on the $(j'-1)$st level. Thus one would need $\sum_{i=1}^j m'_{s_i}$ more levels for the j middle levels of $K_{r,s_1,s_2,\dots,s_j,t}$. It is easy to see that $f(r,t) + \sum_{i=1}^j m'_{s_i} + 1$ consecutive levels do contain an induced copy of $f(r,t) + \sum_{i=1}^j m'_{s_i}$.

To see (ii) let us observe that if $s_{j'} = 1$ and $s_{j'-1}, s_{j'+1} > 1$, then the union U of the sets $F_1^{j'-1}, \dots, F_{s_{j'-1}}^{j'-1}$ on the $(j'-1)$st level strictly contains $F_1^{j'-1}, \dots, F_{s_{j'-1}}^{j'-1}$, and the intersection I of the sets $F_1^{j'+1}, \dots, F_{s_{j'+1}}^{j'+1}$ on the $(j'+1)$st level is strictly contained $F_1^{j'+1}, \dots, F_{s_{j'+1}}^{j'+1}$ and also $U \subset I$. Thus even in the 'most economic' $U = I$ case U can play the role of the set on the j'th level. If $s_{i-1} = s_i = 1$, then the set representing level i of $K_{r,s_1,s_2,\dots,s_j,t}$ requires a new level. □

Theorem 1.12. *(i) For any positive integers $1 \le r, t$ we have $\Sigma(n, 4 + f(r,t)) + (\frac{r+t-2}{n} - O_{r,t}(\frac{1}{n^2}))\binom{n}{\lceil n/2 \rceil} \le La^*(n, K_{r,4,t}) = (4 + f(r,t) + \frac{2(r+t-2)}{n} + O_{r,t}(\frac{1}{n^2}))\binom{n}{\lfloor n/2 \rfloor}$. In particular, $\pi^*(K_{r,4,t}) = 4 + f(r,t)$ holds.*

(ii) For any constant c with $1/2 < c < 1$ there exists an integer s_c such that if $s \ge s_c$ and $s \le c\binom{m'}{\lceil m'/2 \rceil}$, then we have $\Sigma(n, m' + f(r,t)) + (\frac{r+t-2}{n} - O_{r,t}(\frac{1}{n^2}))\binom{n}{\lceil n/2 \rceil} \le La^*(n, K_{r,s,t}) \le (m' + f(r,t) + \frac{2(r+t-2)}{n} + O_{r,t}(\frac{1}{n^2}))\binom{n}{\lfloor n/2 \rfloor}$. In particular, $\pi^*(K_{r,s,t}) = m' + f(r,t)$ holds.

(iii) There exists an integer s_0 such that for any r, s, t with $s \ge s_0$ we have $\Sigma(n, m' + f(r,t)) + (\frac{r+t-2}{n} - O_{r,t}(\frac{1}{n^2}))\binom{n}{\lceil n/2 \rceil} \le La^*(n, K_{r,s,t}) \le (m' + 1 + f(r,t) + \frac{2(r+t-2)}{n} + O_{r,t}(\frac{1}{n^2}))\binom{n}{\lfloor n/2 \rfloor}$.

(iv) For any constant c with $1/2 < c < 1$ there exists an integer s_c such that if all s_i's satisfy that either $s_i = 4$ or $s_i \ge s_c$ and $s \le c\binom{m'}{\lceil m'/2 \rceil}$, then we have $La^*(n, K_{r,s_1,s_2,\dots,s_j,t}) = (e^*(K_{r,s_1,s_2,\dots,s_j,t}) + O_{r,t}(\frac{1}{n}))\binom{n}{\lfloor n/2 \rfloor}$.

Our main technique to prove all four theorems is the chain partition method [8, 7]. The remainder of the paper is organized as follows: in Section 2 we prove some preliminary lemmas that will be used in the proofs of Theorem 1.7, Theorem 1.8, Theorem 1.9, and Theorem 1.12. Then in Section 3 we prove our results.

2 Preliminary lemmas

Let $\mathbf{C}_n$ denote the set of maximal chains in $[n]$. For a family $\mathcal{F} \subseteq 2^{[n]}$ of sets and $A \subseteq [n]$ we define $s^-_{\mathcal{F}}(A)$ to be the maximum size of an antichain in $\mathcal{F} \cap 2^A$ and $s^+_{\mathcal{F}}(A)$ to be the maximum size of an antichain in $\{F \in \mathcal{F} : A \subseteq F\}$.

Lemma 2.1. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a family such that all $F \in \mathcal{F}$ have size in $[n/2 - n^{2/3}, n/2 + n^{2/3}]$. Let $A \subset [n]$ with $s^-_{\mathcal{F}}(A) < k$. Then the number of pairs $(F, \mathcal{C})$ where $\mathcal{C}$ is a maximal chain from $\emptyset$ to A and $F \in \mathcal{F} \cap (\mathcal{C} \setminus \{A\})$ is $\frac{2(k-1)}{n}|A|! + O(\frac{1}{n^2}|A|!)$.*

Proof. The property possessed by A and $\mathcal{F}$ ensures that $\mathcal{F}_A := \{F \in \mathcal{F} : F \subset A\}$ contains at most $k-1$ sets of each possible size. Thus the number of pairs $(F, \mathcal{C})$ in question is at most

$$\sum_{i=n/2-n^{2/3}}^{\min\{n/2+n^{2/3},|A|-1\}(k-1)} i!(|A|-i)! \leq \frac{k-1}{|A|}|A|! + \frac{2(k-1)}{|A|(|A|-1)}|A|! + \frac{12(k-1)n^{2/3}}{|A|(|A|-1)(|A|-2)}|A|!$$

$$\leq \frac{2(k-1)}{n}|A|! + O_k(\frac{1}{n^2}|A|!)$$

if n is large enough and $|A| \geq (1/2 + o(1))n$. If $|A| \leq (1/2-\varepsilon)n$, then $\mathcal{F}$ does not contain any subset F of A. □

Corollary 2.2. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a family such that all $F \in \mathcal{F}$ have size in $[n/2 - n^{2/3}, n/2 + n^{2/3}]$. Let $A \subset [n]$ with $s^-_{\mathcal{F}}(A) \geq k$ and let $\mathbf{C}_{k,A}$ denote the set of those maximal chains $\mathcal{C}$ from $\emptyset$ to A for which every $C \in \mathcal{C} \setminus \{A\}$ we have $s^-_{\mathcal{F}}(C) < k$. Then the number of pairs $(F, \mathcal{C})$ where $\mathcal{C}$ is a maximal chain from $\emptyset$ to A and $F \in \mathcal{F} \cap (\mathcal{C} \setminus \{A\})$ is $(1 + \frac{2(k-1)}{n})|\mathbf{C}_{k,A}| + O_k(\frac{1}{n^2}|\mathbf{C}_{k,A}|)$.*

Proof. Let $A_1, \ldots, A_j, A_{j+1}, \ldots, A_{|A|}$ denote the subsets of A of size $|A|-1$ such that $s^-_{\mathcal{F}}(A_i) < k$ if and only if $1 \leq i \leq j$. (If $s^-_{\mathcal{F}}(A) \geq k$ for all i, then $\mathbf{C}_{k,A}$ is empty and there is nothing to prove.) Note that if $S_1 \subset S_2$, then $s^-_{\mathcal{F}}(S_2) < k$ implies $s^-_{\mathcal{F}}(S_1) < k$. Therefore $\mathbf{C}_{k,A} = \cup_{i=1}^{j} \mathbf{C}_{A_i,A}$, where $\mathbf{C}_{A_i,A}$ denotes the set of those maximal chains from $\emptyset$ to A that contain both A_i and A. Indeed, $\mathbf{C}_{A_i,A} \subset \mathbf{C}_{\mathcal{F},A}$ for $1 \leq i \leq j$ as by the above A is the first set in a chain $\mathcal{C} \in \mathbf{C}_{A_i,A}$ with $s^-_{\mathcal{F}}(A)$ at least k, while for all $i \geq j+1$ we have $s^-_{\mathcal{F}}(A_j) \geq k$ and thus $\mathbf{C}_{A_j,A} \cap \mathbf{C}_{k,A} = \emptyset$.

Let us fix i with $1 \leq i \leq j$ and consider pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_{A_i,A}$. As $s^-_{\mathcal{F}}(A_i) < k$, we can apply Lemma 2.1 to $\mathcal{F}$ and A_i, and obtain that the number of such pairs with $F \subsetneq A_i$ is at most $\frac{2}{n}|A_i|! + O_k(\frac{1}{n^2}|A_i|)$. Even if all A_i's belong to $\mathcal{F}$, then every chain $\mathcal{C} \in \mathbf{C}_{k,A}$ can contain one more set from $\mathcal{F}$, namely one of the A_i's. This completes the proof. □

Lemma 2.3. *(i) Let $\mathcal{G} \subseteq 2^{[k]}$ be a family of sets such that any antichain $\mathcal{A} \subset \mathcal{G}$ has size at most 3. Then the number of pairs $(G, \mathcal{C})$ with $G \in \mathcal{G} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_k$ is at most $4k!$.*

(ii) For any constant c with $1/2 < c < 1$ there exists an integer s_c such that if $s \geq s_c$ and $s \leq c\binom{m'}{\lceil m'/2 \rceil}$, then the following holds: if $\mathcal{G} \subseteq 2^{[k]}$ is a family of sets such that any antichain $\mathcal{A} \subset \mathcal{G}$ has size less than s, then the number of pairs $(G, \mathcal{C})$ with $G \in \mathcal{G} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_k$ is at most $m'k!$.

(iii) There exists an integer s_0 such that if $s \geq s_0$ and $\mathcal{G} \subseteq 2^{[k]}$ is a family of sets such that any antichain $\mathcal{A} \subset \mathcal{G}$ has size at most s, then the number of pairs $(G, \mathcal{C})$ with $G \in \mathcal{G} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_k$ is at most $(m'+1)k!$.

Proof. First we prove (i). We may assume that $\emptyset, [k] \in \mathcal{G}$ holds as adding them will not result in violating the condition of the lemma and the number of pairs to be counted can only increase. These two sets are in $k!$ maximal chains each, thus giving $2k!$ pairs. All other sets belong to $|G|!(k-|G|)! = \frac{k!}{\binom{k}{|G|}}$ chains in $\mathbf{C}_k$. Sets of same size form an antichain, therefore for every $1 \le i \le k-1$ there exist at most 3 sets of size i in $\mathcal{G}$ and thus the total number of pairs $(G, \mathcal{C})$ is at most

$$S(k) = 2k! + 3k! \sum_{i=1}^{k-1} \frac{1}{\binom{k}{i}}.$$

For $k = 3, 4, 5$ the sum $S(k)$ equals $4k!, 4k!, 3.8m!$, respectively (for $k = 1, 2$ the number of pairs counted is $2k!$ and $3k!$, respectively). Furthermore, if k is at least 5, then $\frac{1}{\binom{k}{i}} \ge \frac{1}{\binom{k+1}{i}}$ holds for all i and also the inequality

$$\begin{aligned} \frac{1}{\binom{k}{k-2}} + \frac{1}{\binom{k}{k-1}} &= \frac{2}{k(k-1)} + \frac{1}{k} \\ &\ge \frac{6}{(k+1)k(k-1)} + \frac{2}{(k+1)k} + \frac{1}{k+1} \\ &= \frac{1}{\binom{k+1}{k-2}} + \frac{1}{\binom{k+1}{k-1}} + \frac{1}{\binom{k+1}{k}} \end{aligned}$$

is valid. Thus, $\frac{S(k)}{k!}$ is monotone decreasing for $k \ge 5$ and therefore $\frac{S(k)}{k!} \le 4$ holds for all positive integer k. This completes the proof of (i).

Now we prove (ii). Clearly, as long as $k < m'$ we can have $\mathcal{G} = 2^{[k]}$ and then the number of pairs is $(k+1)k! \le m'k!$. When $k \ge m'$ we again use the observation that for any $0 \le j \le k$ we have $|\{G \in \mathcal{G} \cap \binom{[k]}{j})| < s$ and thus the number of pairs $(G, \mathcal{C})$ is at most $S(k) = \sum_{j=0}^{k} \min\{s-1, \binom{k}{j}\} j!(n-j)!$ We need to show that $R(k) := \frac{S(k)}{k!} = \sum_{j=0}^{k} \min\{\frac{s-1}{\binom{k}{j}}, 1\} \le m'$ holds for all $k \ge m'$. Consider the case $k = m'$. If s is large enough (and thus m' and k), then $\binom{m'}{\lceil m'/2 \rceil} = (1+o(1))\binom{m'}{\lceil m'/2 \rceil + j}$ holds provided $|j| \le \sqrt{m'}/\log m'$. Therefore, by the assumption on s and c we have at least $2\sqrt{m'}/\log m'$ summands in $R(m')$ that are not more than $\frac{1+c}{2}$, a constant smaller than 1. Thus, if m' is large enough, their subsum

$$\sum_{i=\lceil m'/2 \rceil - \sqrt{m'}/\log m'}^{\lceil m'/2 \rceil + \sqrt{m'}/\log m'} \frac{s-1}{\binom{m'}{j}}$$

is less than $\lceil m'/2 \rceil + 2\sqrt{m'}/\log m' - 1$ and since all other summands are not more than 1, we obtain $R(m') < m'$.

To finish the proof of (ii), we prove that if $k \geq m'$ holds, then we have $R(k+1) \leq R(k)$. First note that if $r_{k,j}$ denotes the jth summand in $R(k)$, then we have $r_{k,j} \geq r_{k+1,j}$ and $r_{k,k-j} \geq r_{k+1,k+1-j}$. Thus it is enough to show

$$\sum_{i=-1}^{1} r_{k,\lceil k/2\rceil+i} \geq \sum_{i=-1}^{2} r_{k+1,\lceil k/2\rceil+i}.$$

By the definition of m', we know that $r_{k,\lceil k/2\rceil} < 1$. Since $\binom{k}{\lceil k/2\rceil} = (1/2+o(1))\binom{k+1}{\lceil k/2\rceil}$ we have that the LHS is $(3+o(1))r_{k,\lceil k/2\rceil}$ while the RHS is $(4+o(1))r_{k,\lceil k/2\rceil}/2 = (2+o(1))r_{k,\lceil k/2\rceil}$. This finishes the proof of (ii).

Finally, we prove (iii). Clearly, as long as $k \leq m'$ for any family $\mathcal{G} \subseteq 2^{[k]}$ the number of pairs is $(k+1)k! \leq (m'+1)k!$. We need to show that $R(k) \leq m'+1$ holds for all $k > m'$. As in (ii), the proof of $R(k+1) \leq R(k)$ for $k \geq m'$ did not require the assumption on s and c, we obtain that $R(k) \leq m'+1$ holds for all k. $\square$

Our last auxiliary lemma was proved by Griggs, Li and Lu [8].

Lemma 2.4 (Griggs, Li, Lu, during the proof of Theorem 2.5 in [8])**.** *Let $s \geq 2$, and define $m^* := \lceil \log_2(s+2)\rceil$.*

(1) If $s \in [2^{m^*-1}-1, 2^{m^*} - \binom{m^*}{\lceil \frac{m^*}{2}\rceil} - 1]$, then if $\mathcal{G} \subseteq 2^{[k]}$ a $K_{1,s,1}$-free family of sets, then the number of pairs $(G,\mathcal{C})$ with $G \in \mathcal{G} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_k$ is at most $m^*k!$.

(2) If $s \in [2^{m^*} - \binom{m^*}{\lceil \frac{m^*}{2}\rceil}, 2^{m^*}-2]$, then if $\mathcal{G} \subseteq 2^{[k]}$ a $K_{1,s,1}$-free family of sets, then the number of pairs $(G,\mathcal{C})$ with $G \in \mathcal{G} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_k$ is at most $(m^*+1-\frac{2^{m^*}-s-1}{\binom{m^*}{\lceil \frac{m^*}{2}\rceil}})k!$.

3 Proofs

In this section we prove our main theorems. Let us start with constructions to see the lower bounds. Let us partition $\binom{[n]}{k}$ into n classes: $\mathcal{F}_{n,k,i} = \{F \in \binom{[n]}{k} : \sum_{j\in F} j \equiv i \pmod n\}$. Let $\binom{[n]}{k}_{r,mod}$ denote the union of the r largest classes. Clearly, $|\binom{[n]}{k}_{r,mod}| \geq \frac{r}{n}\binom{n}{k}$. Furthermore, it has the property that for any distinct $r+1$ sets $F_1, F_2, \ldots, F_{r+1} \in \binom{[n]}{k}_{r,mod}$ we have $|\cap_{i=1}^{r+1}| \leq k-2$ and $|\cup_{i=1}^{r+1}| \geq k+2$.

- For Theorem 1.7 consider the family $\mathcal{F} := \binom{[n]}{\lceil n/2\rceil-2}_{r-1,mod} \cup \binom{[n]}{\lceil n/2\rceil-1} \cup \binom{[n]}{\lceil n/2\rceil} \cup \binom{[n]}{\lceil n/2\rceil+1}_{s-1,mod}$. Suppose $A_1, A_2, \ldots, A_r, B_1, B_2, \ldots, B_s \in \mathcal{F}$ form an induced copy of $K_{r,s}$. Then $\cup_{i=1}^r A_i \subseteq \cap_{j=1}^s B_j$ holds, but by the above property of $\binom{[n]}{k}_{r,mod}$ and the inducedness we have $|\cup_{i=1}^r A_i| \geq \lceil n/2\rceil$ and $|\cap_{j=1}^s B_j| \leq \lceil n/2\rceil - 1$ - a contradiction.

- For Theorem 1.8 let k be the index of the level below the $m+f(r,t)$ middle levels, i.e. $k=\lceil\frac{n-m-f(r,t)}{2}\rceil-1$. Write $l=k+m+f(r,t)+1$ and let us consider the family

$$\mathcal{F}:=\binom{[n]}{k}_{(r-2)^+,mod}\cup\bigcup_{i=1}^{m+f(r,t)}\binom{[n]}{k+i}\cup\binom{[n]}{l}_{(t-2)^+,mod}.$$

We claim that $\mathcal{F}$ is $K_{r,s,t}$-free. Assume not and let $A_1,A_2,\dots,A_r,B_1,B_2,\dots,B_s,C_1,C_2,\dots,C_t\in\mathcal{F}$ form a copy of $K_{r,s,t}$. If $r\geq 2$, then $|\cup_{i=1}^r A_i|\geq k+2$ and if $r=1$, then $|A_1|\geq k+1$ (note that if $r=1,2$, then $(r-2)^+=0$ and thus the smallest set size in $\mathcal{F}$ is $k+1$). Similarly, if $t\geq 2$, then $|\cap_{j=1}^t C_j|\leq l-2$ and if $t=1$, then $|C_1|\leq l-1$. In any case, $|\cup_{t=1}^t C_j|-|\cup_{i=1}^r A_i|\leq m-1$ and thus there is no place for $B_1,B_2,\dots,B_s$ - a contradiction.

- For Theorem 1.9 let k be the index of the level below the three middle levels, i.e. $k=\lceil\frac{n-3}{2}\rceil-1$. Write $l=k+4$ and let us consider the family

$$\mathcal{F}:=\binom{[n]}{k}_{(r-2)^+,mod}\cup\bigcup_{i=1}^{3}\binom{[n]}{k+i}\cup\binom{[n]}{l}_{(t-2)^+,mod}.$$

If $f(r,t)=2$, then for any $A_1,A_2,\dots,A_r\in\mathcal{F}$ and $C_1,C_2,\dots,C_t\in\mathcal{F}$ we have $|\cap_{i=1}^t C_i|-|\cup_{j=1}^r A_j|\leq 0$, thus we cannot have two sets in between. While if $f(r,t)=1$, say $t=1$, then for any $A_1,A_2,\dots,A_r\in\mathcal{F}$ and $C\in\mathcal{F}$ we have $|C|-|\cup_{j=1}^r A_j|\leq 1$, thus we cannot have two sets in between them and below C.

- For Theorem 1.12 (i), (ii) and (iii), let k be the index of the level below the $m'+f(r,t)$ middle levels, i.e. $k=\lceil\frac{n-m'-f(r,t)}{2}\rceil-1$. Write $l=k+m'+f(r,t)+1$ and let us consider the family

$$\mathcal{F}:=\binom{[n]}{k}_{r-1,mod}\cup\bigcup_{i=1}^{m'+f(r,t)}\binom{[n]}{k+i}\cup\binom{[n]}{l}_{t-1,mod}.$$

One can see that for any antichains $A_1,A_2,\dots,A_r\in\mathcal{F}$ and $C_1,C_2,\dots,C_t\in\mathcal{F}$ we have $|\cap_{i=1}^t C_i|-|\cup_{j=1}^r A_j|\leq m'-1$ and thus there is no room for an antichain of size s in between. Note that when $s=4$, then $m'=4$ as $\binom{4}{2}=6\geq 4$, but $\binom{3}{2}=3<4$.

Let us now start proving the upper bounds of our results. First of all, from here on every family $\mathcal{F}\subseteq 2^{[n]}$ contains sets only of size from the interval $[n/2-n^{2/3},n/2+n^{2/3}]$. This leaves all our proofs valid as by Chernoff's inequality $|\{F\subseteq[n]:||F|-n/2|\geq n^{2/3}\}|\leq 2^{n+1}e^{-2n^{1/3}}=o(\frac{1}{n^2}\binom{n}{\lceil n/2\rceil})$.

As we mentioned in the Introduction, for all proofs we will use the chain partition method. This works in the following way: for a family $\mathcal{F} \subseteq 2^{[n]}$ suppose we can partition $\mathbf{C}_n$ into $\mathbf{C}_{n,1}, \mathbf{C}_{n,2}, \dots \mathbf{C}_{n,l}$ such that for all $1 \le i \le l$ the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_{n,i}$ is at most $b|\mathbf{C}_{n,i}|$. Then clearly the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_n$ is at most $b|\mathbf{C}_n|$. Since the number of such pairs is exactly $\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$ we obtain the LYM-type inequality

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \le b$$

and thus $|\mathcal{F}| \le b\binom{n}{\lceil n/2 \rceil}$ holds. Therefore, in the proofs below we will end our reasoning whenever we reach a bound on the appropriate partition as mentioned above.

Proof of the upper bound in Theorem 1.7. Let $\mathcal{F}$ be an induced $K_{r,s}$-free family. We can assume that $\mathcal{F}$ contains an antichain of size at least r as otherwise Lemma 2.1 can be applied to $\mathcal{F}$ and $[n]$ to obtain that $|\mathcal{F}| \le o(\binom{n}{\lceil n/2 \rceil})$.

Now we define the $\min_r^*$-partition of $\mathbf{C}_n$ and $\mathcal{F}$. For a set A with $s_{\mathcal{F}}^-(A) \ge r$ we define $\mathbf{C}_{\mathcal{F},A,r} = \{\mathcal{C} \in \mathbf{C}_n : A \in \mathcal{C}, \forall C \subset A, C \in \mathcal{C} : s_{\mathcal{F}}^-(C) < r\}$. Note that every $\mathcal{C} \in \mathbf{C}_n$ belongs to exactly one set $\mathbf{C}_{\mathcal{F},A,r}$ as by our assumption $s_{\mathcal{F}}^-([n]) \ge r$ holds.

We claim that that the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_{\mathcal{F},A,r}$ is at most $(2 + \frac{2(r+s-2)}{n} + O(\frac{1}{n^2}))|\mathbf{C}_{\mathcal{F},A,r}|$. We distinguish three types of pairs:

1. if $A \in \mathcal{F}$, then there are exactly $|\mathbf{C}_{\mathcal{F},A,r}|$ pairs with $F = A$ (otherwise there is none),
2. any chain in $\mathbf{C}_{r,A}$ can be extended to $(n-|A|)!$ chains in $\mathbf{C}_{\mathcal{F},A,r}$, thus by Corollary 2.2 there are $(1 + \frac{2(r-1)}{n} + O_r(\frac{1}{n^2}))|\mathbf{C}_{\mathcal{F},A,r}|$ pairs with $F \subsetneq A$,
3. finally, any maximal chain from A to $[n]$ can be extended to $|\mathbf{C}_{r,A}|$ chains in $\mathbf{C}_{\mathcal{F},A,r}$, thus Lemma 2.1 implies that there are $(\frac{2(s-1)}{n} + O_s(\frac{1}{n^2}))|\mathbf{C}_{\mathcal{F},A,r}|$ pairs with $A \subsetneq F$

This gives us a total of at most $(2 + \frac{2(r+s-2)}{n} + O_{r,s}(\frac{1}{n^2}))|\mathbf{C}_{\mathcal{F},A,r}|$ pairs, which completes the proof. □

Now we turn our attention to complete three level posets.

Proof of the upper bound in Theorem 1.8. Let $\mathcal{F}$ be a $K_{r,s,t}$-free family. We can assume that $\mathcal{F}$ contains an antichain of size at least $\max\{r,t\}$ as otherwise Lemma 2.1 can be applied to $\mathcal{F}$ and $[n]$ to obtain that $|\mathcal{F}| \le o(\binom{n}{\lceil n/2 \rceil})$.

Now we define the $\min_r^* - \max_t^*$ partition of $\mathbf{C}_n$. Let $\mathcal{S} = \{S \in 2^{[n]} : s_{\mathcal{F}}^-(S) \ge r\}$, $\mathcal{S}^- = \{S \in \mathcal{S} : s_{\mathcal{F}}^+(S) < t\}$ and finally $\mathcal{S}^+ = \mathcal{S} \setminus \mathcal{S}^-$. For any set $S \in \mathcal{S}^-$ let $\mathbf{C}_S$ denote the set of those maximal chains $\mathcal{C}$ in $\mathbf{C}_n$ in which

- if $r = 1$, then S is the smallest set in $\mathcal{F} \cap \mathcal{C}$,

- if $r \geq 2$, then S is the smallest set in $\mathcal{C}$ with $s^-_{\mathcal{F}}(S) \geq r$.

For any set $A \in S^+$ and B with $A \subseteq B$ let $\mathbf{C}_{A,B} = \mathbf{C}_{A,r,B,t}$ denote the set of those maximal chains $\mathcal{C}$ in $\mathbf{C}_n$ in which

- if $r = 1$, then A is the smallest set in $\mathcal{F} \cap \mathcal{C}$,
- if $r \geq 2$, then A is the smallest set in $\mathcal{C}$ with $s^-_{\mathcal{F}}(A) \geq r$,
- if $t = 1$, then B is the largest set in $\mathcal{F} \cap \mathcal{C}$,
- if $t \geq 2$, then B is the largest set in $\mathcal{C}$ with $s^+_{\mathcal{F}}(B) \geq t$.

Consider a maximal chain $\mathcal{C} \in \mathbf{C}_n$. By the assumption $s^-_{\mathcal{F}}([n]) \geq \max\{r, t\}$, there is a smallest set H of $\mathcal{C}$ with $s^-_{\mathcal{F}}(H) \geq r$. If $H \in \mathcal{S}^-$, then $\mathcal{C}$ belongs to $\mathbf{C}_H$. If not, then $H \in \mathcal{S}^+$ and thus for the largest set H' of $\mathcal{C}$ with $s^+_{\mathcal{F}} \geq t$ we have $H \subseteq H'$ and therefore $\mathcal{C} \in \mathbf{C}_{H,H'}$ holds. We obtained that the $\min^*_r - \max^*_t$ partition of $\mathbf{C}_n$ is indeed a partition.

We claim that that the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_S$, $\mathcal{C} \in \mathbf{C}_{A,B}$ is at most $b|\mathbf{C}_S|$, $b|\mathbf{C}_{A,B}|$, respectively, where b is the bound stated in Theorem 1.8.

First consider the "degenerate" case of $\mathbf{C}_S$ with $S \in \mathcal{S}^-$. A chain $\mathcal{C} \in \mathbf{C}_S$ goes from $\emptyset$ until one of the subsets $S_1, S_2, \ldots, S_k$ of B with size $|B| - 1$ for which $s^-_{\mathcal{F}}(S_i) < r$. Then $\mathcal{C}$ must go through S, and finally $\mathcal{C}$ must contain a maximal chain from S to $[n]$. Thus $|\mathbf{C}_S| = k(|S| - 1)!(n - |S|)!$. We distinguish two types of pairs to count.

1. If $r \geq 2$, then applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(r-1)}{n} + O_r(\frac{1}{n^2}))|\mathbf{C}_S|$ pairs $(F, \mathcal{C})$ with $F \subsetneq S$. Together with $\{(S, \mathcal{C}) : \mathcal{C} \in \mathbf{C}_S\}$ we have $(2 + \frac{2(r-1)}{n} + O_r(\frac{1}{n^2}))|\mathbf{C}_S|$ pairs. If $r = 1$, then by definition the number of pairs $(F, \mathcal{C})$ with $F \subseteq S$ is at most $|\mathbf{C}_S|$ as for all such pairs we must have $F = S$.

2. Applying Lemma 2.1 we obtain that there are at most $(\frac{2(t-1)}{n} + O_t(\frac{1}{n^2}))|\mathbf{C}_S|$ pairs $(F, \mathcal{C})$ with $S \subsetneq F$.

This gives a total of at most $(2 + \frac{2(r+t-2)}{n} + O_{r,t}(\frac{1}{n^2}))|\mathbf{C}_S|$ pairs.

We now consider the "more natural" $A \in \mathcal{S}^+$, $A \subseteq B$ case. As there are sets in the interval $[A, B]$, this time we distinguish three types of pairs:

1. If $r = 1$, then there is no pair $(F, \mathcal{C})$ with $F \subsetneq A$. If $r \geq 2$, then applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(r-1)}{n})|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $F \subsetneq A$.

2. If $t = 1$, then there is no pair $(F, \mathcal{C})$ with $B \subsetneq F$. If $t \geq 2$, then applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(t-1)}{n})|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $B \subsetneq F$.

3. If $\mathcal{F}$ is a $K_{r,s,t}$-free family, then $\{F \in \mathcal{F} : A \subseteq F \subseteq B\}$ is a $K_{1,s-f(r,t),1}$-free family. Indeed, if $f(r,t) = 2$, then $|\{F \in \mathcal{F} : A \subseteq F \subseteq B\}| \leq s$ as these sets together with the sets of the antichain of size r below A and the sets of the antichain of size t above B would form a copy of $K_{r,s,t}$ in $\mathcal{F}$. If $f(r,t) = 1$, say $r = 1$, then by the definition of the $\min_1^* - \max_t^*$ partition, we have $A \in \mathcal{F}$ and thus $|\{F \in \mathcal{F} : A \subsetneq F \subseteq B\}| \leq s$, in particular together with A they are $K_{1,s-1,1}$-free. If $f(r,t) = 0$, then the $K_{1,s-f(r,t),1}$-free property is the same as the $K_{1,s,1}$-free property which is possessed by $\{F \in \mathcal{F} : A \subseteq F \subseteq B\}$ as it is a subfamily of $\mathcal{F}$.

   By Lemma 2.4, in case (1) of Theorem 1.8 the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $m|\mathbf{C}_{A,B}|$, while in case (2) of Theorem 1.8 the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $(m + 1 - \frac{2^m - s + f(r,t) - 1}{\binom{m}{\lceil m/2 \rceil}})|\mathbf{C}_{A,B}|$.

Adding up the number of three types of pairs we obtain that the total number of pairs is not more than $(m + f(r,t) + \frac{2(r+t-2)}{n})|\mathbf{C}_{A,B}|$ and $(m + 1 + f(r,t) - \frac{2^m - s + f(r,t) - 1}{\binom{m}{\lceil m/2 \rceil}} + \frac{2(r+t-2)}{n})|\mathbf{C}_{A,B}|$ in the two respective cases of Theorem 1.8. □

We continue with the proof of Theorem 1.9.

Proof of Theorem 1.9. Let $\mathcal{F}$ be a $K_{r,2,t}$-free family and let us write $r^{++} = \max\{r, 2\}, t^{++} = \max\{t, 2\}$. We consider the $\min_{r^{++}}^* - \max_{t^{++}}^*$-partition of $\mathbf{C}_n$ defined in the proof of Theorem 1.8. Just as in the proof of Theorem 1.8, we obtain that if $S \in \mathcal{S}^-$ than the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_S$ is at most $(2 + O(\frac{1}{n}))|\mathbf{C}_S|$. Note that if $A \subseteq B$, then $|\mathcal{F} \cap \{G \in 2^{[n]} : A \subseteq G \subseteq B\}| \leq 1$ as by definition of the $\min_{r^{++}}^* - \max_{t^{++}}^*$-partition two such sets would make $\mathcal{F}$ contain a copy of $K_{r,2,t}$.

- Applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(r^{++}-1)}{n} + O_r(\frac{1}{n^2}))|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $F \subsetneq A$.
- Applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(t^{++}-1)}{n} + O_t(\frac{1}{n^2}))|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $B \subsetneq F$.
- By the observation above, the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $|\mathbf{C}_{A,B}|$.

□

Proof ofTheorem 1.12. First we prove (i), (ii), and (iii). Let $\mathcal{F}$ be an induced $K_{r,s,t}$-free family. We can assume that $\mathcal{F}$ contains an antichain of size at least $\max\{r, t\}$ as otherwise Lemma 2.1 can be applied to $\mathcal{F}$ and $[n]$ to obtain that $|\mathcal{F}| \leq o(\binom{n}{\lceil n/2 \rceil})$. We again consider the $\min_r^* - \max_t^*$ partition of $\mathbf{C}_n$ and count the number of pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_n$.

The degenerate case is identical to what we had in the proof of Theorem 1.8, thus we only consider the case when $A \in \mathcal{S}^+$, $A \subseteq B$. The three types of pairs:

1. If $r = 1$, then there is no pair $(F, \mathcal{C})$ with $F \subsetneq A$. If $r \geq 2$, then applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(r-1)}{n})|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $F \subsetneq A$.
2. If $t = 1$, then there is no pair $(F, \mathcal{C})$ with $B \subsetneq F$. If $t \geq 2$, then applying Corollary 2.2 we obtain that there are at most $(1 + \frac{2(t-1)}{n})|\mathbf{C}_{A,B}|$ pairs $(F, \mathcal{C})$ with $B \subsetneq F$.
3. Note that $\{F \in \mathcal{F} : A \subseteq F \subseteq B\}$ cannot contain an antichain of size s as otherwise $\mathcal{F}$ would contain an induced copy of $K_{r,s,t}$.
   (a) If $\mathcal{F}$ is an induced $K_{r,4,t}$-free family, then by Lemma 2.3 (i) the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $4|\mathbf{C}_{A,B}|$.
   (b) If $\mathcal{F}$ is an induced $K_{r,s,t}$-free family with $s \leq c\binom{m'}{\lceil m'/2 \rceil}$ and s large enough, then by Lemma 2.3 (ii) the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $m'|\mathbf{C}_{A,B}|$.
   (c) If $\mathcal{F}$ is an induced $K_{r,s,t}$-free family with s large enough, then by Lemma 2.3 (iii) the number of pairs $(F, \mathcal{C})$ with $A \subseteq F \subseteq B$ is at most $(m' + 1)|\mathbf{C}_{A,B}|$.

Altogether these bounds yield that the total number of pairs is at most

1. $(4 + f(r,t) + \frac{2(r+t-2)}{n} + O(\frac{1}{n^2}))|\mathbf{C}_n|$ if $\mathcal{F}$ is induced $K_{r,4,t}$-free.
2. $(m' + f(r,t) + \frac{2(r+t-2)}{n} + O(\frac{1}{n^2}))|\mathbf{C}_n|$ if $\mathcal{F}$ is induced $K_{r,s,t}$-free, $s \leq c\binom{m'}{\lceil m'/2 \rceil}$ and s large enough.
3. $(m' + 1 + f(r,t) + \frac{2(r+t-2)}{n} + O(\frac{1}{n^2}))|\mathbf{C}_n|$ if $\mathcal{F}$ is induced $K_{r,s,t}$-free and s large enough.

Now we prove (iv). Let $\mathcal{F}$ be an induced $K_{r,s_1,s_2,\dots,s_j,t}$-free family. We can assume that $\mathcal{F}$ contains an antichain of size at least $\max\{r,t\}$ as otherwise Lemma 2.1 can be applied to $\mathcal{F}$ and $[n]$ to obtain that $|\mathcal{F}| \leq o(\binom{n}{\lceil n/2 \rceil})$. We consider the following partition of $\mathbf{C}_n$: for any chain $A_0 \subseteq A_1 \subseteq \dots \subseteq A_i$ with $i \leq j$ we define $\mathbf{C}_{A_0,A_1,\dots,A_i}$ if (1) $i = j$ and $s^+_{\mathcal{F}}(A_{i-1}) \geq t$ or if (2) if $i < j - 1$ and $s^+_{\mathcal{F}}(A_i) < s_{i+1}$ but for all $k < i$ we have $s^+_{\mathcal{F}}(A_k) < s_{k+1}$ or if (3) $i = j - 1$ and $s^+_{\mathcal{F}}(A_i) < t$. In case (1) the set $\mathbf{C}_{A_0,A_1,\dots,A_j}$ consists of those maximal chains $\mathcal{C}$ in $\mathbf{C}_n$ that satisfy

- if $r = 1$, then A_0 is the smallest set in $\mathcal{F} \cap \mathcal{C}$,
- if $r \geq 2$, then A_0 is the smallest set in $\mathcal{C}$ with $s^-_{\mathcal{F}}(A_0) \geq r$,

- for any k with $1 \leq k \leq j-1$ A_k is the smallest set in $\mathcal{F} \cap \mathcal{C}$ such that there exists an antichain of size s_k in $\mathcal{F} \cap [A_{k-1}, A_k]$,
- if $t = 1$, then A_j is the largest set in $\mathcal{F} \cap \mathcal{C}$,
- if $t \geq 2$, then A_j is the largest set in $\mathcal{C}$ with $s^+_{\mathcal{F}}(A_j) \geq t$.

In case (2) and (3) the definition of $\mathbf{C}_{A_0,A_1,\ldots,A_i}$ is modified to the set of maximal chains $\mathcal{C}$ in $\mathbf{C}_n$ that satisfy

- if $r = 1$, then A_0 is the smallest set in $\mathcal{F} \cap \mathcal{C}$,
- if $r \geq 2$, then A_0 is the smallest set in $\mathcal{C}$ with $s^-_{\mathcal{F}}(A_0) \geq r$,
- for any k with $1 \leq k \leq i$ A_k is the smallest set in $\mathcal{F} \cap \mathcal{C}$ such that there exists an antichain of size s_k in $\mathcal{F} \cap [A_{k-1}, A_k]$,
- in case (2) we have $s^+_{\mathcal{F}}(A_i) < s_{i+1}$, while in case (3) we have $s^+_{\mathcal{F}}(A_{j-1}) < t$.

One can verify along the lines of the proof of Theorem 1.8 that the above definition results a partition of $\mathbf{C}_n$. Let us note that a chain $\mathcal{C}$ in $\mathbf{C}_{A_0,\ldots,A_i}$ contains all A_i's and for every $0 \leq k \leq j-1$ it goes through one of the $(|A_k|-1)$-subsets $A^k_1, \ldots A^k_{l_k}$ of A_k for which $[A_{k-1}, A^k_l]$ does not contain an antichain of size s_k where $A_{-1} = \emptyset$ and $s_{-1} = r$.

We now count the pairs $(F, \mathcal{C})$ with $F \in \mathcal{F} \cap \mathcal{C}$ and $\mathcal{C} \in \mathbf{C}_{A_0,\ldots,A_i}$. First we consider the cases (2) and (3)

- Applying Lemma 2.1 and/or Corollary 2.2 we obtain that below A_0 and above A_i there are at most $(f(r,t) + O_{r,t,s_{i+1}}(\frac{1}{n}))|\mathbf{C}_{A_0,\ldots,A_i}|$ such pairs,
- applying Lemma 2.3 to A_k and all $A^k_1, \ldots, A^k_{l_k}$ we obtain that the number of such pairs above A_k (including A_k) and below A_{k+1} there are at most $m'_{s_k}|\mathbf{C}_{A_0,\ldots,A_i}|$ such pairs.

Altogether we obtained that the number of pairs is at most $(f(r,t) + 1 + \sum_{k=1}^{i} m'_{s_k} + O(\frac{1}{n}))|\mathbf{C}_{A_0,\ldots,A_i}| \leq (f(r,t) + \sum_{k=1}^{j} m'_{s_k} + O(\frac{1}{n}))|\mathbf{C}_{A_0,\ldots,A_i}|$.

Finally, in case (1) everything below A_{j-1} and above A_j is as before. As $\mathcal{F}$ is induced $K_{r,s_1,\ldots,s_j,t}$-free the interval $[A_{j-1}, A_j]$ cannot contain an antichain of size s_j. Thus the number of pairs with $F \in [A_{j-1}, A_j]$ is at most $m'_{s_j}|\mathbf{C}_{A_0,\ldots,A_i}|$. Thus in this case the total number of pairs is at most $(f(r,t) + \sum_{k=1}^{j} m'_{s_k} + O(\frac{1}{n}))|\mathbf{C}_{A_0,\ldots,A_j}|$. □

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