Inclusionwise minimal completely separating systems

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Abstract

A set system $\mathcal{H} \subseteq 2^{[n]}$ is said to be *completely separating* if for any $x, y \in [n]$ there exist sets $A, B \in \mathcal{H}$, such that $x \in A \cap \overline{B}$, $y \in B \cap \overline{A}$. Let us denote the maximum size of an inclusionwise minimal completely separating system on the underlying set $[n]$ by $g(n)$. We show that for $2 \leq n \leq 6$, $g(n) = 2n - 2$, and for $n \geq 7$, $g(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.

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1 Introduction

We denote the set of the first n positive integers by $[n]$ and the set of integers $\{a, a+1, a+2, \ldots, b\}$ by $[a, b]$. Let $\mathcal{H} \subseteq 2^{[n]}$ be a set system of cardinality m and consider any linear order of its sets. The *incidence matrix* of $\mathcal{H}$ is a 0-1 matrix $M_{\mathcal{H}} = (m_{ij})_{m,n}$, where m_{ij} is 1 if and only if the i^{th} set of $\mathcal{H}$ contains the element j. It is obvious that any row permutation of an incidence matrix of a set system is also an incidence matrix of the same set system. The *dual* of a set system $\mathcal{H}$ is the (multi)system $\mathcal{H}^*$, whose incidence matrix is $M_{\mathcal{H}}^T$.

The *restriction* of a set system $\mathcal{H} \subseteq 2^{[n]}$ to a subset $X \subseteq [n]$ is the system $\{H \cap X : H \in \mathcal{H}\}$ and is denoted by $\mathcal{H}|_X$. Restrictions to a subset are often called *traces* in the literature (Füredi and Pach, 1991).

A set system $\mathcal{H} \subseteq 2^{[n]}$ is said to be *separating* if for any $x, y \in [n]$ there exists a set $A \in \mathcal{H}$, such that $x \in A$, $y \notin A$ or $y \in A$, $x \notin A$. A set system $\mathcal{H} \subseteq 2^{[n]}$ is said to be *completely separating* if for any $x, y \in [n]$ there exist sets $A, B \in \mathcal{H}$, such that $x \in A$, $y \notin A$, $y \in B$, $x \notin B$.

Minimum size separating systems of sets of at most k elements were studied by Katona (1966) and Wegener (1979). (Without the restriction on the size of the sets the problem is trivial.)

To find the maximum size of an inclusionwise minimal separating system one can apply a theorem of Bondy.

Theorem 1.1. (Bondy, 1972) *Let $\mathcal{H} \subseteq 2^{[n]}$ be a set system of cardinality $m \leq n$. Then there exists $x \in [n]$, such that $\mathcal{H}|_{[n]\setminus\{x\}}$ also has cardinality m.*

Since it is obvious that a set system is separating if and only if its dual is also a set system (that is, the incidence matrix of the system does not contain multiple columns), Bondy's theorem immediately implies that an inclusionwise minimal separating system on the underlying set $[n]$ may have size at most $n-1$. It is easy to see that this result is sharp: consider the system $\{\{1\}, \{2\}, \dots, \{n-1\}\}$.

Completely separating systems of minimum size were investigated first in Dickson (1969). Spencer pointed out that a set system is completely separating if and only if its dual is a Sperner family (that is, no set contains another one) (Spencer, 1970). This observation – together with Sperner's theorem (Sperner, 1928) – determines the size of minimum completely separating systems on the underlying set $[n]$.

Minimum size completely separating systems of sets of exactly or at most k elements were examined by several authors including Ramsay et al. (1998) and Kündgen et al. (2001).

In this paper we deal with the problem of determining the maximum size of an inclusionwise minimal completely separating system.

Definition 1.2. *For an arbitrary positive integer n let $g(n)$ denote the maximum size of completely separating systems $\mathcal{H}$ on the underlying set $[n]$, such that no $\mathcal{G} \subsetneq \mathcal{H}$ is completely separating.*

Since a set system is completely separating if and only if its dual is a Sperner family, $g(n)$ is the same as the maximum number m, for which there exists a Sperner family $\mathcal{F} \subseteq 2^{[m]}$ of cardinality n, such that $\mathcal{F}|_{[m]\setminus\{x\}}$ is not a Sperner family for any $x \in [m]$.

Therefore we discuss Sperner families with this property in the sequel.

In the next section we solve our problem for regular systems (systems having a uniform dual). In the third section we solve the general version, that is, we determine the function $g(n)$ precisely.

2 Regular systems

We show that the maximum size of a regular, inclusionwise minimal completely separating system $\mathcal{H} \subseteq 2^{[n]}$ is $2n-2$.

Theorem 2.1. *Let n be an arbitrary positive integer and let $m \geq 2n-1$. Then for any uniform (Sperner) family $\mathcal{F} \subseteq 2^{[m]}$ of cardinality n there exists an element $x \in [m]$, such that $\mathcal{F}|_{[m]\setminus\{x\}}$ is a Sperner family. Furthermore, the bound on m is best possible.*

Proof. Suppose that for a uniform set system $\mathcal{F} \subseteq 2^{[m]}$ of cardinality n there exists no element $x \in [m]$, such that $\mathcal{F}|_{[m]\setminus\{x\}}$ is a Sperner family. This means that for any element $x \in [m]$ there exist $A_x, B_x \in \mathcal{F}$, such that $A_x \setminus \{x\} \subseteq B_x$. We fix these sets A_x, B_x for every element x. Since $|A_x| = |B_x|$, there exists a unique $y \in [m] \setminus \{x\}$, for which $B_x \setminus \{y\} \subseteq A_x$. We denote this element y by $p(x)$.

Consider now the $m \times n$ incidence matrix M of the set system $\mathcal{F}$ (that is, columns of M are the characteristic vectors of the sets in $\mathcal{F}$). Denote the row of M corresponding to the element $x \in [m]$ by v_x.

Now we define a set $X = \{x_1, x_2, \dots, x_r\} \subseteq [m]$ in the following way. First we choose $x_1 \in [m]$ arbitrarily. If we have already chosen some k elements of X, then we choose x_{k+1} arbitrarily from the set $[m] \setminus \{p(x_i) \ : \ i = 1, 2, \ldots, k\}$ and we continue this procedure till it is possible. Since the rows of M must be different, it is obvious that $r = |X| \geq \frac{m}{2}$. Now let M' be that $r \times n$ submatrix of M whose i^{th} row is v_{x_i} if the first coordinate of v_{x_i} is 0 and $v_{x_i} + \mathbf{1}$ otherwise, where **1** denotes the all 1 vector of length n and + is the addition in the vector space $GF(2)^n$. Thus the first column of M' is the all zero column (of length r).

We show that the rows of M' are linearly independent over $GF(2)$. Assume to the contrary that there is a nontrivial linear combination of these rows that gives the all zero vector. Over $GF(2)$ this means that a nonempty sum of some rows of M' is the all zero vector. Let i be the minimum number such that either v_{x_i} or $v_{x_i} + \mathbf{1}$ appears in this nonempty sum. Then it is easy to see that neither $v_{p(x_i)}$ nor $v_{p(x_i)} + \mathbf{1}$ is in M' and therefore none of them appears in the sum. This is a contradiction, since the columns of M corresponding to the sets A_{x_i} and B_{x_i} differ only in the rows corresponding to x_i and $p(x_i)$, thus the sum cannot give the all zero vector.

This shows that $r \leq rank(M') \leq n-1$, from which $m \leq 2n-2$ follows, contradicting the condition $m \geq 2n-1$ and thus proving the first half of the theorem.

To prove that the bound is best possible, it suffices to give a uniform set system $\mathcal{F} \subseteq 2^{[2n-2]}$ of cardinality n, for which there exists no $x \in [m]$, such that $\mathcal{F}|_{[m]\setminus\{x\}}$ is a Sperner family.

Let $A_0 = \emptyset$ and $A_i = \{i\}$, for $i = 1, 2, \ldots, n-1$. Let furthermore $B_0 = [n, 2n-2]$ and $B_i = [n, 2n-2] \setminus \{n+i\}$, for $i = 1, 2, \ldots, n-1$. Now it is easy to check that $\{A_0 \cup B_0, A_1 \cup B_1, \ldots, A_{n-1} \cup B_{n-1}\} \subseteq 2^{[2n-2]}$ is a uniform set system with the desired property. □

Let us mention that this theorem can also be derived from a theorem of Winter concerning traces of d-distance codes (Winter, 2000).

3 Determining $g(n)$

Now we turn our attention to the general case. First we give lower bounds on $g(n)$ and then we prove that these bounds are tight.

Lemma 3.1. *If* $2 \leq n \leq 6$*, then* $g(n) \geq 2n-2$*. If* $n \geq 7$*, then* $g(n) \geq \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.

Proof. We have to give Sperner families $\mathcal{F}$ containing n sets on an underlying set of cardinality $2n-2$ if $2 \leq n \leq 6$ and $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ if $n \geq 7$, such that no restriction of $\mathcal{F}$ to a proper subset of the underlying set is a Sperner family.

For $2 \leq n \leq 6$ the uniform families given in the proof of the (second part of) Theorem 2.1 fulfil this requirement, proving the first lower bound.

For $n \geq 7$ consider the following sets A_i and B_j on the underlying set $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.

$$A_i = [(i-1)\lfloor \frac{n}{2} \rfloor + 1, i\lfloor \frac{n}{2} \rfloor],$$

for any $i \in \left[\lceil \frac{n}{2} \rceil\right]$ and

$$B_j = \left[\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil\right] \setminus \{x \ : \ x \equiv j \pmod{\lfloor \frac{n}{2} \rfloor}\},$$

for any $j \in \left[\lfloor \frac{n}{2} \rfloor\right]$.

Since all the A_i's are of size $\lfloor \frac{n}{2} \rfloor$ and all the B_j's are of size $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil$, only some B_j could contain some A_i, but for $x = (i-1)\lfloor \frac{n}{2} \rfloor + j$ we have $x \in A_i, x \notin B_j$, so the family $\mathcal{F}$ consisting of all A_i's and B_j's is a Sperner family, and $A_i|_{[\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil]\setminus\{x\}} \subset B_j|_{[\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil]\setminus\{x\}}$. Since $\mathcal{F}$ contains $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ sets, the second lower bound is also proved. □

To show that these lower bounds are tight, we introduce some notations and prove some useful lemmas.

Let $f(n) = \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor + 1$. Note that $f(n)$ is monotone increasing and $f(n) - f(n-2) = n-1$, $f(n) - f(n-3) = \lfloor \frac{3}{2}n \rfloor - 2$ and for $n \geq 12$ $f(n) - f(n-5) > 2n$.

We say that F_1 *almost contains* F_2 if there is an element x, such that $F_2 \setminus \{x\} \subset F_1$. In this case we also say that F_2 is *almost contained* in F_1 or that the pair (F_1, F_2) *ruins* x. For a set system $\mathcal{H}$ the set of elements that are ruined by a pair of sets in $\mathcal{H}$ is denoted by $\mathcal{R}(\mathcal{H})$. Observe that the fact that for any Sperner system $\mathcal{F}$ with $|\mathcal{F}| = n$, $|\mathcal{R}(\mathcal{F})| \leq k$ holds implies $g(n) \leq k$.

A set system is said to be an MRU (maximal recursive uniform) subsystem of a Sperner family $\mathcal{F}$ if it is formed in the following way: let $\mathcal{G}_0 = \{F_1, F_2\}$, where $F_1, F_2 \in \mathcal{F}$ with $|F_1| = |F_2|$ and $|F_1 \setminus F_2| = 1$ and define recursively $\mathcal{G}_{i+1} = \{F \in \mathcal{F} : \exists G \in \mathcal{G}_i, \text{ such that } |F| = |G|, |F \setminus G| = 1\}$. Since $\mathcal{G}_i \subseteq \mathcal{F}$ for all i, there exists a number i^*, such that $\mathcal{G}_{i^*} = \mathcal{G}_{i^*+1}$. Set systems $\mathcal{G}_{i^*}$ obtained in this way are the MRU subsystems of $\mathcal{F}$. The maximum size of an MRU subsystem of a Sperner family $\mathcal{F}$ is denoted by $\gamma(\mathcal{F})$ (if there is no MRU subsystem of $\mathcal{F}$, then $\gamma(\mathcal{F}) = 0$). It is obvious that $\gamma(\mathcal{F})$ cannot be exactly 1.

First we handle the case $\gamma(\mathcal{F}) = 0$, that is, when there is no nonempty MRU subsystem of $\mathcal{F}$.

Lemma 3.2. *If $\gamma(\mathcal{F}) = 0$ (i.e. there are no two sets $F_1, F_2 \in \mathcal{F}$, such that $|F_1| = |F_2|$ and $|F_1 \setminus F_2| = 1$), then $\mathcal{R}(\mathcal{F}) < f(n)$.*

Proof. Suppose to the contrary that there is a Sperner system $\mathcal{F}$ with $|\mathcal{F}| = n$ and $\mathcal{R}(\mathcal{F}) \geq f(n)$. Let us define a graph G on n vertices with vertex set $\mathcal{F}$. We know that for $f(n)$ different elements in $X = \bigcup \mathcal{F}$ there is a pair of two sets $F_1, F_2 \in \mathcal{F}$, such that $F_1|_{X \setminus \{x\}} \subset F_2|_{X \setminus \{x\}}$. Let us pick a pair for each such x and put an edge in our graph G between the members of each such pair. Now we have $|E(G)| = f(n)$. We claim that there are no parallel edges in G. Indeed, if $F_1|_{X \setminus \{x_1\}} \subset F_2|_{X \setminus \{x_1\}}$ and $F_1|_{X \setminus \{x_2\}} \subset F_2|_{X \setminus \{x_2\}}$ for some $x_1 \neq x_2$, then $F_1 \subset F_2$ which contradicts the Sperner property of $\mathcal{F}$, while if $F_1|_{X \setminus \{x_1\}} \subset F_2|_{X \setminus \{x_1\}}$ and $F_2|_{X \setminus \{x_2\}} \subset F_1|_{X \setminus \{x_2\}}$, then $|F_1| = |F_2|$ and $F_1 \setminus F_2 = \{x_1\}$, which contradicts the assumption of the lemma.

We also claim that G does not contain a triangle. Indeed, by the above argument we know that (because of the assumption of the lemma) the endpoints of an edge must have different cardinality, therefore if F_1, F_2, F_3 form a triangle in G, such that (say) $|F_1| < |F_2| < |F_3|$, then there must exist elements x_1, x_2, x_3, such that $F_1 \setminus \{x_1\} \subset F_3$, $F_1 \setminus \{x_2\} \subset F_2$ and $F_2 \setminus \{x_3\} \subset F_3$. But then we have $F_1 \subset F_3$, which contradicts the Sperner property of $\mathcal{F}$.

We obtained that G is a simple triangle free graph on n vertices with $f(n)$ edges, which contradicts Mantel's well-known theorem Mantel (1907) stating that the maximum number of edges that a simple triangle-free graph on n vertices can have is $f(n) - 1$. □

Now we prove some lemmas concerning MRU subsystems.

Lemma 3.3. *If $\mathcal{G}$ is an MRU subsystem of a system $\mathcal{F}$, then*

$$\bigcup \mathcal{G} = \mathcal{R}(\mathcal{G}) \cup \bigcap \mathcal{G}.$$

Proof. We have to prove that if $x \in \bigcup \mathcal{G}$ and there are two sets $G_1, G_2 \in \mathcal{G}$ with $x \in G_1, x \notin G_2$, then there are two sets $G_1', G_2' \in \mathcal{G}$ with $G_1' \setminus G_2' = \{x\}$. This follows from the recursive definition of $\mathcal{G}$ and the observation that we can assume that one of G_1 and G_2 is in $\mathcal{G}_0$ (just consider the pair G, G_2 if $x \in G \in \mathcal{G}_0$, and the pair G_1, G otherwise). □

Lemma 3.4. *If $\mathcal{G}$ is an MRU subsystem of a Sperner system $\mathcal{F}$ and $F \in \mathcal{F}$ then we have $|\mathcal{R}(\mathcal{G} \cup \{F\}) \setminus \mathcal{R}(\mathcal{G})| \leq 1$.*

Proof. If $|F| = |G|$ for some $G \in \mathcal{G}$ then since $F \notin \mathcal{G}$, we have that $\mathcal{R}(\mathcal{G} \cup \{F\}) \setminus \mathcal{R}(\mathcal{G})$ is empty. Let us suppose $|F| > |G|$ for all $G \in \mathcal{G}$ (the case $|F| < |G|$ is similar). So suppose to the contrary that for $x_1, x_2 \notin \mathcal{R}(\mathcal{G})$ we have $G_1 \setminus \{x_1\} \subset F$ and $G_1 \setminus \{x_2\} \subset F$ or $G_1 \setminus \{x_1\} \subset F$ and $G_2 \setminus \{x_1\} \subset F$. But then (as x_1, x_2 should be in $(G_1 \cup G_2) \setminus \mathcal{R}(\mathcal{G}) = \bigcap \mathcal{G} \subseteq G_1 \cap G_2$ by the previous lemma) $G_1 \subset F$, which contradicts the Sperner property of $\mathcal{F}$. □

Lemma 3.5. *If $\mathcal{G}$ is an MRU subsystem of a Sperner system $\mathcal{F}$ and $F_1, F_2 \in \mathcal{F} \setminus \mathcal{G}$ with $\mathcal{R}(\{F_1, F_2\}) \geq 1$ and $\mathcal{R}(\mathcal{G}) \cap \mathcal{R}(\{F_1, F_2\}) = \emptyset$, then we have $|\mathcal{R}(\mathcal{G} \cup \{F_1, F_2\}) \setminus (\mathcal{R}(\mathcal{G}) \cup \mathcal{R}(\{F_1, F_2\}))| \leq 1$.*

Proof. If at least one of F_1, F_2 has the same size as the sets in $\mathcal{G}$, then the lemma follows from the previous one. So we have three cases: either $|G| < |F_1|, |F_2|$ or $|G| > |F_1|, |F_2|$ or $|F_1| < |G| < |F_2|$ for all $G \in \mathcal{G}$. It is easy to see that the first two cases are analogous.

Cases I and II

We just consider Case I. By Lemma 3.3 and Lemma 3.4 we know that if the statement of the lemma does not hold, then there should exist $G_1, G_2 \in \mathcal{G}$ and three different elements $x, y \in \bigcap \mathcal{G} \subseteq G_1 \cap G_2$ and $z \notin \bigcup \mathcal{G}$, such that $G_1 \setminus \{x\} \subset F_1$, $G_2 \setminus \{y\} \subset F_2$ and $F_1 \setminus \{z\} \subset F_2$. But then (as $y \in G_1$, thus $y \in F_1$ and thus $y \in F_2$) we would have $G_2 \subset F_2$ contradicting the Sperner property of $\mathcal{F}$.

Case III

By Lemma 3.3 and Lemma 3.4 we know that if the statement of the lemma does not hold, then there should exist $G_1, G_2 \in \mathcal{G}$ and three different elements $x \notin \bigcup \mathcal{G}$, $y \in \bigcap \mathcal{G} \subseteq G_1 \cap G_2$ and z either in $\bigcup \mathcal{G}$ or in $\bigcap \mathcal{G}$ (as $\mathcal{R}(\mathcal{G}) \cap \mathcal{R}(\{F_1, F_2\}) = \emptyset$), such that $F_1 \setminus \{x\} \subset G_1$, $G_2 \setminus \{y\} \subset F_2$ and $F_1 \setminus \{z\} \subset F_2$. Since $x \neq z$, we have $z \in G_1$ and thus $z \in G_1 \cap G_2$. Now $z \in F_2$ would contradict the Sperner property of $\mathcal{F}$ ($\mathcal{F}_1 \subset \mathcal{F}_2$), while $z \notin F_2$ would contradict $G_2 \setminus \{y\} \subset F_2$. □

Next we determine the values $g(n)$ for $2 \leq n \leq 8$.

Lemma 3.6. *If $2 \leq n \leq 6$, then $g(n) = 2n-2$. If $n = 7$ or $n = 8$, then $g(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.*

Proof. By Lemma 3.1, it suffices to show that $g(n) \leq 2n - 2$ for $2 \leq n \leq 6$ and $g(n) \leq \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ for $n = 7$ and $n = 8$. The cases $2 \leq n \leq 6$ can be handled similarly as (and much easily than) the cases $n = 7$ and $n = 8$ and therefore are left to the reader. In the cases $n = 7$ and $n = 8$ we perform a case by case analysis, where we consider cases according to $\gamma(\mathcal{F})$.

The case $n = 7$

Let $\mathcal{F}$ be a Sperner system of size 7 and let $\mathcal{G}$ be an MRU subsystem of $\mathcal{F}$ (if such a subsytem exists). We have to prove that $|\mathcal{R}(\mathcal{F})| \leq 12$.

If $\gamma(\mathcal{F}) = 7$, then $\mathcal{F}$ is uniform and we are done by Theorem 2.1.

If $\gamma(\mathcal{F}) = 6$, then we have $|\mathcal{R}(\mathcal{G})| \leq 10$ by Theorem 2.1, thus by Lemma 3.4 $|\mathcal{R}(\mathcal{F})| \leq 11$ holds.

If $\gamma(\mathcal{F}) = 5$, then we have $|\mathcal{R}(\mathcal{G})| \leq 8$ by Theorem 2.1 and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 2$ by the first (easy) part of the present lemma, thus by Lemma 3.4 $|\mathcal{R}(\mathcal{F})| \leq 12$ holds.

If $\gamma(\mathcal{F}) = 4$, then we have $|\mathcal{R}(\mathcal{G})| \leq 6$ by Theorem 2.1 and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 4$ by the first part of the present lemma. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G})| \leq 3$, then by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| \leq 12$. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G})| = 4$, then we have a pair of sets $F_1, F_2 \in \mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5 and thus by Lemma 3.4 and Lemma 3.5 we have $|\mathcal{R}(\mathcal{F})| \leq 12$.

If $\gamma(\mathcal{F}) = 3$, then we have $|\mathcal{R}(\mathcal{G})| \leq 4$ by Theorem 2.1 and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 6$ by the first part of the present lemma. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 4$, then we are done by Lemma 3.4. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 5$, then there is a pair of sets $F_1, F_2 \in \mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5, thus we are done by Lemma 3.4 and Lemma 3.5. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 6$, then it is easy to check that $\mathcal{F} \setminus \mathcal{G}$ can be divided into two pairs, both satisfying the assumptions of Lemma 3.5, thus we are done by Lemma 3.5.

If $\gamma(\mathcal{F}) = 2$, then we have $|\mathcal{R}(\mathcal{G})| = 2$ and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 8$ by the first part of the present lemma. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 8$ then it is not difficult to verify that $\mathcal{F} \setminus \mathcal{G}$ is an MRU subsystem of $\mathcal{F}$, so $\gamma(\mathcal{F}) \geq 5$, which is impossible now. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 7$, then it is easy to check that there are two disjoint pairs in $\mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5 and so we are done by Lemma 3.4 and Lemma 3.5. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 6$, then we need just one pair of sets in $\mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5 to finish the proof by Lemma 3.4 and Lemma 3.5 (and if $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| > 0$ then by definition there is such a pair). If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 5$, then we are done by Lemma 3.4.

The case $\gamma(\mathcal{F}) = 0$ is covered by Lemma 3.2.

The case $n = 8$

Let $\mathcal{F}$ be a Sperner system of size 8 and let $\mathcal{G}$ be an MRU subsystem of $\mathcal{F}$ (if such a subsytem exists). We have to prove that $|\mathcal{R}(\mathcal{F})| \leq 16$.

If $\gamma(\mathcal{F}) = 8$, then $\mathcal{F}$ is uniform and we are done by Theorem 2.1.

If $4 \leq \gamma(\mathcal{F}) \leq 7$, then we have $|\mathcal{R}(\mathcal{G})| \leq 2\gamma(\mathcal{F}) - 2$ by Theorem 2.1 and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 2(8 - \gamma(\mathcal{F})) - 2$ by the first part of the present lemma, thus by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| \leq (2\gamma(\mathcal{F}) - 2) + (2(8 - \gamma(\mathcal{F})) - 2) + (8 - \gamma(\mathcal{F})) = 20 - \gamma(\mathcal{F}) \leq 16$.

If $\gamma(\mathcal{F}) = 3$, then we have $|\mathcal{R}(\mathcal{G})| \leq 4$ and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 8$ by the first part of the present lemma and if $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| > 0$, then there is a pair of sets in $\mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5 and thus by Lemma 3.4 and Lemma 3.5 we have $|\mathcal{R}(\mathcal{F})| \leq 16$. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 0$, then by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| \leq 9$.

If $\gamma(\mathcal{F}) = 2$, then we have $|\mathcal{R}(\mathcal{G})| = 2$ and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| \leq 10$ by the first part of the present lemma and if $8 \geq |\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| > 0$, then there is a pair of sets in $\mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5 and thus by Lemma 3.4 and Lemma 3.5 $|\mathcal{R}(\mathcal{F})| \leq 16$ holds. If $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 0$, then by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| \leq 8$. It is easy to check that if $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 9$ (resp. $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 10$), then there are 2 (resp. 3) pairwise disjoint pairs of sets satisfying the assumptions of Lemma 3.5 and therefore by Lemma 3.4 and Lemma 3.5 we have $|\mathcal{R}(\mathcal{F})| \leq 16$.

The case $\gamma(\mathcal{F}) = 0$ is again covered by Lemma 3.2 and the proof of our lemma is finished. □

Now we are in a position to prove our main theorem.

Theorem 3.7. *If* $2 \leq n \leq 6$, *then*

$$g(n) = 2n - 2,$$

if $n \geq 7$, *then*

$$g(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil.$$

Proof. The cases $n \leq 8$ are covered by Lemma 3.6. Now we prove that for any $n \geq 9$ we have $|\mathcal{R}(\mathcal{F})| < f(n) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil + 1$, provided $\mathcal{F}$ is a Sperner family of n sets. We proceed by induction, so let us suppose we have already proved the statement for all $7 \leq n' < n$. Let $\mathcal{G}$ be an MRU subsystem of $\mathcal{F}$, if such a subsytem exists. We consider three cases according to $\gamma(\mathcal{F}) = |\mathcal{G}|$.

If $\gamma(\mathcal{F}) = 0$ (that is, no MRU subsystem of $\mathcal{F}$ exists), then we are done by Lemma 3.2.

The case $\gamma(\mathcal{F}) = 2$

Now we have $|\mathcal{R}(\mathcal{G})| = 2$ and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| < f(n-2)$, by the inductive hypothesis. Since $f(n) - f(n-2) = n - 1$, for $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| > 0$, by definition there is a pair in $\mathcal{F} \setminus \mathcal{G}$ satisfying the assumptions of Lemma 3.5, so we have (using Lemma 3.4 and Lemma 3.5) $|\mathcal{R}(\mathcal{F})| < 2 + f(n-2) + n - 3 = f(n)$. For $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| = 0$, by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| \leq 2 + (n-2) = n < f(n)$, if $n \geq 9$.

The case $\gamma(\mathcal{F}) \geq 3$

Now we have $|\mathcal{R}(\mathcal{G})| \leq 2\gamma(\mathcal{F}) - 2$, by Theorem 2.1 and $|\mathcal{R}(\mathcal{F} \setminus \mathcal{G}) \setminus \mathcal{R}(\mathcal{G})| < \max\{2(n - \gamma(\mathcal{F})) - 1, f(n - \gamma(\mathcal{F})\}$, by the inductive hypothesis (the term $2(n - \gamma(\mathcal{F}) - 1$ is needed when $n - \gamma(\mathcal{F}) \leq 6$). Thus by Lemma 3.4 we have $|\mathcal{R}(\mathcal{F})| < 2\gamma(\mathcal{F}) - 2 + \max\{2(n - \gamma(\mathcal{F})) - 1, f(n - \gamma(\mathcal{F}))\} + n - \gamma(\mathcal{F}) = n + \gamma(\mathcal{F}) - 2 + \max\{2(n - \gamma(\mathcal{F})) - 1, f(n - \gamma(\mathcal{F}))\}$.

If $n = 9$, then $2(n - \gamma(\mathcal{F})) - 1 \geq f(n - \gamma(\mathcal{F}))$, as $\gamma(\mathcal{F}) \geq 3$. Therefore $|\mathcal{R}(\mathcal{F})| < 3n - \gamma(\mathcal{F}) - 3 \leq 21 = f(9)$ as desired.

If $n = 10$, then $2(n - \gamma(\mathcal{F})) - 1 \geq f(n - \gamma(\mathcal{F}))$, as $\gamma(\mathcal{F}) \geq 3$. Therefore $|\mathcal{R}(\mathcal{F})| < 3n - \gamma(\mathcal{F}) - 3 \leq 24 < f(10)$ as desired.

If $n = 11$, then for $\gamma(\mathcal{F}) \geq 4$ we have $2(n - \gamma(\mathcal{F})) - 1 \geq f(n - \gamma(\mathcal{F}))$, thus $|\mathcal{R}(\mathcal{F})| < 3n - \gamma(\mathcal{F}) - 3 \leq 26 < f(11)$, while for $\gamma(\mathcal{F}) = 3$ we have $2(n - \gamma(\mathcal{F})) - 1 < f(n - \gamma(\mathcal{F}))$ thus $|\mathcal{R}(\mathcal{F})| < 11 + 3 - 2 + f(8) = 28 < f(11)$.

For the remaining cases (i.e. $n \geq 12$) we use the fact $\max\{2(n - \gamma(\mathcal{F})) - 1, f(n - \gamma(\mathcal{F}))\} \leq f(n - \gamma(\mathcal{F})) + 2$ (the function $2k - 1 - f(k)$ attains its maximum at $k = 3, 4, 5$ with maximum value 2). Therefore we have $|\mathcal{R}(\mathcal{F})| < 2\gamma(\mathcal{F}) - 2 + f(n - \gamma(\mathcal{F})) + 2 + n - \gamma(\mathcal{F}) = f(n - \gamma(\mathcal{F})) + n + \gamma(\mathcal{F}) \leq f(n)$, where the last inequality follows from the obvious inequalities $f(n) - f(4) \geq f(n) - f(n-3) = \lfloor \frac{3}{2} n \rfloor - 2$ (valid for all n), and $f(n) - f(n - \gamma(\mathcal{F})) \geq f(n) - f(n-5) \geq 2n$, which holds if $n \geq 12$ and $\gamma(\mathcal{F}) \geq 5$. □

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