A Hilton-Milner Theorem for Vector Spaces

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Abstract

We show for $k \geq 2$ that if $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$, then any intersecting family $\mathcal{F}$ of k-subspaces of an n-dimensional vector space over $GF(q)$ with $\bigcap_{F \in \mathcal{F}} F = 0$ has size at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k$. This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding q-Kneser graphs.

1 Introduction

1.1 Sets

Let X be an n-element set and, for $0 \leq k \leq n$, let $\binom{X}{k}$ denote the family of all subsets of X of cardinality k. A family $\mathcal{F} \subset \binom{X}{k}$ is called *intersecting* if for all $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \neq \emptyset$. Erdős, Ko, and Rado [7] determined the maximum size of an intersecting family, and introduced the so-called shifting technique.

Theorem 1.1 (Erdős-Ko-Rado) *Suppose $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Excepting the case $n = 2k$, equality holds only if $\mathcal{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$.*

For any family $\mathcal{F} \subset \binom{X}{k}$, the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. Theorem 1.1 shows that if $\mathcal{F}$ is an intersecting family of maximum size and $n > 2k$, then $\tau(\mathcal{F}) = 1$.

Hilton and Milner [17] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geq 2$. Later, Frankl and Füredi [11] gave an elegant proof of Theorem 1.2 using the shifting technique.

Theorem 1.2 (Hilton-Milner) *Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geq 2$, $n \geq 2k+1$, and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if*

(i) $\mathcal{F} = \{F\} \cup \{G \in \binom{X}{k} : x \in G,\ F \cap G \neq \emptyset\}$ for some k-subset F and $x \in X \setminus F$.

(ii) $\mathcal{F} = \{F \in \binom{X}{3} : |F \cap S| \geq 2\}$ for some 3-subset S if $k = 3$.

1.2 Vector spaces

Theorem 1.1 and Theorem 1.2 have natural extensions to vector spaces. We let V always denote a n-dimensional vector space over the finite field $GF(q)$. For $k \in \mathbb{Z}^+$, we write $\begin{bmatrix} V \\ k \end{bmatrix}_q$ to denote the family of all k-dimensional subspaces of V. For $a, k \in \mathbb{Z}^+$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \leq i < k} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$

A simple counting argument shows that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From now on, we will omit the subscript q.

A family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is called intersecting if for all $F_1, F_2 \in \mathcal{F}$, we have $F_1 \cap F_2 \neq 0$. The maximum size of an intersecting family of k-spaces was first determined by Hsieh [18]. For alternate proofs of Theorem 1.3, see [4] and [13].

Theorem 1.3 (Hsieh) *Suppose $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Equality holds if and only if $\mathcal{F} = \left\{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\right\}$ for some one-dimensional subspace $v \subset V$, unless $n = 2k$.*

We remark that there is as yet no analog of the shifting technique for vector spaces. Indeed, there were two attempts ([5], [6]) to prove Theorem 1.3 using a generalization of the shifting technique. Unfortunately, both these generalizations of shifting are flawed, and these proofs are not valid.

Let the *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be defined as the minimum dimension of a subspace of V that meets all elements of $\mathcal{F}$ nontrivially. Theorem 1.3 shows that, as in the set case, if $\mathcal{F}$ is a maximum intersecting family of k-spaces, then $\tau(\mathcal{F}) = 1$. Families satisfying $\tau(\mathcal{F}) = 1$ are known as *point-pencils*.

In this paper, we will extend Theorem 1.2 to vector spaces, and determine the maximum size of an intersecting family of subspaces with $\tau(\mathcal{F}) \geq 2$. Let us first remark that for a fixed 1-subspace $E \leqslant V$ and a k-subspace U with $E \nleqslant U$ the family

$$\mathcal{F}_{E,U} = \{U\} \cup \{W \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right] : E \leqslant W, \ \dim(W \cap U) \geq 1\}$$

is not maximal as we can add all subspaces in $\left[\begin{smallmatrix} E+U \\ k \end{smallmatrix}\right]$. We will say that $\mathcal{F}$ is an *HM-type family* if

$$\mathcal{F} = \left\{W \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right] : E \leqslant W, \ \dim(W \cap U) \geq 1\right\} \cup \left[\begin{smallmatrix} E+U \\ k \end{smallmatrix}\right]$$

for some $E \in \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix}\right]$ and $U \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ with $E \nleqslant U$. If $\mathcal{F}$ is an HM-type family, then its size is

$$|\mathcal{F}| = f(n,k,q) := \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k. \tag{1.1}$$

The main result of the paper is the following theorem.

Theorem 1.4 *Suppose $k \geq 3$, and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. For any intersecting family $\mathcal{F} \subseteq \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ with $\tau(\mathcal{F}) \geq 2$, we have $|\mathcal{F}| \leq f(n,k,q)$ (with $f(n,k,q)$ as in (1.1)). Equality holds only if*

(i) $\mathcal{F}$ is an HM-type family,

(ii) $\mathcal{F} = \mathcal{F}_3 = \{F \in \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right] : \dim(S \cap F) \geq 2\}$ for some $S \in \left[\begin{smallmatrix} V \\ 3 \end{smallmatrix}\right]$ if $k = 3$.

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of n, k, q) such that if $|\mathcal{F}| \geq (1-\epsilon) f(n,k,q)$, then $\mathcal{F}$ is a subfamily of an HM-type family.

If $k = 2$, then a maximal intersecting family $\mathcal{F}$ of k-spaces with $\tau(\mathcal{F}) > 1$ is the family of all lines in a plane, and the conclusion of the theorem holds.

After proving Theorem 1.4 in Section 2, we apply this result to determine the chromatic number of q-Kneser graphs. The vertex set of the q-Kneser graph $qK_{n:k}$ is $\left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$. Two vertices of $qK_{n:k}$ are adjacent if and only if the corresponding k-subspaces are disjoint (i.e., meet in the zero subspace). In [3], the chromatic number of the q-Kneser graph $qK_{n:2}$ is determined, and the minimum colorings are characterized. In [20], the chromatic number of the q-Kneser graph is determined in general for $q > q_k$. In Section 4, we prove the following theorem.

Theorem 1.5 *If $k \geq 3$, and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$, then the chromatic number of the q-Kneser graph is $\chi(qK_{n:k}) = \left[\begin{smallmatrix} n-k+1 \\ 1 \end{smallmatrix}\right]$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n-k+1)$-dimensional subspace.*

In Section 5, we prove the non-uniform version of the Erdős-Ko-Rado theorem.

Theorem 1.6 *Let $\mathcal{F}$ be an intersecting family of subspaces of V.*

(i) If n is even, then $|\mathcal{F}| \le \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix} + \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}$.

(ii) If n is odd, then $|\mathcal{F}| \le \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}$.

For even n, equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, or if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \begin{bmatrix} U \\ n/2 \end{bmatrix}$ for some $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$. For odd n, equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix}$.

Note that Theorem 1.6 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [14], but the proof we present in Section 5 is simple and direct.

2 Proof of Theorem 1.4

This section contains the proof of Theorem 1.4 which we divide into two cases.

2.1 The case $\tau(\mathcal{F}) = 2$

For any $A \leqslant V$ and $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leqslant F\}$. First, let us state some easy technical lemmas.

Lemma 2.1 *Let $a \ge 0$ and $n \ge k \ge a+1$ and $q \ge 2$. Then*

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-a-1 \\ k-a-1 \end{bmatrix} < \frac{1}{(q-1)q^{n-2k}} \begin{bmatrix} n-a \\ k-a \end{bmatrix}.$$

Proof. The inequality to be proved simplifies to

$$(q^{k-a}-1)(q^k-1)q^{n-2k} < q^{n-a}-1. \qquad \square$$

Lemma 2.2 *Let $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$. If $E \not\leqslant L \le V$, where L is an l-subspace, then the number of k-subspaces of V containing E and intersecting L is at least $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} l \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ (with equality for $l = 2$), and at most $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The k-spaces containing E and intersecting L in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect L in a 2-dimensional space are counted $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q+1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects L in a subspace of dimension $i \ge 3$, then it is counted $\begin{bmatrix} i \\ 1 \end{bmatrix}$ times in the first term and $-q\begin{bmatrix} i \\ 2 \end{bmatrix}$ times in the second term, and hence a negative number of times overall. $\square$

Our next lemma gives bounds on the size of an HM-type family that are easier to work with than the precise formula mentioned in the introduction.

Lemma 2.3 *Let $n \ge 2k+1$, $k \ge 3$ and $q \ge 2$. If $\mathcal{F}$ is an HM-type family, then $(1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} k \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \le f(n,k,q) = |\mathcal{F}| \le \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The first inequality follows immediately from Lemma 2.1 by noting that $q\begin{bmatrix}k\\2\end{bmatrix} = \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix}-1)/(q+1)$ and $n \geq 2k+1$. □

Lemma 2.4 *If a subspace S does not intersect each element of $\mathcal{F}$, then there is a subspace $T > S$ with $\dim T = \dim S + 1$ and $|\mathcal{F}_T| \geq |\mathcal{F}_S|/\begin{bmatrix}k\\1\end{bmatrix}$.*

Proof. There is an $F \in \mathcal{F}$ such that $S \cap F = 0$. Average over all $T = S + E$ where E is a 1-subspace of F. □

Lemma 2.5 *If an s-dimensional subspace S does not intersect each element of $\mathcal{F}$, then $|\mathcal{F}_S| \leq \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}$.*

Proof. There is an $(s+1)$-space T with $\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix} \geq |\mathcal{F}_T| \geq |\mathcal{F}_S|/\begin{bmatrix}k\\1\end{bmatrix}$. □

Corollary 2.6 *Let $\mathcal{F} \subseteq \begin{bmatrix}V\\k\end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq s$. Then for any i-space $L \leqslant V$ with $i \leq s$ we have $|\mathcal{F}_L| \leq \begin{bmatrix}k\\1\end{bmatrix}^{s-i}\begin{bmatrix}n-s\\k-s\end{bmatrix}$.* □

Proof. By $\tau(\mathcal{F}) \geq s$ we know that for any j-space A, $j < s$, there exists an $F \in \mathcal{F}$ disjoint from A. Now apply Lemma 2.4 $s-i$ times. □

Before proving the q-analogue of the Hilton-Milner theorem, we describe the essential part of maximal intersecting families with $\tau(\mathcal{F}) = 2$.

Proposition 2.7 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Define $\mathcal{T}$ to be the family of 2-spaces of V that intersect all subspaces in $\mathcal{F}$. One of the following three possibilities holds:*

(i) $|\mathcal{T}| = 1$ *and* $\begin{bmatrix}n-2\\k-2\end{bmatrix} < |\mathcal{F}| < \begin{bmatrix}n-2\\k-2\end{bmatrix} + (q+1)\left(\begin{bmatrix}k\\1\end{bmatrix}-1\right)\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$;

(ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is an $(l+1)$-space W (with $2 \leq l \leq k$) and a 1-space $E \leqslant W$ so that $\mathcal{T} = \{M : E \leqslant M \leqslant W,\ \dim M = 2\}$. In this case,*
$\begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} - q\begin{bmatrix}l\\2\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix} \leq |\mathcal{F}| \leq \begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix} - \begin{bmatrix}l\\1\end{bmatrix})\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^l\begin{bmatrix}n-l\\k-l\end{bmatrix}$.
For $l = 2$, the upper bound can be strengthened to
$|\mathcal{F}| \leq (q+1)\begin{bmatrix}n-2\\k-2\end{bmatrix} - q\begin{bmatrix}n-3\\k-3\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix})\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^2\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$;

(iii) $\mathcal{T} = \begin{bmatrix}A\\2\end{bmatrix}$ *for some 3-subspace A and $\mathcal{F} = \{U \in \begin{bmatrix}V\\k\end{bmatrix} : \dim(U \cap A) \geq 2\}$. In this case,*
$|\mathcal{F}| = (q^2+q+1)(\begin{bmatrix}n-2\\k-2\end{bmatrix} - \begin{bmatrix}n-3\\k-3\end{bmatrix}) + \begin{bmatrix}n-3\\k-3\end{bmatrix}$.

Proof. Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. By maximality, $\mathcal{F}$ contains all k-spaces containing a $T \in \mathcal{T}$. Since $n \geq 2k$ and $k \geq 2$, two disjoint elements of $\mathcal{T}$ would be contained in disjoint elements of $\mathcal{F}$, which is impossible. Hence, $\mathcal{T}$ is intersecting.

Observe that if $A, B \in \mathcal{T}$ and $A \cap B < C < A + B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space E or a family of 2-subspaces of a 3-space, we get the following:

$(*)$: $\mathcal{T}$ is either a family of all 2-subspaces in a given $(l+1)$-space containing some fixed 1-space E (and $k \geq l \geq 1$), or $\mathcal{T}$ is the family of all 2-subspaces of a 3-space.

(i) : If $|\mathcal{T}| = 1$, then let S denote the only 2-space in $\mathcal{T}$ and let $E \leqslant S$ be any 1-space. Since $\tau(\mathcal{F}) > 1$, there exists an $F \in \mathcal{F}$ with $E \not\leqslant F$, for which we must have $\dim(F \cap S) = 1$. As S is the only element of $\mathcal{T}$, for any 1-subspace E' of F different from $F \cap S$, we have $\mathcal{F}_{E+E'} \le \genfrac{[}{]}{0pt}{}{k}{1}\genfrac{[}{]}{0pt}{}{n-3}{k-3}$ by Lemma 2.5. Hence the number of subspaces containing E but not containing S is at most $(\genfrac{[}{]}{0pt}{}{k}{1} - 1)\genfrac{[}{]}{0pt}{}{k}{1}\genfrac{[}{]}{0pt}{}{n-3}{k-3}$. This gives the upper bound.

(ii) : Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By $(*)$, $\mathcal{T}$ is the set of 2-spaces in an $(l+1)$-space W (with $l \ge 2$) containing some fixed 1-space E. Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects W in a hyperplane. Let L be a hyperplane in W not on E. Then $\mathcal{F}$ contains all k-spaces on E that intersect L. Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from counting the k-spaces of $\mathcal{F}$ that contain E and intersect a given $F \in \mathcal{F}$ (not containing E) in a point of $F \setminus W$. Here Lemma 2.5 is used. If $l \ge 3$, then there are q^l hyperplanes in W not containing E and there are $\genfrac{[}{]}{0pt}{}{n-l}{k-l}$ k-spaces through such a hyperplane. For $l = 2$, we use the tight lower bound in Lemma 2.2 to count the number of k-spaces on E that intersect L. There are q^2 hyperplanes in W, and they cannot be in $\mathcal{T}$, so Lemma 2.5 gives the bound.

(iii) : This is immediate. □

Corollary 2.8 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Suppose $q \ge 3$ and $n \ge 2k+1$, or $q = 2$ and $n \ge 2k+2$. If $\mathcal{F}$ is at least as large as an HM-type family and $k > 3$, then $\mathcal{F}$ is an HM-type family. If $k = 3$, then $\mathcal{F}$ is an HM-type family or an $\mathcal{F}_3$-type family.*

There exists an $\epsilon > 0$ (independent of n, k, q) such that if $k \ge 4$ and $|\mathcal{F}|$ is at least $(1-\epsilon)$ times the size of an HM-type family, then $\mathcal{F}$ is a subfamily of an HM-type family.

Proof. Apply Proposition 2.7. Note that the Hilton-Milner families are precisely those from case (ii) with $k = l$.

Let $n \ge 2k + a$ where $a \ge 1$. We have $|\mathcal{F}|/\genfrac{[}{]}{0pt}{}{n-2}{k-2} < 1 + \frac{q+1}{(q-1)q^a}\genfrac{[}{]}{0pt}{}{k}{1}$ in case (i) of Proposition 2.7 by Lemma 2.1. We have $|\mathcal{F}|/\genfrac{[}{]}{0pt}{}{n-2}{k-2} < (\frac{1}{q} + \frac{1}{(q-1)q^a})\genfrac{[}{]}{0pt}{}{k}{1} + \frac{q^2}{(q-1)q^a}$ in case (ii) when $l < k$. In both cases, for $q \ge 3$ and $k \ge 3$, or $q = 2$, $k \ge 4$, and $a \ge 2$, this is less than $(1-\epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. Using the stronger estimate in Lemma 2.3, we find the same conclusion for $q = 2$, $k = 3$, and $a \ge 2$.

In case (iii), $|\mathcal{F}_3| = \genfrac{[}{]}{0pt}{}{3}{2}\genfrac{[}{]}{0pt}{}{n-2}{k-2} - \frac{q^3-q}{q-1}\genfrac{[}{]}{0pt}{}{n-3}{k-3}$. For $k \ge 4$, this is much smaller than the size of the HM-type families. For $k = 3$, the two families have the same size. □

Proposition 2.9 *Suppose that $k \ge 3$ and $n \ge 2k$. Let $\mathcal{F} \subseteq \genfrac{[}{]}{0pt}{}{V}{k}$ be an intersecting family with $\tau(\mathcal{F}) \ge 2$. Let $3 \le l \le k$. If there is an l-space that intersects each $F \in \mathcal{F}$ and*

$$|\mathcal{F}| > \genfrac{[}{]}{0pt}{}{l}{1}\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l}{k-l}, \tag{2.2}$$

then there is an $(l-1)$-space that intersects each $F \in \mathcal{F}$.

Proof. By averaging there is a 1-space P with $|\mathcal{F}_P| \ge |\mathcal{F}|/\genfrac{[}{]}{0pt}{}{l}{1}$. If $\tau(\mathcal{F}) = l$, then by Corollary 2.6, $|\mathcal{F}| \le \genfrac{[}{]}{0pt}{}{l}{1}\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l}{k-l}$, contradicting the hypothesis. □

Corollary 2.10 *Suppose $k \geq 3$ and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. If $|\mathcal{F}| > \left[{3 \atop 1}\right]\left[{k \atop 1}\right]^2\left[{n-3 \atop k-3}\right]$, then $\tau(\mathcal{F}) = 2$; that is, $\mathcal{F}$ is contained in one of the systems in Proposition 2.7, which satisfy the bound on $|\mathcal{F}|$.*

Proof. By Lemma 2.1 and the conditions on n and q, the right hand side of (2.2) is decreasing in l for $3 \leq l \leq k$. Hence, by Proposition 2.9, we can find a 2-space that intersects each $F \in \mathcal{F}$. □

Remark 2.11 For $n \geq 3k$, all systems described in Proposition 2.7 occur.

2.2 The case $\tau(\mathcal{F}) > 2$

Suppose that $\mathcal{F}$ is an intersecting family and $\tau(\mathcal{F}) = l > 2$. We shall derive a contradiction from $|\mathcal{F}| \geq f(n,k,q)$, and even from $|\mathcal{F}| \geq (1-\epsilon)f(n,k,q)$ for some $\epsilon > 0$ (independent of n,k,q).

For each point P we have $|\mathcal{F}_P| \leq \left[{k \atop 1}\right]^{l-1}\left[{n-l \atop k-l}\right]$, and for each line L we have $|\mathcal{F}_L| \leq \left[{k \atop 1}\right]^{l-2}\left[{n-l \atop k-l}\right]$, by Corollary 2.6.

If there are two l-spaces that meet all $F \in \mathcal{F}$, and these meet in an m-space, where $0 \leq m \leq l-1$, then

$$|\mathcal{F}| \leq \left[{m \atop 1}\right]\left[{k \atop 1}\right]^{l-1}\left[{n-l \atop k-l}\right] + \left(\left[{l \atop 1}\right] - \left[{m \atop 1}\right]\right)^2\left[{k \atop 1}\right]^{l-2}\left[{n-l \atop k-l}\right]. \tag{2.3}$$

2.2.1 The case $k = l$

First consider the case $k = l$. Then $|\mathcal{F}| \leq \left[{k \atop 1}\right]^k$ by Corollary 2.6. On the other hand,

$$|\mathcal{F}| \geq \left(1 - \frac{1}{q^3-q}\right)\left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right] > \left(1 - \frac{1}{q^3-q}\right)\left[{k \atop 1}\right]^{k-1}\left((q-1)q^{n-2k}\right)^{k-2}$$

by Lemma 2.3 and Lemma 2.1. If either $q > 2$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, then either $k \leq 3$, $(n,k,q) = (9,4,3)$, or $(n,k,q) = (10,4,2)$. If $(n,k,q) = (9,4,3)$ then $f(n,k,q) = 3837721$, and $40^4 = 2560000$, which gives a contradiction. If $(n,k,q) = (10,4,2)$, then $f(n,k,q) = 153171$, and $15^4 = 50625$, which again gives a contradiction. Hence $k = 3$. Now $|\mathcal{F}| \geq (1 - \frac{1}{q^3-q})\left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right]$ gives a contradiction for $n \geq 8$, so $n = 7$. Therefore, if we assume that $n \geq 2k+1$ and either $q > 2$, $(n,k) \neq (7,3)$ or $q = 2$, $n \geq 2k+2$ then we are not in the case $k = l$.

It remains to settle the case $n = 7$, $k = l = 3$, and $q > 2$. Pick a 1-space E such that $|\mathcal{F}_E| \geq |\mathcal{F}|/\left[{3 \atop 1}\right]$ and a 2-space S on E such that $|\mathcal{F}_S| \geq |\mathcal{F}_E|/\left[{3 \atop 1}\right]$. Then $|\mathcal{F}_S| > q+1$ since $|\mathcal{F}| > \left[{2 \atop 1}\right]\left[{3 \atop 1}\right]^2$. Pick $F' \in \mathcal{F}$ disjoint from S and define $H := S + F'$. All $F \in \mathcal{F}_S$ are contained in the 5-space H. Since $|\mathcal{F}| > \left[{5 \atop 3}\right]$, there is an $F_0 \in \mathcal{F}$ not contained in H. If $F_0 \cap S = 0$, then each $F \in \mathcal{F}_S$ is contained in $S + (H \cap F_0)$; this implies $|\mathcal{F}_S| \leq q+1$, which is impossible. Thus, all elements of $\mathcal{F}$ disjoint from S are in H.

Now F_0 must meet F' and S, so F_0 meets H in a 2-space S_0. Since $|\mathcal{F}_S| > q+1$, we can find two elements F_1, F_2 of $\mathcal{F}_S$ with the property that S_0 is not contained in the 4-space $F_1 + F_2$. Since any $F \in \mathcal{F}$ disjoint from S is contained in H and meets F_0, it must meet S_0 and also F_1 and F_2. Hence the number of such F's is at most q^5. Altogether $|\mathcal{F}| \le q^5 + \genfrac{[}{]}{0pt}{}{2}{1}\genfrac{[}{]}{0pt}{}{3}{1}^2$; the first term comes from counting $F \in \mathcal{F}$ disjoint from S and the second term comes from counting $F \in \mathcal{F}$ on a given one-dimensional subspace $E < S$. This contradicts $|\mathcal{F}| \ge (1 - \frac{1}{q^3-q})\genfrac{[}{]}{0pt}{}{3}{1}\genfrac{[}{]}{0pt}{}{5}{1}$.

2.2.2 l is small

The upper bound (2.3) is a quadratic in $x = \genfrac{[}{]}{0pt}{}{m}{1}$ and is largest at one of the extreme values $x = 0$ and $x = \genfrac{[}{]}{0pt}{}{l-1}{1}$. The maximum is taken at $x = 0$ only when $\genfrac{[}{]}{0pt}{}{l}{1} - \frac{1}{2}\genfrac{[}{]}{0pt}{}{k}{1} > \frac{1}{2}\genfrac{[}{]}{0pt}{}{l-1}{1}$; that is, when $k = l$. Since we assume that $l < k$, the upper bound in (2.3) is largest for $m = l - 1$. We find

$$|\mathcal{F}| \le \genfrac{[}{]}{0pt}{}{l-1}{1}\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l}{k-l} + (\genfrac{[}{]}{0pt}{}{l}{1} - \genfrac{[}{]}{0pt}{}{l-1}{1})^2\genfrac{[}{]}{0pt}{}{k}{1}^{l-2}\genfrac{[}{]}{0pt}{}{n-l}{k-l}.$$

On the other hand,

$$|\mathcal{F}| \ge (1 - \tfrac{1}{q^3-q})\genfrac{[}{]}{0pt}{}{k}{1}\genfrac{[}{]}{0pt}{}{n-2}{k-2} > (1 - \tfrac{1}{q^3-q})\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l}{k-l}((q-1)q^{n-2k})^{l-2}.$$

Comparing these, and using $k > l$, $n \ge 2k+1$, and $n \ge 2k+2$ if $q = 2$, we find either $(n,k,l,q) = (9,4,3,3)$ or $q = 2$, $n = 2k+2$, $l = 3$, and $k \le 5$. If $(n,k,l,q) = (9,4,3,3)$ then $f(n,k,q) = 3837721$, while the upper bound is 3508960, which is a contradiction. If $(n,k,l,q) = (12,5,3,2)$ then $f(n,k,q) = 183628563$, while the upper bound is 146766865, which is a contradiction. If $(n,k,l,q) = (10,4,3,2)$ then $f(n,k,q) = 153171$, while the upper bound is 116205, which is a contradiction. Under our assumption that there are two distinct l-spaces that meet all $F \in \mathcal{F}$, the case $2 < l < k$ cannot occur.

2.2.3 A unique l-space

We now assume that there is a unique l-space T that meets all $F \in \mathcal{F}$. We can pick a 1-space $E < T$ such that $|\mathcal{F}_E| \ge |\mathcal{F}|/\genfrac{[}{]}{0pt}{}{l}{1}$. Now there is some $F' \in \mathcal{F}$ not on E, so E is in $\genfrac{[}{]}{0pt}{}{k}{1}$ lines such that each $F \in \mathcal{F}_E$ contains at least one of these lines. Suppose L is one of these lines and L does not lie in T; we can enlarge L to an l-space that still does not meet all elements of $\mathcal{F}$, so $|\mathcal{F}_L| \le \genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l-1}{k-l-1}$ by Lemma 2.4 and Lemma 2.5. If L does lie on T, we have $|\mathcal{F}_L| \le \genfrac{[}{]}{0pt}{}{k}{1}^{l-2}\genfrac{[}{]}{0pt}{}{n-l}{k-l}$ by Corollary 2.6. Hence,

$$|\mathcal{F}| \le \genfrac{[}{]}{0pt}{}{l}{1}|\mathcal{F}_E| \le \genfrac{[}{]}{0pt}{}{l}{1}\left(\genfrac{[}{]}{0pt}{}{l-1}{1}(\genfrac{[}{]}{0pt}{}{k}{1}^{l-2}\genfrac{[}{]}{0pt}{}{n-l}{k-l}) + (\genfrac{[}{]}{0pt}{}{k}{1} - \genfrac{[}{]}{0pt}{}{l-1}{1})(\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l-1}{k-l-1})\right).$$

On the other hand, we have $|\mathcal{F}| > \left(1 - \frac{1}{q^3-q}\right)((q-1)q^{n-2k})^{l-2}\genfrac{[}{]}{0pt}{}{k}{1}^{l-1}\genfrac{[}{]}{0pt}{}{n-l}{k-l}$. Under our standard assumptions $n \ge 2k+1$ and $n \ge 2k+2$ if $q = 2$, this implies $q = 2$, $n = 2k+2$, $l = 3$, which gives a contradiction. We showed: If $n \ge 2k+1$ and $n \ge 2k+2$ if $q = 2$, then $\tau(\mathcal{F}) \le 2$. Together with Corollary 2.8, this proves Theorem 1.4.

3 Critical families

A subspace will be called a *hitting subspace* (and we shall say that the subspace intersects $\mathcal{F}$), if it intersects each element of $\mathcal{F}$.

The previous results just used the parameter τ, so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

Definition 3.1 An intersecting family $\mathcal{F}$ of subspaces of V is *critical* if for any two distinct $F, F' \in \mathcal{F}$ we have $F \not\subset F'$, and moreover for any hitting subspace G there is a $F \in \mathcal{F}$ with $F \subset G$.

Lemma 3.2 *For every non-extendable intersecting family $\mathcal{F}$ of k-spaces there exists some critical family $\mathcal{G}$ such that*

$$\mathcal{F} = \{F \in \left[{V \atop k}\right] : \exists\, G \in \mathcal{G},\, G \subseteq F\}.$$

Proof. Extend $\mathcal{F}$ to a maximal intersecting family $\mathcal{H}$ of subspaces of V, and take for $\mathcal{G}$ the minimal elements of $\mathcal{H}$. $\square$

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [8]:

Construction 3.3 *Let $A_1, \ldots, A_k$ be subspaces of V such that* $\dim A_i = i$ *and* $\dim(A_1 + \cdots + A_k) = \binom{k+1}{2}$. *Define*

$$\mathcal{F}_i = \{F \in \left[{V \atop k}\right] : A_i \subseteq F,\ \ \dim A_j \cap F = 1 \text{ for } j > i\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ is a critical, non-extendable, intersecting family of k-spaces, and $|\mathcal{F}_i| = \left[{i+1 \atop 1}\right]\left[{i+2 \atop 1}\right] \cdots \left[{k \atop 1}\right]$ for $1 \le i \le k$.

For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of k-sets cannot have more than k^k members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [12]. Here we prove the following analogous result.

Theorem 3.4 *Let $\mathcal{F}$ be a critical, intersecting family of subspaces of V of dimension at most k. Then $|\mathcal{F}| \le \left[{k \atop 1}\right]^k$.*

Proof. Suppose that $|\mathcal{F}| > \left[{k \atop 1}\right]^k$. By induction on i, $0 \le i \le k$, we find an i-dimensional subspace A_i of V such that $|\mathcal{F}_{A_i}| > \left[{k \atop 1}\right]^{k-i}$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and $\mathcal{F}$ is critical, the subspace A_i is not hitting, and there is an $F \in \mathcal{F}$ disjoint from A_i. Now all elements of $\mathcal{F}_{A_i}$ meet F, and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}| / \left[{k \atop 1}\right]$. For $i = k$ this is a contradiction. $\square$

Remark 3.5 For $l \le k$ this argument shows that there are not more than $\left[{l \atop 1}\right]\left[{k \atop 1}\right]^{l-1}$ l-spaces in $\mathcal{F}$.

If $l = 3$ and $\tau > 2$ then for the size of $\mathcal{F}$ the previous remark essentially gives $\left[{3 \atop 1}\right]\left[{k \atop 1}\right]^2\left[{n-3 \atop k-3}\right]$, which is the bound in Corollary 2.10.

Modifying the Erdős-Lovász construction (see Frankl [9]), one can get intersecting families with many l-spaces in the corresponding critical family.

Construction 3.6 *Let $A_1, \ldots, A_l$ be subspaces with $\dim A_1 = 1$, $\dim A_i = k + i - l$ for $i \geq 2$. Define $\mathcal{F}_i = \{F \in \left[{V \atop k}\right] : A_i \leq F, \dim(F \cap A_j) \geq 1$ for $j > i\}$. Then $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l$ is intersecting and the corresponding critical family has at least $\left[{k-l+2 \atop 1}\right] \cdots \left[{k \atop 1}\right]$ l-spaces.*

For n large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from $(*)$ which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\left[{n \atop k-3}\right])$, if k is fixed and n is large enough. Stronger and more general stability theorems can be found in Frankl [10] for the subset case.

4 Coloring q-Kneser graphs

In this section, we prove Theorem 1.5. We will need the following result of Bose and Burton [2] and its extension by Metsch [19].

Theorem 4.1 (Bose-Burton) *If $\mathcal{E}$ is a family of 1-subspaces of V such that any k-subspace of V contains at least one element of $\mathcal{E}$, then $|\mathcal{E}| \geq \left[{n-k+1 \atop 1}\right]$. Furthermore, equality holds if and only if $\mathcal{E} = \left[{H \atop 1}\right]$ for some $(n-k+1)$-subspace H of V.*

Proposition 4.2 (Metsch) *If $\mathcal{E}$ is a family of $\left[{n-k+1 \atop 1}\right] - \varepsilon$ 1-subspaces of V, then the number of k-subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k)}$.*

Proof of Theorem 1.5. Suppose that we have a coloring with at most $\left[{n-k+1 \atop 1}\right]$ colors. Let G (the good colors) be the set of colors that are point-pencils and let B (the bad colors) be the remaining set of colors. Then $|G| + |B| \leq \left[{n-k+1 \atop 1}\right]$. Suppose $|B| = \varepsilon > 0$. By Proposition 4.2, the number of k-spaces with a color in B is at least $\varepsilon q^{(k-1)(n-k)}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k)}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k)} \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$

For $k \geq 3$ and $q \geq 3$, $n \geq 2k + 1$ or $q = 2$, $n \geq 2k + 2$, this is a contradiction. (The weaker form of Proposition 4.2 suffices unless $q = 2$, $n = 2k + 2$.)

If $|B| = 0$, all color classes are point-pencils, and we are done by Theorem 4.1. □

5 Proof of Theorem 1.6

Let $a+b=n, a<b$ and let $\mathcal{F}_a = \mathcal{F} \cap \begin{bmatrix} V \\ a \end{bmatrix}$ and $\mathcal{F}_b = \mathcal{F} \cap \begin{bmatrix} V \\ b \end{bmatrix}$. We prove

$$|\mathcal{F}_a| + |\mathcal{F}_b| \leq \begin{bmatrix} n \\ b \end{bmatrix} \tag{5.4}$$

with equality only if $\mathcal{F}_a = \emptyset$ and $\mathcal{F}_b = \begin{bmatrix} V \\ b \end{bmatrix}$.

Adding up (5.4) for $n/2 < b \leq n$ gives the bound on $|\mathcal{F}|$ in Theorem 1.6 if n is odd; adding the result of Greene and Kleitman [16] that states $|\mathcal{F}_{n/2}| \leq \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix}$ proves it for even n. For the uniqueness part of Theorem 1.6, we only have to note that if n is even then, by results of Godsil and Newman [15], we must have $\mathcal{F}_{n/2} = \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ or $\mathcal{F}_{n/2} = \begin{bmatrix} U \\ n/2 \end{bmatrix}$ for some $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$.

Now we prove (5.4). Consider the bipartite graph with vertex set $(\begin{bmatrix} V \\ a \end{bmatrix}, \begin{bmatrix} V \\ b \end{bmatrix})$ and join $A \in \begin{bmatrix} V \\ a \end{bmatrix}$ and $B \in \begin{bmatrix} V \\ b \end{bmatrix}$ if $A \cap B = 0$. Observe that $\mathcal{F}_a \cup \mathcal{F}_b$ is an independent set in this graph. Now, this graph is regular with degree q^{ab}. Therefore any independent set in this graph has size at most $\begin{bmatrix} n \\ b \end{bmatrix}$ by König's Theorem. Moreover, independent sets of size $\begin{bmatrix} n \\ b \end{bmatrix}$ can only be $\begin{bmatrix} V \\ a \end{bmatrix}$ or $\begin{bmatrix} V \\ b \end{bmatrix}$, but the former is not an intersecting family. This proves (5.4). □

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References

[1] C. Bey. Polynomial LYM inequalities. *Combinatorica*, 25(1):19–38, 2005.

[2] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combinatorial Theory*, 1:96–104, 1966.

[3] A. Chowdhury, C. Godsil, and G. Royle. Colouring lines in projective space. *J. Combin. Theory Ser. A*, 113(1):39–52, 2006.

[4] A. Chowdhury and B. Patkós. Shadows and intersections in vector spaces. *J. Combin. Theory Ser. A*, to appear.

[5] E. Czabarka and L. Székely. An alternative shifting proof to Hsieh's theorem. In *Proceedings of the Twenty-ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1998)*, volume 134, pages 117–122, 1998.

[6] M. Deza and P. Frankl. The Erdős-Ko-Rado theorem—22 years later. *SIAM J. Algebraic Discrete Methods*, 4(4):419–431, 1983.

[7] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[8] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[9] P. Frankl. On families of finite sets no two of which intersect in a singleton. *Bull. Austral. Math. Soc.*, 17(1):125–134, 1977.

[10] P. Frankl. On intersecting families of finite sets. *J. Combinatorial Theory Ser. A*, 24(2):146–161, 1978.

[11] P. Frankl and Z. Füredi. Nontrivial intersecting families. *J. Combin. Theory Ser. A*, 41(1):150–153, 1986.

[12] P. Frankl, K. Ota, and N. Tokushige. Covers in uniform intersecting families and a counterexample to a conjecture of Lovász. *J. Combin. Theory Ser. A*, 74(1):33–42, 1996.

[13] P. Frankl and R. M. Wilson. The Erdős-Ko-Rado theorem for vector spaces. *J. Combin. Theory Ser. A*, 43(2):228–236, 1986.

[14] D. Gerbner and B. Patkós. Profile vectors in the lattice of subspaces. *Discrete Math.*, 309(9):2861–2869, 2009.

[15] C.D. Godsil and M.W. Newman. Independent sets in association schemes. *Combinatorica*, 26(4):431–443, 2006.

[16] C. Greene and D.J. Kleitman. Proof techniques in the theory of finite sets. In *Studies in combinatorics*, volume 17 of *MAA Stud. Math.*, pages 22–79. Math. Assoc. America, Washington, D.C., 1978.

[17] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 18:369–384, 1967.

[18] W.N. Hsieh. Intersection theorems for systems of finite vector spaces. *Discrete Math.*, 12:1–16, 1975.

[19] K. Metsch. How many s-subspaces must miss a point set in $\mathrm{PG}(d,q)$. *J. Geom.*, 86(1-2):154–164, 2006.

[20] T. Mussche. *Extremal combinatorics in generalized Kneser graphs.* PhD thesis, Technical University Eindhoven, 2009.