Induced and non-induced poset saturation problems

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Extremal problems vs Saturation problem

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Triangle free graphs:

- Most number of edges: $\lfloor \frac{n^2}{4} \rfloor$ (Mantel 1908),
- Least number of edges in unextendable triangle-free graphs: $n-1$.

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For graphs:

- Turán number: Erdős-Stone Simonovits theorem $\rightarrow$ $ex(n, F) = \Theta(n^2)$ unless F is bipartite
- $sat(n, G) =$ least number of edges in maximal/unextendable n-vertex G-free graphs $= O(n)$ Kászonyi, Tuza, 1986.

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k-graphs:

- Turán number: ???
- $sat(n,H)=O(n^{k-1})$ Pikhurko, 1999.

Forbidden subposet problems

Theorem (Sperner, 1928)

If $\mathcal{F} \subseteq 2^{[n]}$ *does not contain* F, F' *with* $F \subsetneq F'$, *then* $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Theorem (Erdős, 1945)

If $\mathcal{F} \subseteq 2^{[n]}$ *does not contain any* $(k+1)$*-chain* $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k+1}$, *then* $|\mathcal{F}| \leq \sum_{i=1}^{k} \binom{n}{\lfloor (n-k)/2 \rfloor + i}$.

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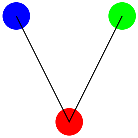
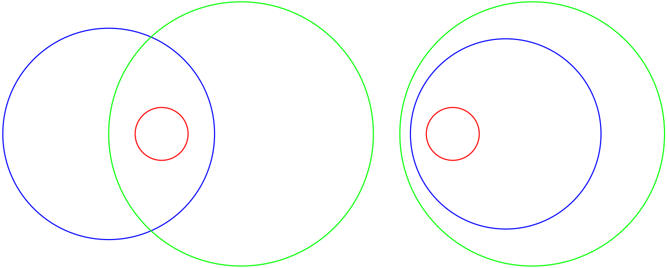
Katona and Tarján in 1983 introduced forbidden containment patterns described by posets.

Definition

Let P be a partially ordered set. We say that a family $\mathcal{F}$ of sets contains P if there exists an injection $i : P \to \mathcal{F}$ such that $p \leq_P q$ implies $i(p) \subset i(q)$.



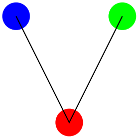
the poset $\vee$



Definition

Let P be a partially ordered set. We say that a subfamily $\mathcal{G} \subseteq \mathcal{F}$ of sets is

- a non-induced copy of P if there exists an injection $i : P \to \mathcal{G}$ such that $p \leq_P q$ implies $i(p) \subset i(q)$,
- an induced copy of P if there exists an injection $i : P \to \mathcal{G}$ such that $p \leq_P q$ if and only if $i(p) \subset i(q)$.



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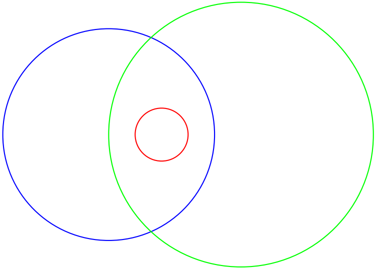
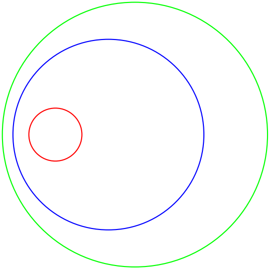


an induced copy of $\vee$



a non-induced copy of $\vee$

- If $\mathcal{F}$ does not contain a non-induced copy of P, then we say that $\mathcal{F}$ is P-free.
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$La(n, P)$ denotes the maximum size of a P-free family $\mathcal{F} \subseteq 2^{[n]}$.
$La^*(n, P)$ denotes the maximum size of an induced P-free family $\mathcal{F} \subseteq 2^{[n]}$.

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Erdős's theorem from 1945 about k-Sperner families states that

$$La(n, C_{k+1}) = La^*(n, C_{k+1}) = \sum_{i=1}^{k} \binom{n}{\lfloor (n-k)/2 \rfloor + i},$$

where C_{k+1} is the total ordering or chain on $k+1$ elements.

Erdős's result implies that $La(n, P) \leq (|P| - 1)\binom{n}{\lfloor n/2 \rfloor}$.

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Still unknown: do

$$\pi(P) = \lim_n \frac{La(n,P)}{\binom{n}{\lceil n/2 \rceil}} \qquad \pi^*(P) = \lim_n \frac{La^*(n,P)}{\binom{n}{\lceil n/2 \rceil}}$$

exist for all finite posets P?

Conjecture

- *For any poset P let $e(P)$ denote the most number of middle levels without creating a non-induced copy of P. Then $\pi(P)$ exists and is equal to $e(P)$.*
- *For any poset P let $e^*(P)$ denote the most number of middle levels without creating a induced copy of P. Then $\pi^*(P)$ exists and is equal to $e^*(P)$.*

Saturation forbidden subposet problems

$sat(n,P)$ = minimum size of a P-free $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F} \cup \{G\}$ contains a non-induced copy of P for any $G \in 2^{[n]} \setminus \mathcal{F}$,

$sat^*(n,P)$ = minimum size of an induced P-free $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F} \cup \{G\}$ contains an induced copy of P for any $G \in 2^{[n]} \setminus \mathcal{F}$.

Saturation forbidden subposet problems - History

$G6$ =Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi, Patkós (2013)

Construction (G6)

For C_k: for $k \geq 3$, the family

$$\mathcal{F} = 2^{[k-3]} \cup \{[n] \setminus F : F \in 2^{[k-3]}\}$$

is C_k-saturating, so $sat(n, C_k) = sat^*(n, C_k) \leq 2^{k-2}$

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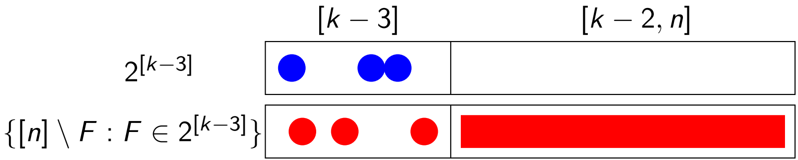

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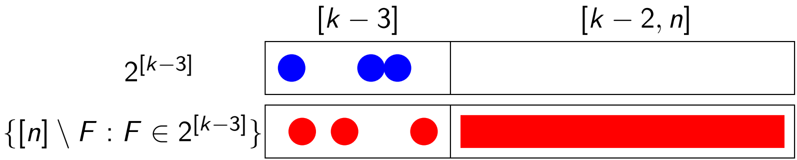

$\mathcal{F}$ is C_k-free as it is poset-isomorphic to $2^{[k-2]}$.

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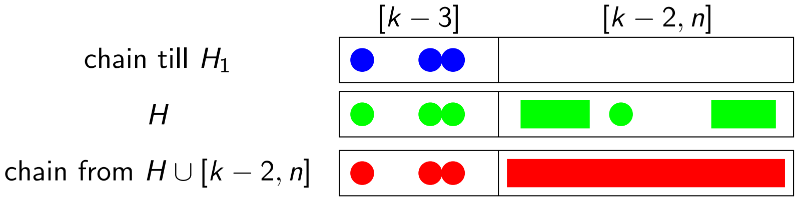
Adding a set $H = H_1 \cup H_2$ with $H_1 \subseteq [k-3]$ and $\emptyset \subsetneq H_2 \subsetneq [k-2, n]$ creates a k-chain:



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This is sharp if $k \leq 6$. On the other hand

Theorem (G6, 2013)

If $k \geq 7$, then $2^{\lfloor \frac{k-3}{2} \rfloor} \leq sat(n, C_k) \leq \frac{15}{16} 2^{k-2}$.

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Theorem (Morrison, Noel, Scott, 2014)

As k tends to infinity, we have $sat(n, C_k) \leq 2^{(0.98+o(1))k}$.

Saturation forbidden subposet problems - History II

$F7 =$ Ferrara, Kay, Kramer, Martin, Reiniger, Smith, Sullivan (2017)

Found specific posets and classes of posets for which $sat^*(n, P) \to \infty$ as n tends to infinity.

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Ivan (2020+)
Linear lower bound on $sat^*(n, \bowtie)$ and $\sqrt{n} \leq sat^*(n, N)$.

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F7 & Martin, Smith, Walker & Ivan are “right” not to consider non-induced versions as:

Theorem (KLMPP, 2020+)

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This is relatively sharp in general, because of the G6 result: $2^{\frac{k-3}{2}} \leq sat(n, C_k)$.

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Conjecture (KLMPP, 2020+)

For any poset P on k elements, we have $sat(n, P) \leq sat(n, C_k)$.

The proof

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GREEDY COLEX ALGORITHM

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GREEDY COLEX ALGORITHM

Greedy: consider sets of $2^{[n]}$ in some order $F_1, F_2, \dots, F_{2^n}$. Let $\mathcal{F}_0 = \emptyset$.

$$\mathcal{F}_{i+1} = \begin{cases} \mathcal{F}_i \cup \{F_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\}, \text{does not contain any copy of} P \\ \mathcal{F}_i & \text{otherwise} \end{cases}$$

$\mathcal{F} := \mathcal{F}_{2^n}$ is clearly P-saturating.

The proof - II

Theorem (KLMPP, 2020+)

For any poset P, we have $sat(n, P) \leq 2^{|P|-2}$.

Colex: the co-lexicographic ordering of $Fin(\mathbb{Z}^+)$:
$A < B$ if and only if $\max(A \setminus B) \cup (B \setminus A)$ belongs to B.

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$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{1, 4\} \ldots$

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The greedy colex algo is NOT what you would think!

The proof - III

Theorem (KLMPP, 2020+)

For any poset P, we have $sat(n, P) \leq 2^{|P|-2}$.

Let $F_1, F_2, \ldots, F_{2^{n-1}}$ be the enumeration of all sets in $2^{[n-1]}$ and let $G_i = [n] \setminus F_i$.

So the G_j's contain n, the F_i's do not.

The proof - III

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The greedy colex algorithm considers the sets of $2^{[n]}$ in the order $F_1, G_1, F_2, G_2, \ldots, F_{2^{n-1}}, G_{2^{n-1}}$.

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$\emptyset, [n], \{1\}, [n] \setminus \{1\}, \{2\}, [n] \setminus \{2\}, \{1, 2\}, [3, n], \dots, [n-1], \{n\}$.

$\mathcal{F}_{i+1} =$

$$\begin{cases} \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} \text{ is } P\text{-free} \\ \mathcal{F}_i \cup \{F_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\} \text{ is } P\text{-free, } \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} \text{ not} \\ \mathcal{F}_i \cup \{G_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\} \text{ not } P\text{-free, } \mathcal{F}_i \cup \{G_{i+1}\} \text{ is } P\text{-free,} \\ \mathcal{F}_i & \text{otherwise.} \end{cases}$$

$\mathcal{F} := \mathcal{F}_{2^n-1}$ is the output of the greedy colex algorithm.

Theorem (KLMPP, 2020+)

For $1 \leq k \leq n$*, let* P *be a* k*-element poset and let* $\mathcal{F} := \mathcal{F}_{2^{n-1}}$ *be the output of the greedy colex process. Then,* $\mathcal{F}$ *is* P*-saturating,* $\mathcal{F} = \mathcal{F}_{2^{k-3}}$ *and therefore* $|\mathcal{F}| \leq 2^{k-2}$*. In particular,* $sat(n, P) \leq 2^{k-2}$ *holds.*

Theorem (KLMPP, 2020+)

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Remark

Oh my God! Oh one God! O1G!

$$F_1, G_2, F_2, G_2, \ldots, F_{2^{k-3}}, G_{2^{k-3}}$$

is exactly the construction

$$2^{[k-3]} \cup \{[n] \setminus F : F \in 2^{[k-3]}\}$$

of the G6 guys!

What if $\{\ell+1\}$ is not added in the greedy colex process?

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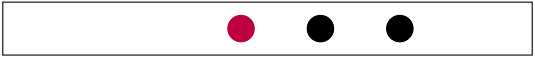
Can we say something about $\{\ell+2\}$ or $\{\ell+5, \ell+17\}$?



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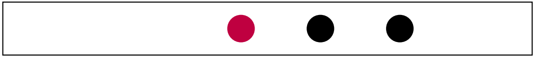
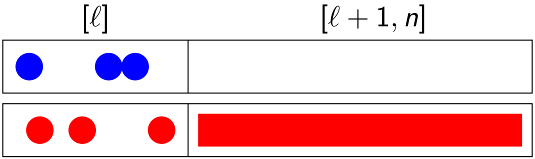


Sets added so far:



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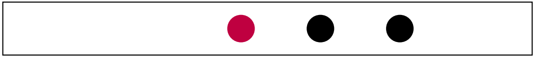

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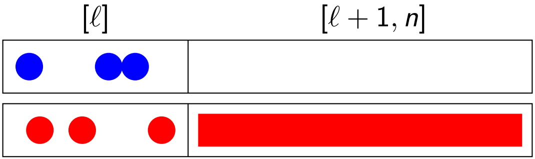


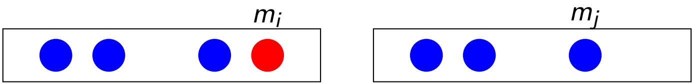
So if $\{\ell+1\}$ is not added, then later on the other two cannot be added either.

Let $m_i = \max F_i$.

Lemma

For any $i \leq 2^{n-1}$ we have the following.

1. *$F_i \in \mathcal{F}_i$ implies $F_i \setminus \{m_i\} \in \mathcal{F}_j$ for some $j < i$.*



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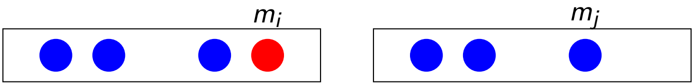

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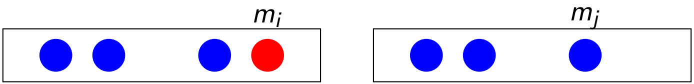

Let j be defined such that $F_j = F_i \backslash \{m_i\}$ and observe that $j < i$. We claim that for any $H \in \mathcal{F}_{j-1}$, the pair (H, F_j) has the same containment relation / non-relation as the pair (H, F_i).

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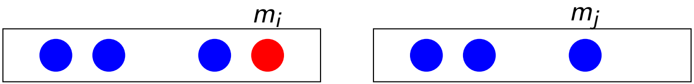

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We claim that for any $H \in \mathcal{F}_{j-1}$, the pair (H, F_j) has the same containment relation / non-relation as the pair (H, F_i).

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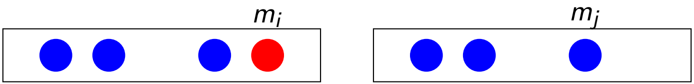

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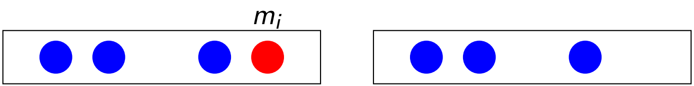

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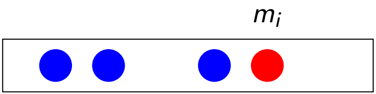

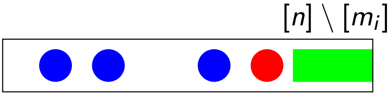

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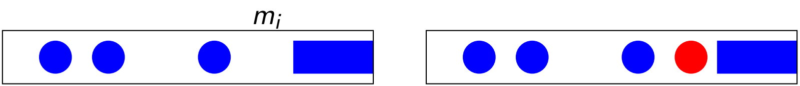

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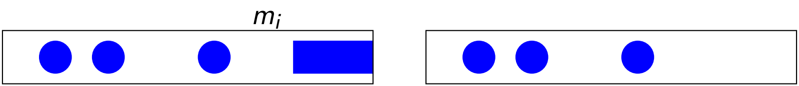

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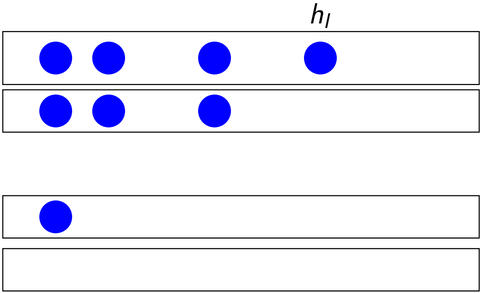

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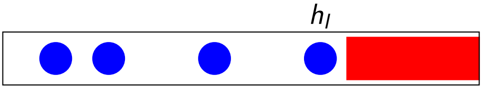

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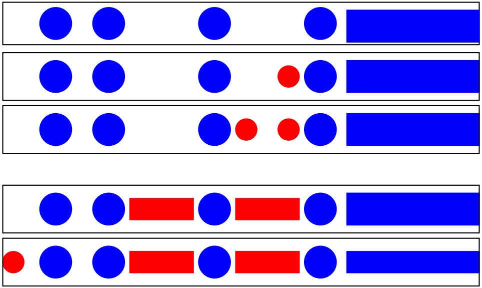

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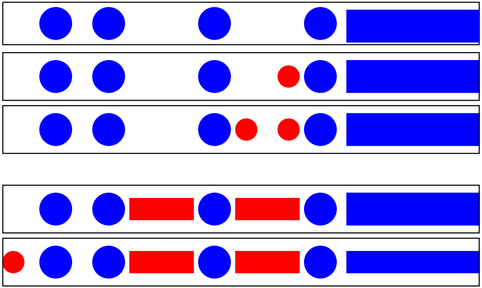

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Together: chain of length $l + 1 + m_i - l + 1 = m_i + 2 \geq k$

The case of H being a G_i for some $i > 2^{k-3}$ is very similar.

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What can the algo do for poset classes with $sat^*(n, P) \to \infty$? (not much)

The following is implicitly in the work of F7

Lemma

For any poset P, the following are equivalent:

1. *There exists a constant C_P such that $sat^*(n, P) \leq C_P$ holds for all n.*
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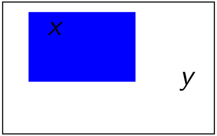

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$$\mathcal{F}_n = \mathcal{F}_0 \cup \{F \cup ([n] \setminus [m]) : F \in \mathcal{F}_{xy}\}$$

is P-saturating in $2^{[n]}$ with $|\mathcal{F}| = |\mathcal{F}_n|$. □

Consequences of the lemma:

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For any poset P,

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We conjecture the following strengthening.

Conjecture

For any poset P,

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Is it recursive?

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Yet, the greedy colex algo can be useful even if $sat^*(n, P) \to \infty$.

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$$\mathcal{T}_{2k} = \{\{1,4,6,\dots,2k\}, \{2,4,6,\dots,2k\}, \{1,2,4,6,\dots,2k\}\}$$

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For any $1 \leq j < n$, let $\mathcal{T}_{j,n} = \{[n] \setminus T : T \in \mathcal{T}_j\}$. Realize that the output is $\mathcal{H}_n = \cup_{j=1}^{n-1}(\mathcal{T}_j \cup \mathcal{T}_{j,n})$. So $|\mathcal{H}_n| = 6n - 10$.

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- Prove this mathematically.

Observe that together with Ivan's recent result you (well, we...) obtained:

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But we do not have an example of a poset P with bounded saturation number, and the greedy colex giving an unbounded family.

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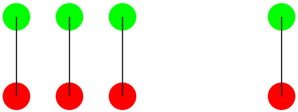

Conjecture

Let k be a positive integer. There exists a constant c_k such that $sat^*(n,kC_2) \leq c_k$ if and only if k is odd.

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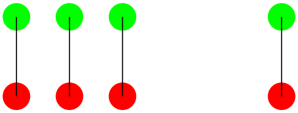
kC_2 

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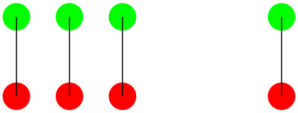

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For even values of k we were only able to prove the conjecture for $k = 2$.

Theorem (KLMPP, 2020+)

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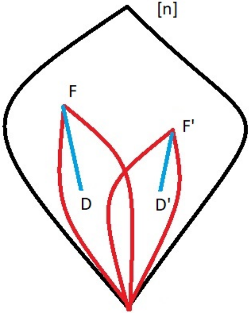
Proof: $\mathcal{F} \subseteq 2^{[n]}$ $2C_2$-saturating, $\mathcal{D}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \subsetneq G\}$.
For any $F, F' \in \mathcal{F}$: $\mathcal{D}_{\mathcal{F}}(F) \subseteq \mathcal{D}_{\mathcal{F}}(F')$ or $\mathcal{D}_{\mathcal{F}}(F) \supseteq \mathcal{D}_{\mathcal{F}}(F')$.



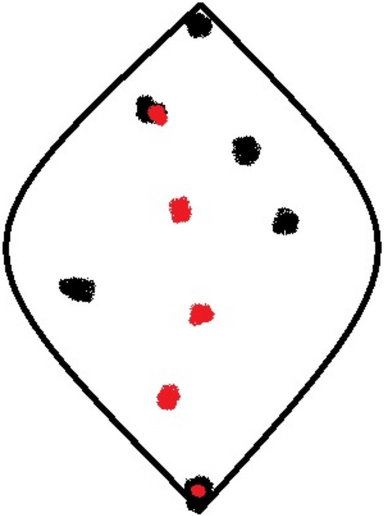
(Indeed, if $D \in \mathcal{D}_{\mathcal{F}}(F) \setminus \mathcal{D}_{\mathcal{F}}(F')$ and $D' \in \mathcal{D}_{\mathcal{F}}(F') \setminus \mathcal{D}_{\mathcal{F}}(F)$, then $D \subsetneq F$ and $D' \subsetneq F'$ are two incomparable pairs, i.e. they form a copy of $2C_2$.)

Theorem (KLMPP, 2020+)

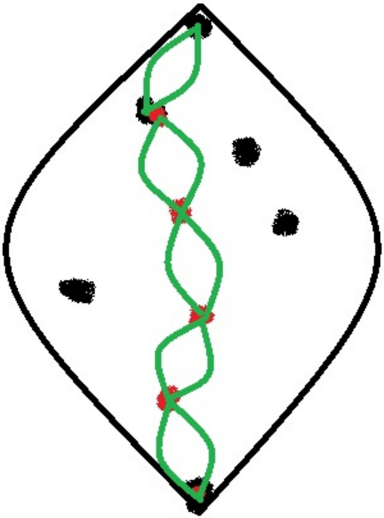
If $\mathcal{F} \subseteq 2^{[n]}$ is saturating induced $2C_2$-free, then $\mathcal{F}$ contains a maximal chain in $[n]$. In particular, $sat^*(n, 2C_2) \geq n+2$ holds.

Therefore, we can enumerate $\mathcal{F}$ as $[n] = F_1, F_2, \ldots, F_m = \emptyset$ such that $\mathcal{D}_{\mathcal{F}}(F_1) \supseteq \mathcal{D}_{\mathcal{F}}(F_2) \supseteq \cdots \supseteq \mathcal{D}_{\mathcal{F}}(F_m)$ holds.

Let $G_j = \cap_{i=1}^{j} F_i$.



To show: if $X \in (G_{j+1}, G_j]$, then H must belong to $\mathcal{F}$.



Claim

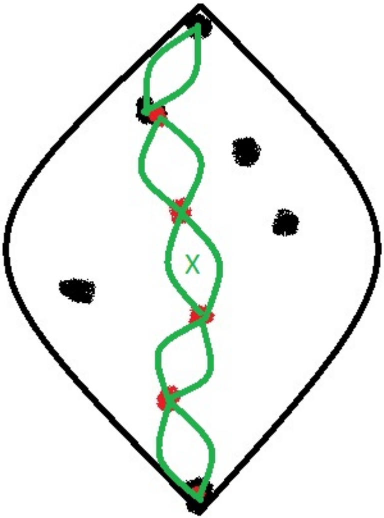
For any $h = 1, 2, \ldots, m$ *we have*
$\mathcal{D}_{\mathcal{F}}(G_h) \subseteq \mathcal{D}_{\mathcal{F}}(F_h) \subseteq \mathcal{D}_{\mathcal{F}}(G_h) \cup \{G_h\}$.

Proof of Claim.

$G_h = \cap_{i=1}^{h} F_i \subseteq F_h$. This implies $\mathcal{D}_{\mathcal{F}}(G_h) \subseteq \mathcal{D}_{\mathcal{F}}(F_h)$.

Also, $\mathcal{D}_{\mathcal{F}}(F_h) \subseteq \mathcal{D}_{\mathcal{F}}(F_i)$ for all $1 \leq i < h$, so
$\mathcal{D}_{\mathcal{F}}(F_h) = \cap_{i=1}^{h} \mathcal{D}_{\mathcal{F}}(F_i) \subseteq \mathcal{D}_{\mathcal{F}}(G_h) \cup \{G_h\}$. □

To show: $G_{j+1} \subsetneq X \subseteq G_j$, then X must belong to $\mathcal{F}$.



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If $\ell \geq j+1$, applying Claim to $h = j+1$ shows that

$$A \in \mathcal{D}_{\mathcal{F}}(F_\ell) \subseteq \mathcal{D}_{\mathcal{F}}(F_{j+1}) \subseteq \mathcal{D}_{\mathcal{F}}(G_{j+1}) \cup \{G_{j+1}\} \subseteq \mathcal{D}(X),$$

which contradicts the assumption that A and X are incomparable.

Thank you for your attention!

And for Dömötör's sake, please do look at the table included in our manuscript on ArXiv!