

On the number of maximal intersecting k -uniform families and further applications of Tuza's set pair method

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Abstract

We study the function $M(n, k)$ which denotes the number of maximal k -uniform intersecting families $\mathcal{F} \subseteq \binom{[n]}{k}$. Improving a bound of Balogh et al. on $M(n, k)$, we determine the order of magnitude of $\log M(n, k)$ by proving that for any fixed k , $M(n, k) = n^{\Theta(\binom{2k}{k})}$ holds. Our proof is based on Tuza's set pair approach.

The main idea is to bound the size of the largest possible point set of a cross-intersecting system. We also introduce and investigate some related functions and parameters.

1 Introduction

Many problems in extremal combinatorics ask for the maximum possible size that a combinatorial structure can have provided it satisfies some prescribed property P . Questions about the size of the 'underlying set' of the combinatorial structure are much less frequently asked. (In many cases, this size is part of property P .) This note is devoted to an application of Tuza's set pair method [11] which provides good bounds for problems of the first type through results on problems of the second type.

The starting point of Tuza's method is the following celebrated theorem of Bollobás.

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Theorem 1.1 (Bollobás, [2]). *Let A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m be sets such that $|A_i| \leq k$ and $|B_i| \leq l$ hold for all $1 \leq i \leq m$. Let furthermore these sets satisfy*

- (1) $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$,
- (2) $A_i \cap B_j \neq \emptyset$ for all $1 \leq i, j \leq m, i \neq j$.

Then $\sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$, in particular $m \leq \binom{k+l}{l}$ holds.

Pairs satisfying the condition of Theorem 1.1 will be called *cross intersecting set pairs* and if we want to emphasize the size condition of the A_i 's and B_j 's, then we call the system (k, l) -*cross intersecting*.

Modifying Lovász's proof [9] of Theorem 1.1, Frankl [6] and later Kalai [7] obtained the following skew version of the result.

Theorem 1.2 (Frankl). *Let A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m be sets such that $|A_i| \leq k$ and $|B_i| \leq l$, satisfying the conditions*

- (1) $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$,
- (2') $A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq m$.

Still the bound $m \leq \binom{k+l}{l}$ remains valid.

Pairs satisfying the condition of Theorem 1.2 will be called *skew cross intersecting set pairs*.

The vertex set of a (skew) cross intersecting system of set pairs is $V = \bigcup_{i=1}^m (A_i \cup B_i)$. Tuza was interested in the maximum possible size of the vertex set of a (k, l) -cross intersecting system. Let us write

$$n(k, l) = \max \left\{ \left| \bigcup_{i=1}^m (A_i \cup B_i) \right| : (A_1, B_1), \dots, (A_m, B_m) \text{ is a } (k, l)\text{-cross intersecting system} \right\}.$$

Obviously, by Theorem 1.1, we have $n(k, l) \leq (k+l)\binom{k+l}{l}$, but the following upper bound was obtained in [11].

Theorem 1.3 (Tuza [11]). *For positive integers $k \leq l$ we have*

$$\frac{1}{4} \binom{k+l+1}{k+1} < n(k, l) \leq \sum_{i=1}^{2k-2} \binom{i}{\lfloor i/2 \rfloor} + \sum_{i=2k-1}^{k+l-1} \binom{i}{l} < \binom{k+l+1}{k+1}.$$

Section 2 is devoted to prove another application of the set pair method, the main result of this note. Apart from antichains the most studied set families are *intersecting* families. We say that $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if $F_1 \cap F_2 \neq \emptyset$ holds for all $F_1, F_2 \in \mathcal{F}$. It is well-known that all *maximal* (unextendable) intersecting families have size 2^{n-1} . (Here and thereafter $[n]$ stands for the set $\{1, 2, \dots, n\}$.) The investigation of $\lambda(n)$ and $\Lambda(n)$, the number of intersecting and maximal intersecting families, respectively, was started in [4]. The exact values are known for small n [3] and determining the order of magnitude of $\log \lambda(n)$ and $\log \Lambda(n)$ is an easy exercise.

Recently, Balogh, Das, Delcourt, Liu, and Sharifzadeh [1] studied the uniform version of the problem. The famous Erdős-Ko-Rado theorem states that an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ can have size at most $\binom{n-1}{k-1}$ if $2k \leq n$ holds. Furthermore, intersecting families achieving the extremal size consist of all k -sets containing a fixed element of $[n]$ provided $2k < n$. Balogh et al determine the threshold function $N(k)$ with the property that if $n \geq N(k)$, then the number of k -uniform intersecting families is $2^{(1+o(1))\binom{n-1}{k-1}}$. In their proof they obtain an upper bound on the number $M(n, k)$ of maximal k -uniform intersecting families. Here we improve on this bound and by using a construction given in [11] we determine the order of magnitude of the exponent of n in $M(n, k)$ for any fixed k .

Theorem 1.4. *For any fixed integer k , as n tends to infinity the function $M(n, k)$ satisfies*

$$M(n, k) = n^{\Theta\left(\binom{2k}{k}\right)}.$$

Moreover,

$$\frac{1}{8} \leq \limsup_n \frac{\log M(n, k)}{\binom{2k}{k} \log n} \leq 1.1 \quad \text{and} \quad \limsup_k \limsup_n \frac{\log M(n, k)}{\binom{2k}{k} \log n} \leq 1$$

holds.

The proof of Theorem 1.4 uses the upper bound in Theorem 1.3. In Section 3 we first prove an upper bound on $n(k, l)$ that is weaker than that of Theorem 1.3, but its proof technique is completely different: it involves skew cross intersecting systems. Therefore it is natural to introduce the following analog of the function $n(k, l)$:

$$n_1(k, l) = \max \left\{ \left| \bigcup_{i=1}^m (A_i \cup B_i) \right| : (A_1, B_1), \dots, (A_m, B_m) \text{ is a } (k, l)\text{-skew cross intersecting system} \right\}.$$

We finish the section by presenting lower and upper bounds on $n_1(k, l)$.

2 Proof of the main theorem

We start with the lower bound of Theorem 1.4. For a family \mathcal{F} of sets its *covering number* $\tau(\mathcal{F})$ is the minimum size that a *transversal* G of \mathcal{F} can have. A transversal of \mathcal{F} is a set meeting all $F \in \mathcal{F}$. Clearly, $\tau(\mathcal{F}) \leq k$ holds for all intersecting k -uniform families as any set in \mathcal{F} is a transversal. Let us define the function $f(k)$ by

$$f(k) = \max\{|\cup_{F \in \mathcal{F}} F| : \mathcal{F} \text{ is } k\text{-uniform intersecting with } \tau(\mathcal{F}) = k\}.$$

Note that $f(k)$ is finite (see [5]), while the condition $\tau(\mathcal{F}) = k$ is essential in the sense that $|\cup_{F \in \mathcal{F}} F|$ could be arbitrarily large if \mathcal{F} was k uniform intersecting with $\tau(\mathcal{F}) < k$. Many similar functions concerning k -uniform intersecting families with covering number k were introduced and studied in [5] (and later by many other researchers). The following example is due to Tuza [11].

Construction 2.1. *Let $|Y| = 2k - 4$. If $E \cup E' = Y$ and $|E| = |E'| = k - 2$ then take four new points x, x', y, y' and set $E \cup \{x, y\}$, $E \cup \{x', y'\}$, $E' \cup \{x, x'\}$, $E' \cup \{y, y'\}$. In this way we obtain $2\binom{2k-4}{k-2}$ k -element sets forming an intersecting family with covering number k , such that the union of these sets consists of $2k - 4 + 2\binom{2k-4}{k-2}$ points.*

Corollary 2.2. $\frac{1}{8}\binom{2k}{k} < 2k - 4 + 2\binom{2k-4}{k-2} \leq f(k)$.

The following proposition finishes the proof of the lower bound of Theorem 1.4.

Proposition 2.3. *For any positive integers k and n we have $\binom{n}{f(k)} \leq M(n, k)$.*

Proof. Consider a k -uniform intersecting family \mathcal{F} with $\tau(\mathcal{F}) = k$ and $|\cup_{F \in \mathcal{F}} F| = f(k)$. As adding more sets to \mathcal{F} can only increase the size of the union, we may assume that \mathcal{F} is maximal intersecting. Every set $X \in \binom{[n]}{f(k)}$ contains at least one family \mathcal{F}_X isomorphic to \mathcal{F} . As $\mathcal{F}_X \neq \mathcal{F}_Y$ whenever $\cup_{F \in \mathcal{F}_X} F = X \neq Y = \cup_{F \in \mathcal{F}_Y} F$, we have at least $\binom{n}{f(k)}$ different maximal intersecting k -uniform subfamilies of $\binom{[n]}{k}$. \square

As we mentioned in the proof, the value of $f(k)$ is attained at a maximal intersecting family that we could not extend even if we do not make any restrictions on the underlying set. This kind of maximal intersecting set systems were studied a lot, the best known upper bound on $f(k)$ is due to Majumder [10].

We now turn our attention to the upper bound of Theorem 1.4. We start by describing the basic ideas of Balogh, Das, Delcourt, Liu, and Sharifzadeh [1]. For a family $\mathcal{F} \subseteq \binom{[n]}{k}$ of sets let $\mathcal{I}(\mathcal{F}) = \{G \in \binom{[n]}{k} : \forall F \in \mathcal{F} : F \cap G \neq \emptyset\}$, that is if \mathcal{F} is intersecting, then $\mathcal{I}(\mathcal{F})$ denotes the family of those sets that can be added to \mathcal{F} preserving the intersecting property. Clearly, \mathcal{F} is maximal intersecting if and only if $\mathcal{I}(\mathcal{F}) = \mathcal{F}$. For any maximal

intersecting family we can assign a subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ that is minimal with respect to the property $\mathcal{I}(\mathcal{F}_0) = \mathcal{F}$ (note that \mathcal{F}_0 is not necessarily unique). Then by definition, for every $F \in \mathcal{F}_0$ there exists a $G \in \mathcal{I}(\mathcal{F}_0 \setminus \{F\}) \setminus \mathcal{F}$, thus this G intersects all sets in \mathcal{F} but F . Therefore the sets of \mathcal{F}_0 and their pairs G satisfy the condition of Theorem 1.1 and thus by the above, we obtain that $|\mathcal{F}_0| \leq \binom{2k}{k}$. Moreover, if $\mathcal{F}_0 = \{F_1, F_2, \dots, F_s\}$ and G_i is a set in $\mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$, then the set of pairs $\{(A_i, B_i)\}_{i=1}^{2s}$ with $A_i = F_i, B_i = G_i$ for $1 \leq i \leq s$ and $A_i = G_{i-s}, B_i = F_{i-s}$ for $s < i \leq 2s$ is skew cross intersecting and thus by Theorem 1.2 the inequality $|\mathcal{F}_0| \leq \frac{1}{2} \binom{2k}{k}$ holds. Since the mapping of maximal intersecting families via $\mathcal{F} \mapsto \mathcal{F}_0$ is bijective, Balogh et al obtained $M(n, k) \leq \sum_{j=1}^{\frac{1}{2} \binom{2k}{k}} \binom{n}{j} = O(n^{\frac{k}{2} \binom{2k}{k}})$. Comparing this to our lower bound, we see that the exponent is off only by a factor of $4k$. In what follows we show how to improve the previous upper bound.

In order to obtain our upper bound, we will use the function $n(k, l)$. As the argument of Balogh et al. yields a cross intersecting system in which sets of the first co-ordinate form an intersecting family on their own, we introduce the following:

$$g(k) = \max\{|\cup_{i=1}^s A_i| : \{(A_i, B_i)\}_{i=1}^s \text{ is } (k, k)\text{-cross intersecting and } \{A_i\}_{i=1}^s \text{ is intersecting}\}.$$

By definition, we have $g(k) \leq n(k, k)$. The following lemma and proposition complete the proof of the upper bound of Theorem 1.4.

Lemma 2.4. $M(n, k) \leq 2^{2g(k)} \binom{n}{g(k)}$.

Proof. Let us consider a function f that maps any maximal intersecting family \mathcal{F} to one of its subfamily \mathcal{F}_0 that is minimal with respect to the property that $\mathcal{I}(\mathcal{F}_0) = \mathcal{F}$. As mentioned earlier, f is injective, \mathcal{F}_0 is intersecting and the set of pairs $\{(F_i, G_i)\}_{i=1}^{|\mathcal{F}_0|}$ is cross intersecting. Thus by definition $|\cup_{F \in \mathcal{F}_0} F| \leq g(k)$ holds. Therefore the set families that can be the image of a maximal intersecting family with respect to f are subfamilies of 2^X for some $X \in \binom{[n]}{g(k)}$. The number of such families is not more than $2^{2g(k)} \binom{n}{g(k)}$. \square

Though it was not mentioned in [11], the summation form of the upper bound of Theorem 1.3 provides much better estimation in the case $k = l$.

Proposition 2.5. Let $S(k)$ denote Tuza's upper bound on $n(k, k)$ in Theorem 1.3, that is, $S(k) = \sum_{i=1}^{2k-1} \binom{i}{\lfloor i/2 \rfloor}$. Then

$$(i) \quad g(k) \leq n(k, k) \leq S(k) \leq 1.1 \cdot \binom{2k}{k},$$

$$(ii) \quad s(k) := \frac{S(k)}{\binom{2k}{k}} \rightarrow 1 \text{ if } k \rightarrow \infty.$$

Proof. Statement (i) can be confirmed easily for $k \leq 4$, and for $k > 4$ simple inductive argument works. For statement (ii), one can easily check that $s(k) > 1$ holds, and the sequence $s(k)$ is monotone decreasing from $k = 4$. Moreover the limit cannot be greater than 1, since if $s(k) > (1 + \varepsilon)$ held with a fixed $\varepsilon > 0$ for all k , that would imply $\frac{s(k+1)}{s(k)} \leq \frac{4k+3}{4k+2} \frac{1}{(1+\varepsilon)}$, a contradiction. \square

3 Bounds on the size of the vertex set

In the forthcoming section we present lower and upper bounds on $n(k, l)$ and $n_1(k, l)$, that is, on the maximal size of the underlying set of a (skew-)cross intersecting system.

Construction 3.1 (Erdős-Lovász, [5]). *Let Y be a set of $2k - 2$ elements. For each subset $A' \subset Y$, $|A'| = k - 1$, we assign a set pair (A, B) such that $|A| = k = |B|$ holds, $A' \subset A$, $(Y \setminus A') \subset B$ and the one element sets $A \setminus Y$, $B \setminus Y$ are disjoint. In this way we obtain $\binom{2k-2}{k-1}$ set pairs such that the union of these sets consists of $2k - 2 + 2 \binom{2k-2}{k-1}$ points.*

This construction slightly improves the general lower bound of Theorem 1.3 on $n(k, l)$ in the special case $k = l$. Thus in view of Proposition 2.5, this provides

Proposition 3.2. $2k - 2 + 2 \binom{2k-2}{k-1} \leq n(k, k) \leq 1.1 \cdot \binom{2k}{k}$.

In the spirit of Tuza's approach, the following upper bound is obtained on $n(k, l)$.

Lemma 3.3. $n(k, l) \leq \binom{k+l}{l+1} + \binom{k+l}{k+1}$.

Proof. Let $\{(A_i, B_i)\}_{i=1}^s$ be a set of cross intersecting pairs with $|A_i| \leq k$ and $|B_i| \leq l$ for all $1 \leq i \leq s$. Let $\alpha_t = |\{i : |A_i \setminus (\cup_{j=1}^{i-1} (A_j \cup B_j))| \geq t\}|$ and $\beta_t = |\{i : |B_i \setminus (\cup_{j=1}^{i-1} (A_j \cup B_j))| \geq t\}|$. Clearly, we have

$$|\bigcup_{i=1}^s (A_i \cup B_i)| = \sum_{t=1}^k (\alpha_t + \beta_t).$$

Let us bound β_t . Observe that if we write $B'_i = B_i \cap (\cup_{j=1}^{i-1} (A_j \cup B_j))$, then the set of pairs $\{(A_i, B'_i)\}_{i=1}^s$ is skew cross intersecting. Moreover

$$|B_i \setminus (\cup_{j=1}^{i-1} (A_j \cup B_j))| \geq t$$

holds for i if and only if $|B'_i| \leq l - t$. Hence β_t is equal to the number of skew cross intersecting set pairs $\{(A_i, B'_i)\}$ where $|A_i| \leq k$ and $|B'_i| \leq l - t$. Applying Theorem 1.2 we obtain $\beta_t \leq \binom{k+l-t}{k}$, and as the role of α_t and β_t is similar we also have $\alpha_t \leq \binom{k+l-t}{l}$. Consequently,

$$|\bigcup_{i=1}^s (A_i \cup B_i)| \leq \sum_{t=1}^l \binom{k+l-t}{l} + \sum_{t=1}^k \binom{k+l-t}{k} = \binom{k+l}{l+1} + \binom{k+l}{k+1}.$$

\square

Although this slightly improves the bound $\binom{k+l+1}{l+1}$ of Theorem 1.3 when $k = l$, but essentially it does not in view of Proposition 2.5.

Recall that

$$n_1(k, l) = \max \left\{ \left| \bigcup_{i=1}^m (A_i \cup B_i) \right| : (A_1, B_1), \dots, (A_m, B_m) \text{ is a } (k, l)\text{-skew cross intersecting system} \right\}.$$

Our second result gives lower and upper bounds on $n_1(k, l)$. In order to do this, we recall what a reverse lexicographic order (or sometimes called *colex order*) is.

Definition 3.4. A reverse lexicographic order of the k -element subsets of $[n]$ is defined by the relation

$$C < D \text{ for } C, D \in \binom{[n]}{k} \Leftrightarrow \text{the largest element of the symmetric difference } C \Delta D \text{ is in } D.$$

Construction 3.5. Let Y be the set $Y = [k + l]$. Consider the reverse lexicographic order of all the k -element subsets of Y . Let $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,k}\}$ ($i = 1 \dots \binom{k+l}{k}$) be the i th set in this order with the $a_{i,j}$'s enumerated in increasing order, and let B_i be defined as follows. $B_i \cap Y = [a_{i,k}] \setminus A_i$ and let all the sets $B_i \setminus Y$ be pairwise disjoint for all i such that $|B_i| = k$.

Proposition 3.6. $k + l + \binom{k+l}{k+1} \leq n_1(k, l)$.

Proof. Construction 3.5 provides a (k, l) -skew cross intersecting set system. Indeed, $A_i \cap B_i = \emptyset$, while $A_i \cap B_j \neq \emptyset$ for $i < j$, since $A_i \subseteq [a_{i,k}] \subseteq [a_{j,k}]$, hence $A_i \cap B_j \supseteq A_i \cap ([a_{j,k}] \setminus A_j) \neq \emptyset$. Observing that the number of k -sets A_j with $a_{j,k} = k + c$ is $\binom{k+c-1}{k-1}$ (assuming $c \leq l$), we get

$$\left| \bigcup_{i=1}^m (A_i \cup B_i) \right| = k + l + \sum_{c=0}^l (l - c) \binom{k + c - 1}{k - 1}.$$

Next notice that

$$\sum_{c=0}^l (l - c) \binom{k + c - 1}{k - 1} = \sum_{x=0}^{l-1} \sum_{c=0}^x \binom{k + c - 1}{k - 1} = \sum_{x=0}^{l-1} \binom{k + x}{k} = \binom{k + l}{k + 1},$$

hence the result follows. □

Note that Construction 3.5 shows that the calculation in Lemma 3.3 to bound β_t is tight and thus to obtain better bounds on $n_1(k, l)$ one has to use further ideas.

The proof below of the upper bound on $n_1(k, l)$ is based on Tuza's approach [11] to determine $n(k, l)$.

Proposition 3.7. Let $k \leq l$ be positive integers. Then $n_1(k, l) \leq \binom{k+l+2}{k+1} - \binom{k+l}{k} - 2$ holds.

Proof. Let $\{(A_i, B_i)\}_{i=1}^s$ be a set of skew cross intersecting pairs with $|A_i| \leq k$ and $|B_i| \leq l$ for all $1 \leq i \leq s$ and let us define $S_0 = [s]$ and $\mathcal{M}_0 = \{(A_i^0, B_i^0)\}_{i=1}^s$ with $A_i^0 = A_i$ and $B_i^0 = B_i$. If S_j and \mathcal{M}_j are defined for some $j \leq k+l-2$, then let $S_{j+1} \subset S_j$ be an index set minimal with respect to the property that

$$\bigcup_{i \in S_j} (A_i^j \cup B_i^j) = \bigcup_{i \in S_{j+1}} (A_i^j \cup B_i^j).$$

By minimality for every $i \in S_{j+1}$ there exists a point $x_i \in (A_i^j \cup B_i^j) \setminus \bigcup_{l \in S_{j+1} \setminus \{i\}} (A_l^j \cup B_l^j)$. Let us define $A_i^{j+1} = A_i^j \setminus \{x_i\}$, $B_i^{j+1} = B_i^j \setminus \{x_i\}$ for all $i \in S_{j+1}$ and put $\mathcal{M}_{j+1} = \{(A_i^{j+1}, B_i^{j+1}) : i \in S_{j+1}\}$. Observe that \mathcal{M}_j is skew intersecting for all $1 \leq j \leq k+l-1$ with $|A_i^j \cup B_i^j| \leq k+l-j$ for all $i \in S_j$ and furthermore

$$\bigcup_{i=1}^s (A_i \cup B_i) = \sum_{j=1}^{k+l-1} |\mathcal{M}_j|.$$

In Tuza's original proof the \mathcal{M}_j 's are cross intersecting and therefore he can use Bollobás's inequality to obtain $|\mathcal{M}_j| \leq \binom{k+l-j}{\lceil \frac{k+l-j}{2} \rceil}$ for any j and $|\mathcal{M}_j| \leq \binom{k+l-j}{k}$ if $j \leq l-k$. As Bollobás's inequality is not valid for skew intersecting pairs, therefore we partition \mathcal{M}_j into some subsystems indexed by the pairs $(|A_i \setminus A_i^j|, |B_i \setminus B_i^j|)$. Note that by the construction of the \mathcal{M}_j 's for the index pairs (a, b) we have $0 \leq a, b \leq j$, $a+b=j$, $a \leq k$ and $b \leq l$. For such a subsystem of \mathcal{M}_j , indexed by (a, b) , we can apply Theorem 1.2 and obtain the upper bound $\binom{k+l-j}{k-a}$. Thus adding these up for all \mathcal{M}_j , $j \in [1, k+l-1]$, we get

$$\sum_{j=1}^{k+l-1} \sum_{a=0}^j \binom{k+l-j}{k-a} = \sum_{\substack{\beta \leq k \\ \alpha \leq \beta+l}} \binom{\alpha}{\beta} - \binom{0}{0} - \binom{k+l}{k}.$$

Here

$$\sum_{\substack{\beta \leq k \\ \alpha \leq \beta+l}} \binom{\alpha}{\beta} = \sum_{\beta=0}^k \sum_{\gamma=0}^l \binom{\beta+\gamma}{\beta} = \sum_{\beta=0}^k \binom{\beta+l+1}{\beta+1} = \binom{k+l+2}{k+1} - 1,$$

confirming the statement. □

In [12], Tuza proposed the investigation of the so-called *weakly cross-intersecting set pair systems*, which are closely related to the cross-intersecting set pair systems.

Definition 3.8. Let A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m be sets such that $|A_i| = k$ and $|B_i| = l$ holds for all $1 \leq i \leq m$. Let furthermore these sets satisfy

- (1) $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$,
- (2) $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for all $1 \leq i, j \leq m$, $i \neq j$.

Then the system $(A_i, B_i)_{i=1}^m$ is called a (k, l) -weakly cross-intersecting set pair system. Let $m_{max}(k, l)$ denote the largest $m \in \mathbb{Z}$ for which a (k, l) -weakly cross-intersecting set pair system $(A_i, B_i)_{i=1}^m$ exists.

Surprisingly, much less is known about the maximum size of a weakly cross-intersecting set pair system compared to the original case. Concerning the upper bound, Tuza showed [12] that $m_{max}(k, l) \leq \frac{(k+l)^{k+l}}{k^k l^l}$. Király, Nagy, Pálvölgyi and Visontai gave a construction [8] that provides $\liminf_{k+l \rightarrow \infty} m_{max}(k, l) \geq (2 - o(1)) \binom{k+l}{k}$. Moreover, they conjectured the latter result to be sharp:

Conjecture 3.9 ([8]).

$$m_{max}(k, l) \leq 2 \binom{k+l}{k}.$$

These questions motivate the investigation of

$$n_2(k, l) = \max \left\{ \left| \bigcup_{i=1}^m (A_i \cup B_i) \right| : (A_1, B_1), \dots, (A_m, B_m) \text{ is a } (k, l)\text{-weakly cross intersecting system} \right\}.$$

First, observe that the idea of the proof of Proposition 3.7 works smoothly to obtain an upper bound on $n_2(k, l)$, since we may define weakly cross intersecting set pair systems \mathcal{M}_j similarly from a given (k, l) -weakly cross intersecting set pair system. Thus the exact upper bound only depends on $m_{max}(k, l)$. Hence, assuming that Conjecture 3.9 holds, we get the double of the upper bound of Proposition 3.7.

A lower bound follows from

Construction 3.10. Let Y be a set of $k+l-1$ elements. Assign a subset $B'_i \subset (Y \setminus A'_i)$ of size $l-1$ to each $k-1$ element subset $A'_i \subset Y$ in such a way that the sets B'_i are distinct. This can be done due to the König-Hall theorem and the fact that $k \leq l$. For each A'_i , assign furthermore three distinct elements $x_i, y_i, z_i \notin Y$. Take the set pairs $(A'_i \cup \{x_i\}, B'_i \cup \{y_i\})$, $(A'_i \cup \{y_i\}, B'_i \cup \{z_i\})$, $(A'_i \cup \{z_i\}, B'_i \cup \{x_i\})$ for all i . This way we obtain $3 \binom{k+l-1}{k-1}$ set pairs such that the union of these sets consists of $k+l-1 + 3 \binom{k+l-1}{k-1}$ points.

Proposition 3.11. $k+l-1 + 3 \binom{k+l-1}{k-1} \leq n_2(k, l)$.

Proof. The proposition follows from the fact that Construction 3.10 provides a weakly cross intersecting set pair system, which is easy to see. \square

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