Shadows and Intersections in Vector Spaces

Ameera Chowdhury * Balázs Patkós †

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Abstract

We prove a vector space analog of a version of the Kruskal-Katona theorem due to Lovász. We apply this result to extend Frankl's theorem on r-wise intersecting families to vector spaces. In particular, we obtain a short new proof of the Erdős-Ko-Rado theorem for vector spaces.

1 Introduction

Let X be an n-element set and, for $0 \leq k \leq n$, let $\binom{X}{k}$ denote the family of all subsets of X of cardinality k. For a family $\mathcal{F} \subset \binom{X}{k}$, we define the shadow of $\mathcal{F}$, denoted $\partial\mathcal{F}$, to consist of those $(k-1)$-subsets of X contained in at least one member of $\mathcal{F}$,

$$\partial\mathcal{F} := \left\{ E \in \binom{X}{k-1} : E \subset F \in \mathcal{F} \right\}.$$

Kruskal [15] and Katona [13] determined the minimum size of the shadow of $\mathcal{F}$ as a function of k and the size of $\mathcal{F}$. Recall that the binomial coefficient

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!}$$

can be defined for all $n \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. Lovász [16, Ex 13.31(b)] proved the following weaker but more convenient version of the Kruskal-Katona theorem.

Theorem 1.1 (Lovász) *Let $\mathcal{F} \subset \binom{X}{k}$ and let $y \geq k$ be the real number defined by $|\mathcal{F}| = \binom{y}{k}$. Then $|\partial\mathcal{F}| \geq \binom{y}{k-1}$. If equality holds, then $y \in \mathbb{Z}^+$ and $\mathcal{F} = \binom{Y}{k}$, where Y is a y-subset of X.*

*Department of Mathematics, University of California San Diego, La Jolla, CA, 92093, USA. E-mail: anchowdh@math.ucsd.edu

†Department of Computer Science, The University of Memphis, Memphis, TN, 38152, USA. Supported by NSF Grant #: CCF-0728928. E-mail: bpatkos@memphis.edu

The Kruskal-Katona theorem can be used to prove many theorems about so-called intersecting families of sets. A family $\mathcal{F} \subset \binom{X}{k}$ is called *r-wise intersecting* if for all $F_1, \ldots, F_r \in \mathcal{F}$ we have $\bigcap_{i=1}^r F_i \neq \emptyset$. When $r = 2$, then r-wise is omitted. The maximum size of an intersecting family was determined by Erdős, Ko, and Rado [6].

Theorem 1.2 (Erdős-Ko-Rado) *Suppose $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Moreover, excepting the case $r = 2$ and $n = 2k$, equality holds if and only if $\mathcal{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$.*

Daykin [4] gave a proof of Theorem 1.2 that essentially only uses Theorem 1.1. Frankl [7] generalized Theorem 1.2 and found the maximum size of an r-wise intersecting family.

Theorem 1.3 (Frankl) *Suppose that $\mathcal{F} \subset \binom{X}{k}$ is r-wise intersecting and $(r-1)n \geq rk$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Moreover, excepting the case $r = 2$ and $n = 2k$, equality holds if and only if $\mathcal{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$.*

Theorem 1.1, Theorem 1.2, and Theorem 1.3 have natural extensions to vector spaces. We let V always denote a n-dimensional vector space over a finite field of order q. For $k \in \mathbb{Z}^+$, we write $\begin{bmatrix} V \\ k \end{bmatrix}_q$ to denote the family of all k-dimensional subspaces of V. For $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \leq i < k} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$

A simple counting argument shows that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. If k and q are fixed, then $\begin{bmatrix} a \\ k \end{bmatrix}_q$ is a continuous function of a which is positive and strictly increasing when $a \geq k$. From now on, we will omit the subscript q.

The definition of the shadow of a family extends naturally to vector spaces. For a family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$, we define the shadow of $\mathcal{F}$, denoted $\partial\mathcal{F}$, to consist of those $(k-1)$-dimensional subspaces of V contained in at least one member of $\mathcal{F}$,

$$\partial\mathcal{F} := \left\{ E \in \begin{bmatrix} V \\ k-1 \end{bmatrix} : E \subset F \in \mathcal{F} \right\}.$$

In this paper, we will prove the following analog of Theorem 1.1.

Theorem 1.4 *Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ and let $y \geq k$ be the real number defined by $|\mathcal{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$. Then*

$$|\partial\mathcal{F}| \geq \begin{bmatrix} y \\ k-1 \end{bmatrix}.$$

If equality holds, then $y \in \mathbb{Z}^+$ and $\mathcal{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$, where Y is a y-dimensional subspace of V.

Not much is known about shadows in vector spaces. In [2], a partial analog of the Kruskal-Katona theorem is given when V is a vector space over the field $\mathbb{F}_2$. The only other result on shadows in vector spaces, which is known to the authors, appears in [8].

We will use Theorem 1.4 to extend Theorem 1.3 to vector spaces. A family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is called r-wise intersecting if for all $F_1, \ldots, F_r \in \mathcal{F}$ we have $\bigcap_{i=1}^r F_i \neq \{0\}$.

Theorem 1.5 *Suppose $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is r-wise intersecting and $(r-1)n \geq rk$. Then*

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Moreover, equality holds if and only if $\mathcal{F} = \left\{ F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F \right\}$ for some one-dimensional subspace $v \subset V$, unless $r = 2$ and $n = 2k$.

The case $r = 2$ of Theorem 1.5 is the Erdős-Ko-Rado theorem for vector spaces, which has been extensively studied. Hsieh [12] first proved the Erdős-Ko-Rado theorem for vector spaces, but not for all relevant n and his proof involves many lengthy calculations. Later, Frankl and Wilson [9] proved the Erdős-Ko-Rado theorem for vector spaces, essentially by computing the eigenvalues of the so-called q-Kneser graph; the *q-Kneser graph* has the k-dimensional subspaces of V as its vertices, where two subspaces α, β are adjacent if $\alpha \cap \beta = \{0\}$. While Frankl and Wilson's method is less computational than Hsieh's, finding the eigenvalues of the q-Kneser graph still requires some calculations. One nice feature of Theorem 1.5's proof is that it hardly involves any calculations.

It is unclear where the characterization of equality in the case $n = 2k$ of the Erdős-Ko-Rado theorem for vector spaces first appeared in the literature. Recently, Godsil and Newman [10, 17] gave a characterization of equality in this case using techniques similar to those of Frankl and Wilson [9]. A second nice feature of Theorem 1.5's proof is that it gives a simple proof of the characterization of equality when $(r-1)n > rk$.

Greene and Kleitman [11] gave a very elegant proof to the Erdős-Ko-Rado theorem for vector spaces when $k|n$. Deza and Frankl [5] sketched an inductive proof of the Erdős-Ko-Rado theorem for vector spaces using Greene and Kleitman's proof for the base case $n = 2k$ and a generalization of the shifting technique. Czabarka and Székely [3] assert that there are counterexamples to Deza and Frankl's proof and attempt a new inductive proof, again using a generalization of shifting. We believe, however, that their definition of shifting is also flawed, and that their proof is not valid. We remark that Theorem 1.5's proof proceeds by induction.

The rest of the paper is organized as follows. Section 2 gives a proof of Theorem 1.4. In Section 3 we prove the bound in Theorem 1.5 and characterize equality when $(r-1)n > rk$. Finally, in Section 4, we characterize equality when $(r-1)n = rk$ for completeness.

2 Proof of Theorem 1.4

Keevash [14] recently gave a short new proof of Theorem 1.1. In this section, we adapt his argument to prove Theorem 1.4. We first collect some definitions and facts that will be used in the proof of Theorem 1.4. If $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$, then

$$K^k_{k+1}(\mathcal{F}) := \left\{ T \in \begin{bmatrix} V \\ k+1 \end{bmatrix} : \begin{bmatrix} T \\ k \end{bmatrix} \subset \mathcal{F} \right\}$$

is the family of $(k+1)$-dimensional subspaces in V all of whose k-dimensional subspaces lie in $\mathcal{F}$. If $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, then

$$K^k_{k+1}(v) := \{T \in K^k_{k+1}(\mathcal{F}) : v \subset T\}$$

is the family of $(k+1)$-dimensional subspaces in $K_{k+1}^{k}(\mathcal{F})$ that contain v. For $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, define the *degree* of v, which is denoted by $d(v)$, to be the number of elements of $\mathcal{F}$ that contain v. If $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \subset V$ is an $(n-1)$-dimensional subspace not containing v then

$$L_U(v) := \left\{ A \in \begin{bmatrix} U \\ k-1 \end{bmatrix} : A \vee v \in \mathcal{F} \right\}$$

is the family of $(k-1)$-dimensional spaces in U whose linear span with v is an element of $\mathcal{F}$. Observe that $d(v) = |L_U(v)|$.

Finally, we collect some notation and facts regarding the Gaussian binomial coefficients. When $k = 1$, we will write the Gaussian binomial coefficient $\begin{bmatrix} a \\ 1 \end{bmatrix}$ as $[a]$. For $a \in \mathbb{Z}^+$, we define $[a]! = \prod_{j=1}^{a} [j]$. A familiar relation involving binomial coefficients is Pascal's identity. We note two similar relations involving Gaussian binomial coefficients.

Lemma 2.1 *If $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$, then*

$$\begin{bmatrix} a \\ k \end{bmatrix} = q^{a-k} \begin{bmatrix} a-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} a-1 \\ k \end{bmatrix} = \begin{bmatrix} a-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} a-1 \\ k \end{bmatrix}.$$

Keevash observed that the analog of Theorem 2.2 for sets implies Theorem 1.1.

Theorem 2.2 *Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ and let $y \geq k$ be the real number defined by $|\mathcal{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$. Then*

$$|K_{k+1}^{k}(\mathcal{F})| \leq \begin{bmatrix} y \\ k+1 \end{bmatrix}.$$

Equality holds if and only if $y \in \mathbb{Z}^+$ and $\mathcal{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$ for some y-dimensional subspace $Y \subset V$.

We observe that Theorem 2.2 implies Theorem 1.4. Let $\mathcal{F}$ be as in Theorem 1.4, and let $x \geq k-1$ be the real number defined by $|\partial\mathcal{F}| = \begin{bmatrix} x \\ k-1 \end{bmatrix}$. By Theorem 2.2,

$$\begin{bmatrix} y \\ k \end{bmatrix} = |\mathcal{F}| \leq |K_k^{k-1}(\partial\mathcal{F})| \leq \begin{bmatrix} x \\ k \end{bmatrix}$$

because $\mathcal{F} \subset K_k^{k-1}(\partial\mathcal{F})$. Hence $x \geq y$ so $|\partial\mathcal{F}| = \begin{bmatrix} x \\ k-1 \end{bmatrix} \geq \begin{bmatrix} y \\ k-1 \end{bmatrix}$. If $|\partial\mathcal{F}| = \begin{bmatrix} y \\ k-1 \end{bmatrix}$ then $x = y$. Hence, $|K_k^{k-1}(\partial\mathcal{F})| = \begin{bmatrix} y \\ k \end{bmatrix}$ and $\mathcal{F} = K_k^{k-1}(\partial\mathcal{F})$. By Theorem 2.2, this implies $y \in \mathbb{Z}^+$ and $\partial\mathcal{F} = \begin{bmatrix} Y \\ k-1 \end{bmatrix}$ for some y-dimensional subspace $Y \subset V$. Clearly, $\begin{bmatrix} Y \\ k \end{bmatrix} = K_k^{k-1}(\partial\mathcal{F}) = \mathcal{F}$.

Proof of Theorem 2.2: We argue by induction on k. The base case $k = 1$ is easy: Suppose $\mathcal{F} \subset \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $|\mathcal{F}| = [y]$. Since there are $q+1$ one-dimensional spaces in a two-dimensional space, $|K_2^1(v)| \leq (1/q)([y]-1)$ if $v \in \mathcal{F}$ and $|K_2^1(v)| = 0$ otherwise. Now

$$(q+1)|K_2^1(\mathcal{F})| = \sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} |K_2^1(v)| \leq \frac{[y]([y]-1)}{q}, \tag{2.1}$$

which implies that $|K_2^1(\mathcal{F})| \le \begin{bmatrix} y \\ 2 \end{bmatrix}$.

Suppose $T \in K_{k+1}^k(v)$. Observe that the q^k k-dimensional subspaces in T that do not contain v are elements of $\mathcal{F}$ that do not contain v. Moreover, if $U \subset V$ is an $(n-1)$-dimensional subspace that does not contain v, then $T \cap U$ is a k-dimensional subspace in $K_k^{k-1}(L_U(v))$. The first condition implies that

$$q^k |K_{k+1}^k(v)| = \left| \left\{ S \in \begin{bmatrix} V \\ k \end{bmatrix} : v \not\subset S \subset T \in K_{k+1}^k(v) \right\} \right| \le |\mathcal{F}| - d(v),$$

and hence that $|K_{k+1}^k(v)| \le (1/q^k)(|\mathcal{F}| - d(v))$. The second condition implies that $|K_{k+1}^k(v)| \le |K_k^{k-1}(L_U(v))|$ because if T_1, T_2 are distinct elements of $K_{k+1}^k(v)$ then $T_1 \cap U$ and $T_2 \cap U$ are distinct elements of $K_k^{k-1}(L_U(v))$.

We claim that $|K_{k+1}^k(v)| \le ([y-k]/[k])d(v)$ for all $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, which is trivial if $d(v) = 0$. Furthermore, if $d(v) \ne 0$, then equality is possible only when $d(v) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$. To see this, suppose first that $d(v) \ge \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$. Then by the first condition and Lemma 2.1, it suffices to observe that $(1/q^k)(\begin{bmatrix} y \\ k \end{bmatrix} - d(v)) \le ([y-k]/[k])d(v)$. On the other hand, if $d(v) \le \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$, then define the real number $y_v \ge k$ by $d(v) = \begin{bmatrix} y_v-1 \\ k-1 \end{bmatrix}$. Since $d(v) = |L_U(v)|$, the second condition and the induction hypothesis imply that

$$|K_{k+1}^k(v)| \le |K_k^{k-1}(L_U(v))| \le \begin{bmatrix} y_v - 1 \\ k \end{bmatrix} = \frac{[y_v - k]}{[k]} d(v) \le \frac{[y-k]}{[k]} d(v).$$

The equality conditions are clear so the claim holds in either case. Now

$$\begin{aligned}[k+1]|K_{k+1}^k(\mathcal{F})| &= \sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} |K_{k+1}^k(v)| \le \frac{[y-k]}{[k]} \sum_{v \in \begin{bmatrix} V \\ 1 \end{bmatrix}} d(v) = \frac{[y-k]}{[k]} [k] |\mathcal{F}| \\ &= [y-k] \begin{bmatrix} y \\ k \end{bmatrix} = [k+1] \begin{bmatrix} y \\ k+1 \end{bmatrix}.\end{aligned}$$

Therefore, $|K_{k+1}^k(\mathcal{F})| \le \begin{bmatrix} y \\ k+1 \end{bmatrix}$, and equality holds only when all one-dimensional subspaces v with non-zero degree satisfy $d(v) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$.

We now characterize the case of equality. Again the proof proceeds by induction on k. The base case $k=1$ is easy: Suppose $\mathcal{F} \subset \begin{bmatrix} V \\ 1 \end{bmatrix}$, $|\mathcal{F}| = [y]$, and $|K_2^1(\mathcal{F})| = \begin{bmatrix} y \\ 2 \end{bmatrix}$. Then (2.1) implies that $|K_2^1(v)| = (1/q)([y]-1)$ for all $v \in \mathcal{F}$. Hence, if v, w are distinct elements of $\mathcal{F}$, then every one-dimensional space in the two-dimensional space spanned by v and w lies in $\mathcal{F}$. It is easy to see by induction that if A is a subspace of dimension $1 \le d < \lceil y \rceil$ such that $\begin{bmatrix} A \\ 1 \end{bmatrix} \subset \mathcal{F}$, then there exists a subspace B of dimension $d+1$ that contains A and for which $\begin{bmatrix} B \\ 1 \end{bmatrix} \subset \mathcal{F}$. In particular, this proves that $y \in \mathbb{Z}^+$ and $\mathcal{F} = \begin{bmatrix} Y \\ 1 \end{bmatrix}$ for some y-dimensional subspace Y.

Now suppose $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$, $|\mathcal{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$, and $|K_{k+1}^k(\mathcal{F})| = \begin{bmatrix} y \\ k+1 \end{bmatrix}$. Choose $v \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ for which $d(v) \ne 0$. Since $|K_{k+1}^k(\mathcal{F})| = \begin{bmatrix} y \\ k+1 \end{bmatrix}$, we have $d(v) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$ and $|K_{k+1}^k(v)| = \begin{bmatrix} y-1 \\ k \end{bmatrix}$. Let U be an $(n-1)$-dimensional subspace not containing v. We have $|L_U(v)| = d(v) = \begin{bmatrix} y-1 \\ k-1 \end{bmatrix}$ so

$$\begin{bmatrix} y-1 \\ k \end{bmatrix} = |K_{k+1}^k(v)| \le |K_k^{k-1}(L_U(v))| \le \begin{bmatrix} y-1 \\ k \end{bmatrix},$$

which implies that $|K_k^{k-1}(L_U(v))| = \begin{bmatrix} y-1 \\ k \end{bmatrix}$. By the induction hypothesis, $L_U(v) = \begin{bmatrix} W \\ k-1 \end{bmatrix}$ for some $(y-1)$-dimensional space W, which implies $y \in \mathbb{Z}^+$. Moreover, for every k-dimensional subspace A in $K_k^{k-1}(L_U(v)) = \begin{bmatrix} W \\ k \end{bmatrix}$, we have $A \vee v \in K_{k+1}^k(v)$. Hence all k-dimensional subspaces in $Y := W \vee v$ lie in $\mathcal{F}$. Since $|\mathcal{F}| = \begin{bmatrix} y \\ k \end{bmatrix}$ and $\dim(Y) = y$, we must have $\mathcal{F} = \begin{bmatrix} Y \\ k \end{bmatrix}$. □

3 Proof of Theorem 1.5

We will prove the bound in Theorem 1.5 and characterize equality when $(r-1)n > rk$. The proof proceeds by induction on $(r-1)n - rk \in \mathbb{N}$. For the base case $(r-1)n - rk = 0$, we generalize Greene and Kleitman's argument in [11]. A family $\mathcal{S}$ of t-dimensional subspaces of V is called a *t-spread* if every one-dimensional subspace of V is contained in exactly one t-dimensional subspace in $\mathcal{S}$. If the elements in $\mathcal{S}$ that lie in a subspace U form a t-spread of U then we say that $\mathcal{S}$ *induces a t-spread* on U. A t-spread $\mathcal{S}$ is called *geometric* if $\mathcal{S}$ induces a t-spread on each $2t$-dimensional subspace generated by a pair of elements in $\mathcal{S}$. It is well-known [1] that V possesses a geometric t-spread if and only if $t|n$. In the base case $(r-1)n - rk = 0$, we have $n = r(n-k)$ so V has a geometric $(n-k)$-spread. The following facts concerning geometric t-spreads are easy to establish.

Lemma 3.1 *If $\mathcal{S}$ is a geometric t-spread of V, then $\mathcal{S}$ induces a geometric t-spread on any subspace of V that is generated by elements of $\mathcal{S}$.* □

Lemma 3.2 *If $\mathcal{S}$ is a geometric t-spread of V, then for any isomorphism $\pi \in GL(V)$, the family $\pi(\mathcal{S}) := \{\pi(S) : S \in \mathcal{S}\}$ is also a geometric t-spread of V.* □

Suppose $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$ and let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be an r-wise intersecting family. Endow V with the usual inner product, and consider the family

$$\mathcal{F}^\perp := \{F^\perp : F \in \mathcal{F}\} \subset \begin{bmatrix} V \\ n-k \end{bmatrix}.$$

Let $\mathcal{B}$ be a geometric $(n-k)$-spread of V. We want to determine the maximum number of elements of $\mathcal{B}$ that lie in $\mathcal{F}^\perp$. Since $\mathcal{F}$ is r-wise intersecting, we have that $\mathcal{F}^\perp$ is *r-wise co-intersecting*; that is, any r elements of $\mathcal{F}^\perp$ are contained in a common $(n-1)$-dimensional space. If $r = 2$ and $n = 2k$, the family $\mathcal{F}^\perp$ is both intersecting and co-intersecting; hence only one element of the spread $\mathcal{B}$ can lie in $\mathcal{F}^\perp$ in this case. Lemma 3.3 determines the maximum number of elements of $\mathcal{B}$ that lie in $\mathcal{F}^\perp$ whenever $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$.

Lemma 3.3 *Let $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$. Let $\mathcal{B}$ be a geometric $(n-k)$-spread of V. If $\mathcal{B}' \subset \mathcal{B}$ is a r-wise co-intersecting subfamily, then*

$$|\mathcal{B}'| \leq \frac{q^{(r-1)(n-k)} - 1}{q^{n-k} - 1}.$$

If equality holds, $\mathcal{B}'$ is a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space.

Proof. Let $B_1, \ldots, B_m$ be a maximum subfamily of $\mathcal{B}'$ such that $\dim\left(\bigvee_{i=1}^m B_i\right) = m(n-k)$. Hence, if $B \in \mathcal{B}'$ then $B \cap \bigvee_{i=1}^m B_i \neq \{0\}$. Since $\mathcal{B}$ is geometric, $\mathcal{B}$ induces a spread on $\bigvee_{i=1}^m B_i$ by Lemma 3.1. As $B \cap \bigvee_{i=1}^m B_i \neq \{0\}$ for every B in $\mathcal{B}'$, all elements in $\mathcal{B}'$ lie in $\bigvee_{i=1}^m B_i$. Since $\mathcal{B}'$ is r-wise co-intersecting, we must have $m \leq r-1$. Therefore,

$$|\mathcal{B}'| \leq \frac{q^{(r-1)(n-k)} - 1}{q^{n-k} - 1},$$

which is the number of elements in a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space. Also, if equality holds, $\mathcal{B}'$ is a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space. □

Now we prove the base case of Theorem 1.5; the case $r = 2$ of Lemma 3.4 is a result of Greene and Kleitman [11].

Lemma 3.4 *Suppose $r, n, k \in \mathbb{Z}^+$ satisfy $(r-1)n - rk = 0$. If $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is r-wise intersecting, then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.*

Proof. Let $\mathcal{B}$ be a geometric $(n-k)$-spread of V and let $\pi \in GL(V)$ be an isomorphism. By Lemma 3.2, the spread $\pi(\mathcal{B})$ is also geometric. Consider the family $\mathcal{F}^\perp \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$. Since $\mathcal{F}$ is r-wise intersecting, $\mathcal{F}^\perp$ is r-wise co-intersecting. By Lemma 3.3,

$$|\mathcal{F}^\perp \cap \pi(\mathcal{B})| \leq \frac{q^{(r-1)(n-k)} - 1}{q^{n-k} - 1} = \frac{q^k - 1}{q^{n-k} - 1} \tag{3.2}$$

because $\mathcal{F}^\perp \cap \pi(\mathcal{B})$ is a r-wise co-intersecting subfamily of $\pi(\mathcal{B})$ and because we have $k = (r-1)(n-k)$ when r, n, k satisfy $(r-1)n - rk = 0$.

As $|GL(V)| = q^{n(n-1)/2}(q-1)^n[n]!$, we have

$$\sum_{\pi \in GL(V)} |\mathcal{F}^\perp \cap \pi(\mathcal{B})| \leq \frac{q^k - 1}{q^{n-k} - 1} \cdot q^{n(n-1)/2}(q-1)^n[n]!.$$

Now, given $F^\perp \in \mathcal{F}^\perp$ and $B \in \mathcal{B}$ there are $q^{n(n-1)/2}(q-1)^n[n-k]![k]!$ isomorphisms $\pi \in GL(V)$ such that $\pi(B) = F^\perp$. Consequently,

$$\begin{aligned}
& \left(\frac{q^n - 1}{q^{n-k} - 1}\right) |\mathcal{F}^\perp|\, q^{n(n-1)/2}(q-1)^n[n-k]![k]! \\
= \;& |\mathcal{B}|\, |\mathcal{F}^\perp|\, |\{\pi \in GL(V) : \pi(B) = F^\perp\}| \\
= \;& \sum_{\pi \in GL(V)} |\mathcal{F}^\perp \cap \pi(\mathcal{B})| \\
\leq \;& \frac{q^k - 1}{q^{n-k} - 1} \cdot q^{n(n-1)/2}(q-1)^n[n]!.
\end{aligned}$$

Since $|\mathcal{F}| = |\mathcal{F}^\perp|$, we have

$$\begin{aligned}
|\mathcal{F}| \;&\leq\; \left(\frac{q^{n(n-1)/2}(q-1)^n[n]!}{q^{n(n-1)/2}(q-1)^n[n-k]![k]!}\right)\left(\frac{q^{n-k} - 1}{q^n - 1}\right)\left(\frac{q^k - 1}{q^{n-k} - 1}\right) \\
&=\; \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \quad \square
\end{aligned}$$

Proof of Theorem 1.5. The proof proceeds by induction on $(r-1)n - rk \in \mathbb{N}$. The base case $(r-1)n - rk = 0$ was proved in Lemma 3.4. Suppose Theorem 1.5 holds when r, n, k satisfy $(r-1)n - rk = p$ for $p \geq 0$. We will prove Theorem 1.5 holds when r, n, k satisfy $(r-1)n - rk = p+1$. Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family. Now the family $\mathcal{P} := \{P \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset P\}$, where $v \subset V$ is some one-dimensional subspace, is r-wise intersecting so $|\mathcal{F}| \geq |\mathcal{P}| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Let W be an $(n+1)$-dimensional space over $\mathbb{F}_q$ that contains V. Define the family

$$\mathcal{A} := \left\{ A \in \begin{bmatrix} W \\ k+1 \end{bmatrix} : \exists\, F \in \mathcal{F} \text{ with } F \subset A \right\}$$

to be the family of all $(k+1)$-dimensional spaces in W that contain some $F \in \mathcal{F}$. We will partition $\mathcal{A}$ into the following subfamilies:

$$\mathcal{A}_1 := \{A \in \mathcal{A} : A \not\subset V\}, \quad \mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1.$$

First let us compute the size of $\mathcal{A}_1$. Observe that if $A \in \begin{bmatrix} W \\ k+1 \end{bmatrix}$ and A does not lie in V, then A intersects V in exactly a k-dimensional space. Therefore, A cannot contain two distinct k-dimensional spaces in $\mathcal{F}$. Any $F \in \mathcal{F}$ can be extended to a $(k+1)$-dimensional space in $\mathcal{A}_1$ in q^{n-k} ways. Therefore, $|\mathcal{A}_1| = q^{n-k}|\mathcal{F}| \geq q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

Now we will compute the size of $\mathcal{A}_2$. Observe that, by duality, we have $F \subset A \in \mathcal{A}_2$ for some $F \in \mathcal{F}$ if and only if $F^\perp \supset A^\perp \in \begin{bmatrix} V \\ n-k-1 \end{bmatrix}$. Therefore, $|\mathcal{A}_2| = |\partial \mathcal{F}^\perp|$. Since

$$|\mathcal{F}^\perp| = |\mathcal{F}| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}, \tag{3.3}$$

by applying Theorem 1.4 we obtain

$$|\mathcal{A}_2| = |\partial \mathcal{F}^\perp| \geq \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{3.4}$$

As $\mathcal{A} = \mathcal{A}_1 \dot\cup \mathcal{A}_2$, we have by Lemma 2.1 that

$$|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| \geq q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{3.5}$$

Now $\mathcal{F}$ is r-wise intersecting so $\mathcal{A}$ is an r-wise intersecting family of $(k+1)$-dimensional spaces in W. Observe that $r, n+1, k+1$ satisfy

$$(r-1)(n+1) - r(k+1) = (r-1)n - rk - 1 = (p+1) - 1 = p.$$

By the induction hypothesis $|\mathcal{A}| \leq \begin{bmatrix} n \\ k \end{bmatrix}$, which implies equality everywhere in (3.3), (3.4), and (3.5). As a result, $q^{n-k}|\mathcal{F}| = |\mathcal{A}_1| = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, which implies $|\mathcal{F}| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Moreover, $|\mathcal{F}^\perp| = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}$ and $|\partial \mathcal{F}^\perp| = |\mathcal{A}_2| = \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix}$. Therefore $\mathcal{F}^\perp$ satisfies equality in Theorem 1.4, which implies that $\mathcal{F}^\perp = \begin{bmatrix} Y \\ n-k \end{bmatrix}$ for some $(n-1)$-dimensional subspace $Y \subset V$. By duality, $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$. □

4 Characterizing Equality in the Base Case

We characterize equality in Theorem 1.5 when $(r-1)n - rk = 0$. Godsil and Newman [10, 17] recently characterized equality in the Erdős-Ko-Rado theorem for vector spaces using the methods of [9]. Recall that the Erdős-Ko-Rado theorem for vector spaces is the case $r = 2$ of Theorem 1.5. In particular, they showed

Theorem 4.1 (Godsil and Newman) *If $n = 2k$ and $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is a maximum size intersecting family, then $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$ or $\mathcal{F} = \begin{bmatrix} U \\ k \end{bmatrix}$ for some $(2k-1)$-dimensional subspace $U \subset V$.*

We use their result to characterize equality in Theorem 1.5 when $(r-1)n - rk = 0$ and $r \geq 3$. The proof proceeds by induction on r; the base case $r = 2$ and $n = 2k$ is Theorem 4.1. Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family. In this section, it will be more natural to state results in terms of $\mathcal{F}^\perp \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$ so we make the following simple observation.

Lemma 4.2 *We have $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is a maximum size r-wise intersecting family if and only if $\mathcal{F}^\perp \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$ is a maximum size r-wise co-intersecting family.* □

Lemma 4.5 allows us to use induction. We first state two simple corollaries of Lemma 3.4 that will be used in the proof of Lemma 4.5. Recall that V is $r(n-k)$-dimensional since r, n, k satisfy $(r-1)n - rk = 0$.

Corollary 4.3 *Suppose r, n, k satisfy $(r-1)n - rk = 0$. Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be r-wise intersecting. If there is a geometric $(n-k)$-spread $\mathcal{B}$ of V such that equality holds in (3.2) for all $\pi \in GL(V)$, then $\mathcal{F}$ has maximum size.* □

Corollary 4.4 *Suppose r, n, k satisfy $(r-1)n - rk = 0$. If $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is a maximum size r-wise intersecting family, then equality holds in (3.2) for every geometric $(n-k)$-spread $\mathcal{B}$ of V and for every $\pi \in GL(V)$.* □

Lemma 4.5 *Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family. Fix $F^\perp$ in $\mathcal{F}^\perp$ and let $U \subset V$ be an $(r-1)(n-k)$-dimensional space that intersects $F^\perp$ trivially; that is $F^\perp \cap U = \{0\}$. Then*

$$\mathcal{F}^\perp|_U := \{E \in \mathcal{F}^\perp : E \subset U\}$$

is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$.

Proof. Let $\mathcal{S}$ be a geometric $(n-k)$-spread of V. Choose $S_1, \ldots, S_r$ in $\mathcal{S}$ such that $\bigvee_{i=1}^r S_i = V$. Since $F^\perp \cap U = \{0\}$, there exists an isomorphism $\rho \in GL(V)$ such that $\rho(S_1) = F^\perp$ and $\rho\left(\bigvee_{i=2}^r S_i\right) = U$. The $(n-k)$-spread $\mathcal{B} := \rho(\mathcal{S})$ is geometric by Lemma 3.2, and $F^\perp \in \mathcal{B}$; moreover $U = \bigvee_{i=2}^r \rho(S_i)$ so $\mathcal{B}$ induces a geometric $(n-k)$-spread $\mathcal{B}'$ on U by Lemma 3.1.

Observe that $\mathcal{F}^\perp|_U$ is $(r-1)$-wise co-intersecting since $F^\perp \cap U = \{0\}$. To prove that $\mathcal{F}^\perp|_U \subset \begin{bmatrix} U \\ n-k \end{bmatrix}$ is a maximum size $(r-1)$-wise co-intersecting family, we will apply

Lemma 4.2 and Corollary 4.3. That is, we will show that if $\alpha \in GL(U)$ then equality holds in (3.2):

$$\left|\mathcal{F}^{\perp}|_U \cap \alpha(\mathcal{B}')\right| = \frac{q^{(r-2)(n-k)}-1}{q^{n-k}-1}.$$

Let $\pi \in GL(V)$ be an isomorphism such that $\pi(F^{\perp}) = F^{\perp}$, $\pi(U) = U$, and $\pi|_U = \alpha$. Since $\mathcal{F}^{\perp}$ is a maximum size r-wise co-intersecting family, $\mathcal{F}^{\perp} \cap \pi(\mathcal{B})$ is a $(n-k)$-spread of a $(r-1)(n-k)$-dimensional space W_π by Lemma 3.3 and Corollary 4.4. Consider the subspace $W_\pi \cap U$ and observe that $\dim(W_\pi \cap U) = (r-2)(n-k)$ since $F^{\perp}$ is contained in W_π and intersects U trivially.

The spread $\pi(\mathcal{B})$ induces the spread $\mathcal{F}^{\perp} \cap \pi(\mathcal{B})$ on W_π and induces the spread $\alpha(\mathcal{B}')$ on U. Consider the elements of $\alpha(\mathcal{B}')$ that intersect $W_\pi \cap U$ non-trivially; as these elements are in $\pi(\mathcal{B})$ and intersect W_π, they must lie in W_π and hence in $W_\pi \cap U$. Hence, the elements of $\alpha(\mathcal{B}')$ that intersect $W_\pi \cap U$ non-trivially form a spread of $W_\pi \cap U$. Moreover, these elements lie in $\mathcal{F}^{\perp} \cap \pi(\mathcal{B})$ so

$$\mathcal{F}^{\perp}|_U \cap \alpha(\mathcal{B}') = (\mathcal{F}^{\perp} \cap \pi(\mathcal{B})) \cap \alpha(\mathcal{B}')$$

is the spread $\pi(\mathcal{B})$ induces on $W_\pi \cap U$. Since $W_\pi \cap U$ is $(r-2)(n-k)$-dimensional, $\left|\mathcal{F}^{\perp}|_U \cap \alpha(\mathcal{B}')\right|$ satisfies (3.2) with equality. By Lemma 4.2 and Corollary 4.3, $\mathcal{F}^{\perp}|_U$ is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$. □

Characterizing Equality in Theorem 1.5 when $(r-1)n - rk = 0$ and $r \geq 3$: We characterize equality in Theorem 1.5 when $(r-1)n - rk = 0$ and $r \geq 3$. The proof proceeds by induction on r; the base case $r = 2$ and $n = 2k$ is Theorem 4.1.

Let $r \geq 3$ and suppose the statement is proved for any $2 \leq r' < r$. Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum size r-wise intersecting family and observe that $\mathcal{F}^{\perp} \subset \begin{bmatrix} V \\ n-k \end{bmatrix}$ is a maximum size r-wise co-intersecting family. We will show that $\mathcal{F}^{\perp} = \begin{bmatrix} H \\ n-k \end{bmatrix}$ where H is a $(n-1)$-dimensional space of V. By duality, this implies that $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$, which is the desired conclusion.

Fix some $F^{\perp} \in \mathcal{F}^{\perp}$. By Lemma 4.5, if U is a $(r-1)(n-k)$-dimensional subspace that intersects $F^{\perp}$ trivially, then $\mathcal{F}^{\perp}|_U$ is a maximum size $(r-1)$-wise co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$. When $r = 3$, then $\dim U = 2(n-k)$ and $\mathcal{F}^{\perp}|_U$ is a maximum size intersecting and co-intersecting family in $\begin{bmatrix} U \\ n-k \end{bmatrix}$; hence by Theorem 4.1

1. $\mathcal{F}^{\perp}|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$ or
2. $\mathcal{F}^{\perp}|_U = \begin{bmatrix} U' \\ n-k \end{bmatrix}$ for some $(2(n-k)-1)$-dimensional subspace $U' \subset U$.

If $r > 3$ then, by the induction hypothesis and duality, $\mathcal{F}^{\perp}|_U = \begin{bmatrix} U' \\ n-k \end{bmatrix}$, where $U' \subset U$ is some $((r-1)(n-k)-1)$-dimensional subspace.

Our first task is to eliminate the possibility that $\mathcal{F}^{\perp}|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$ in the case $r = 3$. We now show that if $\mathcal{F}^{\perp}|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then every element of $\mathcal{F}^{\perp}$ must intersect $F^{\perp} \vee u$ non-trivially.

Claim 4.6 If $\mathcal{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then for all $G \in \mathcal{F}^\perp$ we have $G \cap (F^\perp \vee u) \neq \{0\}$.

Proof. Suppose, for a contradiction, that there exists $G \in \mathcal{F}^\perp$ such that G intersects $F^\perp \vee u$ trivially. We have $\dim((F^\perp \vee G) \cap U) = n-k$ because $F^\perp$ intersects both G and U trivially. Since u does not lie in $F^\perp \vee G$ and $\mathcal{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$, we can find $E' \in \mathcal{F}^\perp|_U$ that intersects $F^\perp \vee G$ trivially. Hence $F^\perp \vee G \vee E' = V$, which contradicts the fact that $\mathcal{F}^\perp$ is 3-wise co-intersecting. $\square$

We now show that if $\mathcal{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$, then any $(n-k)$-dimensional space that meets $F^\perp$ trivially but meets $F^\perp \vee u$ non-trivially must lie in $\mathcal{F}^\perp$.

Claim 4.7 Suppose $\mathcal{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$. If $G \in \begin{bmatrix} V \\ n-k \end{bmatrix}$, $G \cap F^\perp = \{0\}$, and $G \cap (F^\perp \vee u) \neq \{0\}$, then $G \in \mathcal{F}^\perp$.

Proof. There exists a geometric $(n-k)$-spread $\mathcal{B}$ of V that contains both G and $F^\perp$ because $G \cap F^\perp = \{0\}$. Since $\mathcal{B}$ is a spread, all subspaces in $(\mathcal{F}^\perp \cap \mathcal{B}) \setminus \{F^\perp\}$ meet $F^\perp \vee u$ in a one-dimensional subspace that does not lie in $F^\perp$ by Claim 4.6. Lemma 3.3 and Corollary 4.4 imply that $\mathcal{F}^\perp \cap \mathcal{B}$ is a spread of a $2(n-k)$-dimensional space so $|(\mathcal{F}^\perp \cap \mathcal{B}) \setminus \{F^\perp\}| = q^{n-k}$. There are q^{n-k} one-dimensional subspaces in $F^\perp \vee u$ that do not lie in $F^\perp$. Hence, each one-dimensional subspace in $(F^\perp \vee u) \setminus F^\perp$ meets a unique subspace in $(\mathcal{F}^\perp \cap \mathcal{B}) \setminus \{F^\perp\}$. Since G meets $F^\perp \vee u$ in a one-dimensional subspace that does not lie in $F^\perp$ and $G \in \mathcal{B}$, we must have $G \in \mathcal{F}^\perp \cap \mathcal{B} \subset \mathcal{F}^\perp$. $\square$

We now eliminate the possibility that $\mathcal{F}^\perp|_U = \{E \in \begin{bmatrix} U \\ n-k \end{bmatrix} : u \subset E\}$ for some one-dimensional subspace $u \subset U$. We will construct three $(n-k)$-dimensional subspaces that together span V, and intersect $F^\perp \vee u$ in a one-dimensional subspace not lying in $F^\perp$. By Claim 4.7, these three spaces lie in $\mathcal{F}^\perp$, which contradicts $\mathcal{F}^\perp$ being 3-wise co-intersecting. To build these three subspaces, we first choose three one-dimensional subspaces v_1^1, v_2^1, v_3^1 in $(F^\perp \vee u) \setminus F^\perp$ such that $v_3^1 \not\subset v_1^1 \vee v_2^1$. These one-dimensional subspaces exist because $\dim(F^\perp \vee u) = (n-k)+1 \geq 3$ so, after picking v_1^1 and v_2^1, any one-dimensional subspace of $F^\perp \vee u$ not in $F^\perp \cup (v_1^1 \vee v_2^1)$ will do. Since the number of one-dimensional subspaces in $(F^\perp \vee u) \setminus (F^\perp \cup (v_1^1 \vee v_2^1))$ is $q^{n-k} - q > 0$, we can indeed choose v_3^1.

We construct a family of one-dimensional subspaces

$$\{v_i^j : i \in \{1,2,3\}, j \in \{1,\ldots,n-k\}\}$$

such that, for each $i \in \{1,2,3\}$, the subspace $V_i = \bigvee_{j=1}^{n-k} v_i^j$ intersects $F^\perp \vee u$ in the one-dimensional subspace $v_i^1 \not\subset F^\perp$ and $\bigvee_{i=1}^{3} V_i = V$. The subspaces V_1, V_2, V_3 are the desired three $(n-k)$-dimensional subspaces. We pick the one-dimensional subspaces one after the other; we have to show that at each step there is a possible one-dimensional subspace to pick. When picking the last one-dimensional subspace v_3^{n-k} we must choose a one-dimensional subspace from V that is not in $V_1 \vee V_2 \vee \bigvee_{j=1}^{n-k-1} v_3^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_3^j$. By inclusion-exclusion, there are $q^{3(n-k)-1} - q^{2(n-k)-2} > 0$ one-dimensional subspaces in V that do not lie in $V_1 \vee V_2 \vee \bigvee_{j=1}^{n-k-1} v_3^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_3^j$; thus it is indeed possible to construct the desired three $(n-k)$-dimensional subspaces. Therefore, we have eliminated the possibility that $\mathcal{F}^\perp|_U = \{E \in \left[{U \atop n-k}\right] : u \subset E\}$ for some one-dimensional subspace $u \subset U$ in the case $r = 3$.

We may now assume that $r \geq 3$ and that if U is a $(r-1)(n-k)$-dimensional space that intersects $F^\perp$ trivially then $\mathcal{F}^\perp|_U = \left[{U' \atop n-k}\right]$ for some $((r-1)(n-k)-1)$-dimensional subspace $U' \subset U$. Our ultimate goal is to prove that $\mathcal{F}^\perp = \left[{F^\perp \vee U' \atop n-k}\right]$. Naturally, we first show that if U_1, U_2 are two $(r-1)(n-k)$-dimensional subspaces that intersect $F^\perp$ trivially, then $F^\perp \vee U_1' = F^\perp \vee U_2'$.

Claim 4.8 Let U_1, U_2 be two $(r-1)(n-k)$-dimensional subspaces of V that intersect $F^\perp$ trivially. Let U_1', U_2' be the $((r-1)(n-k)-1)$-dimensional subspaces of U_1 and U_2 such that $\mathcal{F}^\perp|_{U_1} = \left[{U_1' \atop n-k}\right]$ and $\mathcal{F}^\perp|_{U_2} = \left[{U_2' \atop n-k}\right]$. Then $F^\perp \vee U_1' = F^\perp \vee U_2'$.

Proof. Suppose, for a contradiction, that $F^\perp \vee U_1' \neq F^\perp \vee U_2'$. Choose subspaces $W_1, \ldots, W_{r-2}$ in $\left[{U_1' \atop n-k}\right]$ such that W_1 is not contained in $F^\perp \vee U_2'$ and $\dim\left(\bigvee_{i=1}^{r-2} W_i\right) = (r-2)(n-k)$.

The subspace $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ is $(r-1)(n-k)$-dimensional because U_1 intersects $F^\perp$ trivially. The subspace U_2' is $((r-1)(n-k)-1)$-dimensional and intersects $F^\perp$ trivially so

$$(r-2)(n-k) - 1 \leq \dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) \leq (r-2)(n-k).$$

Suppose that $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k)$ for a contradiction. By definition of W_1, we can choose a one-dimensional subspace $w \subset W_1$ that does not lie in $F^\perp \vee U_2'$. The subspace $F^\perp \vee w$ is $(n-k+1)$-dimensional. The subspace $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ is $(r-1)(n-k)$-dimensional and contains $F^\perp \vee w$. If $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k)$, then $F^\perp \vee w$ must intersect U_2' non-trivially. This is a contradiction because w does not lie in $F^\perp \vee U_2'$ by construction. Therefore, $\dim\left(U_2' \cap \left(F^\perp \vee \bigvee_{i=1}^{r-2} W_i\right)\right) = (r-2)(n-k) - 1$.

Since U_2' is $((r-1)(n-k)-1)$-dimensional, this implies that there exists a subspace Z in $\left[{U_2' \atop n-k}\right]$ that intersects $F^\perp \vee \bigvee_{i=1}^{r-2} W_i$ trivially. Now $F^\perp, W_1, \ldots, W_{r-2}, Z$ lie in $\mathcal{F}^\perp$ since $\mathcal{F}^\perp|_{U_1} = \left[{U_1' \atop n-k}\right]$ and $\mathcal{F}^\perp|_{U_2} = \left[{U_2' \atop n-k}\right]$. By construction, $F^\perp \vee \bigvee_{i=1}^{r-2} W_i \vee Z = V$, which contradicts $\mathcal{F}^\perp$ being r-wise co-intersecting. This proves $F^\perp \vee U_1' = F^\perp \vee U_2'$. □

Now we show that any $(n-k)$-dimensional subspace in $F^\perp \vee U'$ that intersects $F^\perp$ trivially must lie in $\mathcal{F}^\perp$.

Claim 4.9 If $G \in \left[{F^\perp \vee U' \atop n-k}\right]$ and $G \cap F^\perp = \{0\}$, then $G \in \mathcal{F}^\perp$.

Proof. Since $G \cap F^\perp = \{0\}$, there exists a $(r-1)(n-k)$-dimensional subspace $U(G)$ that contains G and intersects $F^\perp$ trivially. Let $U(G)'$ be the $((r-1)(n-k)-1)$-dimensional subspace of $U(G)$ such that $\mathcal{F}^\perp|_{U(G)} = \left[{U(G)' \atop n-k}\right]$. By Claim 4.8,

$$G \subset (F^\perp \vee U') \cap U(G) = (F^\perp \vee U(G)') \cap U(G) = U(G)'.$$

Hence $G \in \left[{U(G)' \atop n-k}\right] \subset \mathcal{F}^\perp$. □

Now we are ready to prove $\mathcal{F}^\perp = \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. Suppose, for a contradiction, that there exists a subspace $H \in \mathcal{F}^\perp$ that is not in $\begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. We will construct $r-1$ subspaces in $\begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$ that each intersect $F^\perp$ trivially and that together with H span V. By Claim 4.9, these $r-1$ subspaces lie in $\mathcal{F}^\perp$ which contradicts $\mathcal{F}^\perp$ being r-wise co-intersecting.

To build these $r-1$ subspaces, we construct a family of one-dimensional subspaces

$$\{v_i^j : i \in \{1, \dots, r-1\}, j \in \{1, \dots, n-k\}\}$$

such that for each $i \in \{1, \dots, r-1\}$, the subspace $G_i = \bigvee_{j=1}^{n-k} v_i^j$ lies in $F^\perp \vee U'$, intersects $F^\perp$ trivially, and $\bigvee_{i=1}^{r-1} G_i \vee H = V$. The subspaces $G_1, \dots, G_{r-1}$ are the desired $r-1$ subspaces. We pick the one-dimensional subspaces one after the other; we have to show that at each step there is a possible one-dimensional subspace to pick. When picking the last one-dimensional subspace v_{r-1}^{n-k} we must pick a one-dimensional subspace from $F^\perp \vee U'$ that is not in $H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$. Since H is not contained in $F^\perp \vee U'$, we have

$$\dim \left(\left(F^\perp \vee U'\right) \cap \left(H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j \right) \right) = r(n-k) - 2.$$

Hence, there are at least

$$q^{r(n-k)-2} - (q^{2(n-k)-2} + q^{2(n-k)-3} + \cdots + 1) > 0$$

one-dimensional subspaces of $F^\perp \vee U'$ that do not lie in $H \vee \bigvee_{i=1}^{r-2} G_i \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$ nor in $F^\perp \vee \bigvee_{j=1}^{n-k-1} v_{r-1}^j$; thus it is indeed possible to construct the desired $r-1$ subspaces. This proves that $\mathcal{F}^\perp \subseteq \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$, and since $|\mathcal{F}^\perp| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ we have $\mathcal{F}^\perp = \begin{bmatrix} F^\perp \vee U' \\ n-k \end{bmatrix}$. The subspace $F^\perp \vee U'$ is $(n-1)$-dimensional; by duality, $\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\}$ for some one-dimensional subspace $v \subset V$, which is the desired conclusion. □

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