Continuum percolation with steps in line segments

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October 1, 2009

Abstract

In the model of continuum percolation points are placed in the plane according to a Poisson process of density one and two of them P_1, P_2 are joined if P_1 lie within the fixed symmetric open set A centered at P_2. In this paper we define a model in which a line segment l_P of length $2L$ is centered at each point P and the angle of l_P and the x-axis is uniformly distributed over the interval $[0, \pi]$ independently from the other line segments. Two points P_1, P_2 are joined if $l_{P_1} \cap l_{P_2} \neq \emptyset$. We derive bounds on L_c, the critical length for percolation in this model. We also consider some related models, where further conditions are posed on either the angles of the line segments or the placement of the points or both.

1 Introduction

In this paper we address percolation problems defined by Poisson processes in the plane. In Section 2, we consider the following model: let us consider a Poisson process Λ of density 1 and at each point P of Λ let us draw a line segment l_P (which we will call *stick*) of length $2L$ centered at the point such that the angle θ of the line and the x-axis is uniformly distributed over the interval $[0, \pi]$. We address percolation problems in the graph $G = G(L)$ corresponding to this process, i.e. $V(G) = \Lambda$ and two vertices are joint by an edge if and only if the line segments belonging to them intersect each other. Models where line segments are replaced by discs or other symmetric open sets in $\mathbb{R}^2$ were introduced by Gilbert [3] and widely studied ever since (for references see [4],[7]).

The main motivation for changing discs to line segments is that one can consider the points of the Poisson process as sensors. It is natural to calculate with a circular sensing region, however laser sensors or guards looking in one direction should be modeled in the way described above. Note that centering the sticks at the points of the Poisson process or considering Λ as endpoints of the sticks does not make any difference, we chose to present our theorems in this way is easier comparison to results about the original model where bounds are established on the radius of the disc.

The rest of the paper is organized as follows: in section 2, we will obtain lower and upper bounds on the *critical length*

$$L_c = \inf\{L : \mathbb{P}(\,G(L) \text{ contains an infinite component}) > 0\}$$

and we derive an asymptotic formula for the expected area of the *hole around the origin*, the component of the origin in $\mathbb{R}^2 \setminus \bigcup_{P\in\Lambda} l_P$. Note that in the scenario of laser sensors, the latter can be interpreted as the expected area where an object may move without being noticed by any of the sensors.

In Section 3, we discuss similar but more restricted models, where the sticks must be either horizontal or vertical or the Poisson process takes place only in the 2-dimensional grid, i.e. in lines of the form $x = a$ or $y = b$ $(a, b \in \mathbb{N})$.

2 Sticks in the plane

In this section we consider problems involving the process Λ. In the first subsection, we derive bounds on L_c, while in the second subsection, we show how large is the component of the origin in $\mathbb{R}^2 \setminus \bigcup_{P\in\Lambda} l_P$.

2.1 Bounds on L_c

The aim of this subsection is to prove the following theorem.

Theorem 2.1. *The following inequalities hold:*

$$0.62665 \leq L_c \leq 5.7135$$

Proof: To obtain the upper bound on L_c let us define a bond percolation measure μ_L^t on $\mathbb{Z}^2$. For any vertex $(a, b) \in \mathbb{Z}^2$ and any $0 < t < 1/\sqrt{2}$, let us define $S_{a,b}^t = [(a - 1/2)tL, (a + 1/2)tL] \times [(b - 1/2)tL, (b + 1/2)tL]$. The bond $((a, b), (a + 1, b))$ is open if an only if there are two points P_1, P_2 of the Poisson process with $P_1 \in S_{a,b}^t \cup S_{a+1,b}^t, P_2 \in S_{a,b}^t$ such that the following two conditions hold: **(a)** the stick centered at P_1 crosses both the left side of $S_{a,b}^t$ and the right side of $S_{a+1,b}^t$ (we call this the *long stick* of the edge) and **(b)** the stick centered at P_2 crosses both the upper and lower side of $S^t_{a,b}$ (which we call the short stick of the edge).

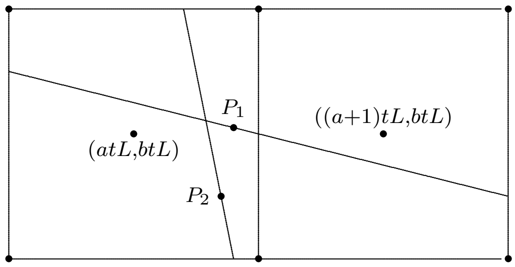


Figure: configuration showing a bond is open

Similarly, the bond $((c, d), (c, d + 1))$ is open if an only if there are two points Q_1, Q_2 of the Poisson process with $Q_1 \in S^t_{c,d} \cup S^t_{c,d+1}, Q_2 \in S^t_{c,d}$ such that the following two conditions hold: **(a)** the stick centered at Q_1 crosses both the lower side of $S^t_{c,d}$ and the upper side of $S^t_{c,d+1}$ and **(b)** the stick centered at Q_2 crosses both the left and right side of $S^t_{c,d}$.

Proposition 2.2. ***(i)*** *The percolation measure μ^t_L is 1-dependent, i.e. the state of any set $\mathcal{S}$ of bonds is independent of the states of any other set $\mathcal{T}$ of bonds if no bond in $\mathcal{T}$ shares a common vertex with any bond in S.*

(ii) *If $\mathbb{Z}^2$ contains an infinite open cluster, then $G(L)$ contains an infinite component.*

Proof: By definition the state of an edge $((a, b), (a + 1, b))$ depends only on $\Lambda \cap (S_{(a,b)} \cup S_{(a+1,b)})$. The squares S_u partition the plane from which **(i)** follows. To see **(ii)** note that if $e_1, e_2, ...$ is an infinite open path in $\mathbb{Z}^2$, then the corresponding long sticks are in the same component of $G(L)$ as if e_i, e_{i+1} are perpendicular, then the long sticks intersect, while if they are parallel, then they intersect the same short stick. □

We will use the following theorem of Balister, Bollobás and Walters [2].

Theorem 2.3. *In any 1-dependent bond percolation measure on $\mathbb{Z}^2$ in which every bond is open with probability at least 0.8639, there exists an infinite open cluster with positive probability.* □

Let us compute the probabilities that conditions **(a)** or **(b)** are not satisfied, thus the bond $((a, b), (a + 1, b))$ is closed. Clearly, to satisfy condition **(a)** for the angle θ of the stick at P_1 and the x-axis we must have $-\arctan(1/2) \leq \theta \leq \arctan(1/2)$. Also the distance of P_1 from both the left side of $S^t_{a,b}$ and the right side of $S^t_{a+1,b}$ cannot exceed $L \cos \theta$. Therefore by symmetry the probability that condition **(a)** is not satisfied is

$$p_1^t(L) = \exp\left(-\frac{2}{\pi}\int_0^{\arctan 1/2} 2L(\cos\theta - t)L(t - 2t\tan\theta)d\theta\right) =$$

$$\exp\Big(- \frac{4tL^2}{\pi}\Big(\sin(\arctan 1/2) + 2\cos(\arctan 1/2) - t\arctan(1/2) -$$

$$2t\log(\cos(\arctan 1/2)) - 2\Big)\Big) =$$

$$\exp\left(-\frac{4tL^2}{\pi}\left(\sqrt{5} - 2 - t\left(\arctan(1/2) + 2\log(2/\sqrt{5})\right)\right)\right).$$

To satisfy condition **(b)** the angle θ of the stick centered at P_2 and the y-axis must be in $[-\pi/4, \pi/4]$. Thus by symmetry the probability that condition **(b)** does not hold is

$$p_2^t(L) = \exp\left(-\frac{2}{\pi}\int_0^{\pi/4} tL\left(tL - tL\tan\theta\right)d\theta\right) =$$

$$\exp\left(-\frac{2t^2L^2}{\pi}\left(\pi/4 + \log\frac{\sqrt{2}}{2}\right)\right)$$

By Theorem 2.3 and Proposition 2.2 we obtain that if $p_1^t(L) + p_2^t(L) \leq 1 - 0.8639 = 0.1361$ holds, then with positive probability $G(L)$ has an infinite component an thus $L_c \leq L$. Choosing $t = 0.63$ gives the bound $L_c \leq 5.7135$.

For the lower bound note that the expected value of the degree of any point $p \in \Lambda = V(G(L))$ is

$$\frac{1}{\pi}\int_0^{\pi} 4L^2 \sin\theta d\theta = \frac{8L^2}{\pi},$$

which is smaller than 1 if $L < 0.62665$. If L is such, then we can couple the graph process to a branching process in a standard way. Let D_u denote the random variable that counts the number of neighbors of $u \in \Lambda$ in $G(L)$. Clearly, $D_u = D_v$ for any $u, v \in \Lambda$ and by our assumption on L we have $\mathbb{E}(D_u) < 1$. The branching process for the coupling is as follows: we pick an arbitrary point p in $\Lambda = V(G(L)$ and then we "expose" its neighbors, i.e. consider the random variable D_p. We declare p *dead* and its D_p neighbors *living*. At each step we expose the neighbors of a living vertex, declare the neighbors living and the vertex dead. Since two vertices in Λ might have common neighbors, we have that if this branching process dies out after finitely many steps, then the component of p in $G(L)$ is finite. It is well-known that a branching process dies out after finitely many steps if and only if the expected number of the children of an individual is at most 1, which proves the theorem. □

2.2 Parameters of cells

In this subsection we derive lower and upper bounds on the area and diameter of cells. Either we will consider the cell of the origin and determine the expected value of the parameter or we will consider a square of area A (or more precisely, a torus of area A as we do not want to consider boundary effects) and address the problem of finding the maximum value that the parameter takes over the cells in the square. In order to avoid situations where a stick crosses the whole square it is reasonable to pose the assumption $A = \omega(L^2)$. Before getting into the details, we establish some lemmas that will be helpful during the proofs.

2.2.1 Basic lemmas

Lemma 2.4. *The probability that a line segment of length l is not crossed by any stick is $e^{-\frac{4lL}{\pi}}$.*

Proof: Conditioning first on the angle of the stick, then seeing where the center must be, we obtain that the expected number of crossings is

$$\frac{1}{2\pi}\int_0^{2\pi} 2lL|\cos\alpha|d\alpha = \frac{4lL}{\pi}.$$

The statement follows from the fact that the number of crossings has Poisson distribution. □

In the following 2 lemmas we prove that for some random variable X we have $\mathbb{E}(X^2) = o(\mathbb{E}^2(X))$ and thus by Chebyshev's inequality X is concentrated around its mean. Intersections of sticks will be called *stick intersection points.* The following lemma states that their number is concentrated around its mean.

Lemma 2.5. *With high probability the number of stick intersection points is $\frac{4+o(1)}{\pi}AL^2$.*

Proof: Let Y denote the number of sticks and X denote the number of stick intersection points. Clearly, we have $X \le \binom{Y}{2}$. By the proof of Lemma 2.4, the expected number of stick intersection points on one fixed stick is $8L^2/\pi$. Thus we have $(1+o(1))\mathbb{E}(X) = \mathbb{E}(X|Y \ge (1-o(1))A) \ge \frac{4+o(1)}{\pi}AL^2$. On, the other hand, for any $\epsilon > 0$ we have $e^{-A}\sum_{k=(1+\epsilon)A}^{\infty}\frac{k^2A^k}{k!} \to 0$, thus $\mathbb{E}(X) = \frac{4+o(1)}{\pi}AL^2$.

To conclude the lemma we have to show that $\mathbb{E}(X^2) = (1+o(1))\mathbb{E}^2(X)$. As for any $\epsilon > 0$ we have $e^{-A}\sum_{k=(1+\epsilon)A}^{\infty}\frac{k^4A^k}{k!} \to 0$, we have to prove $\mathbb{E}(X^2|Y =$

$(1+o(1))A) = (1+o(1))\mathbb{E}^2(X)$. Observe that

$$X = \sum_{1\le i<j\le Y} X_{i,j},$$

where $X_{i,j}$ denotes the indicator variable of the event that the ith and jth sticks intersect. Now the lemma follows as $X_{i,j}$ and $X_{i',j'}$ are independent unless $\{i,j\} = \{i',j'\}$. □

Intersection points of sticks and circles of radius R around stick intersection points will be called *R-circle-stick intersection points*. The following lemma states that their number is also concentrated around its mean.

Lemma 2.6. *Let $R = \frac{\pi}{4}\frac{\log(AL^2)}{L}$. Then there exists a constant c such that if $\log(AL^4) = o(L^2)$, then with high probability the number of R-circle-stick intersection points is at most $cAL^2\log(AL^4)$.*

Proof: First note that by Lemma 2.5 we may only consider circle-stick intersection points defined by 3 different sticks. Let Z denote the number of circle-stick intersection points and Y denote the number of sticks. Clearly, we have $Z = O(Y^3)$ and thus as $e^{-A}\sum_{k=(1+\epsilon)A}^{\infty}\frac{k^3A^k}{k!} \to 0$, we might suppose that $Y = (1+o(1))A$.

Let us write $Z = \sum_{1\le i,j,k\le Y} Z^i_{j,k}$, where $Z^i_{j,k}$ is the indicator variable of the event that the ith stick crosses the circle around the (existing!) intersection point of the jth and kth stick. It is easily seen that $\mathbb{P}(Z^i_{j,k} = 1) = \Theta(\frac{L^2\log(AL^2)}{A^2})$ and thus $\mathbb{E}(Z) = \Theta(AL^2\log(AL^2))$. Clearly, if $|\{i,j,k\}\cap\{i',j'.k'\}| \le 1$, then $Z^i_{j,k}$ and $Z^{i'}_{j',k'}$ are independent, while if the intersection has size 2, then $\mathbb{P}(Z^i_{i,j}Z^{i'}_{j',k'}) = O(\frac{L^4\log^2(AL^2)}{A^3}$ as $Z^i_{j,k} = 1$ must hold and the center of the stick of which the index is in $\{i',j',k'\}$ but not in $\{i,j,k,\}$ must lie within distance $c_2(L+\log(AL^2))$ from the centers of the other sticks involved. The number of such pairs $Z^i_{j,k}, Z^{i'}_{j',k'}$ is $O(A^4)$ and thus the sum of the summands belonging to nonindependent products in $\mathbb{E}(Z^2)$ is $o(\mathbb{E}^2(Z))$. □

2.2.2 The maximum diameter and area of cells in a large square

Let $d_{A,L}$ denote the largest diameter and $a_{A,L}$ denote the largest area that a cell has in a square of area A when sticks have length $2L$.

Theorem 2.7. *If $A = \omega(L^{2+\delta})$ for some $\delta > 0$, then with high probability there exists 2 stick intersection points at distance at least*

$$\frac{\pi}{4}\left(\frac{\log(AL^2)}{L} - 4\frac{\log\log(AL^2)}{L}\right)$$

such that the line segment between them is not crossed by any stick.

Proof: By Lemma 2.4, the probability that the line segment between 2 stick intersection points at distance between $D := \frac{\pi}{4}\left(\frac{\log(AL^2)}{L} - 4\frac{\log\log(AL^2)}{L}\right)$ and $D' := \frac{\pi}{4}\left(\frac{\log(AL^2)}{L} - \frac{3\log\log(AL^2)}{L}\right)$ is not crossed by any stick is $\frac{\log/3(AL^2)}{AL^2}$. Let us denote the number of such pairs by W. By Lemma 2.5 the number of stick intersection points is w.h.p $\frac{4+o(1)}{\pi}AL^2$ and thus (as the expected number of stick intersection points in a set S depends only on the area of $|S|$) we have

$$\mathbb{E}(W) \geq \frac{8-o(1)}{\pi^2}\log^4(AL^2)\log\log(AL^2).$$

Lemma 2.8. *Let P_1, P_2 and P_3, P_4 be 2 pairs of stick intersection points both at distance between D and D' such that at least one of P_3 and P_4 is at distance $\epsilon = \omega(\frac{\log\log(AL^2)}{L})$ from the line segment P_1P_2. Let $\mathcal{E}_{1,2}$ ($\mathcal{E}_{3,4}$) denote the events that the line segment P_1P_2 (P_3P_4) is not crossed by any stick. Then*

(i) *there exists a constant c such that $\mathbb{P}(\mathcal{E}_{1,2}|\mathcal{E}_{3,4}) \leq e^{-c\epsilon L}$ if $\epsilon = o(D)$,*

(ii)) *if $\epsilon = KD$ for some constant K $\mathbb{P}(\mathcal{E}_{1,2}|\mathcal{E}_{3,4}) \leq e^{-(1-\delta(K))\frac{4}{\pi}DL}$ such that $\delta(K) \to 0$ as $K \to \infty$,*

(iii) $\mathbb{P}(\mathcal{E}_{1,2}|\mathcal{E}_{3,4}) = e^{-\frac{4}{\pi}D'L}$*if $\epsilon > D + 2L$.*

Proof of Lemma: First note that **(iii)** is trivial as the centers of all sticks that can have any effect on a line segment should lie within distance L of the line segment.

Observe the following:

(a) if one of P_3, P_4 is at least x away from the line segment e, then at least 1/3 of the line segment P_1P_2 is $x/2$ away from e (e and P_1P_2 might cross!),

(b) if $\text{dist}(P, e) = x$, then the viewing angle of e from P is at least $2\arctan(\frac{d}{2x})$ and if $\mathcal{E}_{3,4}$ holds, then on any line f containing P not meeting e there should not be any point $Q \in \Lambda$ with l_Q parallel with f and $\text{dist}(P, Q) < L$. Note again that the nonexistence of such Q is not assured by $\mathcal{E}_{1,2}$.

Now to prove **(i)** let us use **(a)** with $x = \epsilon$ and **(b)** with $x = \epsilon/2$ to obtain that the expected number of sticks crossing P_3P_4 even if $\mathcal{E}_{1,2}$ holds is at least

$$\int_0^{\arctan(\frac{\epsilon}{D})} \frac{2}{3}\sin\alpha DL \geq c\epsilon L.$$

Then the statement of **(i)** follows as (just as in Lemma 2.4) the number of such sticks has a Poisson distribution.

The proof of **(ii)** follows the same lines, to show that $\delta(K) \to 0$ note that as $K \to \infty$ we have $\text{dist}(P_3P_4, e) \geq (K-1)D$ and thus the viewing angle of e from any point of P_3P_4 tends to 0 (uniformly). $\square$

To finish the proof of Theorem 2.7 let us fix 2 stick intersection points P_1, P_2 at distance between D and D' and let us denote the line segment between P_1 and P_2 by e. Furthermore, let W_1 denote the number of pairs P_3, P_4 of stick intersection points both at distance between D and D' such that both of them within distance $\frac{\log^2 \log(AL^2)}{L}$ from e and their line segment not crossed by any stick, let W_2 denote the number of pairs P_3, P_4 of stick intersection points both at distance between D and D' with $\frac{\log^2 \log(AL^2)}{L} < \text{dist}(P_3P_4, e) < D + 4L$ such that their line segment not crossed by any stick, and let W_3 denote number of pairs P_3, P_4 of stick intersection points both at distance between D and D' with $\text{dist}(P_3P_4, e) \geq D + 4L$ such that their line segment not crossed by any stick.

Clearly, we have $W = W_1 + W_2 + W_3$. Let us bound $\mathbb{E}(W_i|\mathcal{E}_{1,2})$ $i = 1, 2, 3$. Just by the expected number of pairs of stick intersection points at the right distance, we have $\mathbb{E}(W_1|\mathcal{E}_{1,2}) \leq \log(AL)^2 \log^2 \log(AL^2)$. By the last part of Lemma 2.8, we know that $\mathcal{E}_{1,2}$ and $\mathcal{E}_{3,4}$ are independent if both P_3, P_4 are further than $D + 2L$ from e and thus $\mathbb{E}(W_3|\mathcal{E}_{1,2}) \leq \mathbb{E}(W)$.

To bound $\mathbb{E}(W_2|\mathcal{E}_{1,2})$, we distinguish 2 cases. First, let us assume that $L < 10D$. Then for any $\epsilon < D + 2L$ we have $2\epsilon(D + \epsilon) \leq 50D\epsilon$ and thus by Lemma 2.8 we have

$$\mathbb{E}(W_2|\mathcal{E}_{1,2}) \leq \int_0^{D+2L} 50DL^2 \log(AL)^2 e^{-c\epsilon L} d\epsilon = O(\log^2(AL)^2).$$

Now assume $L > 10D$. Then choosing a large enough K and using the assumption $A = \omega(L^{2+\delta})$ for some positive δ and Lemma 2.8 we obtain

$$\mathbb{E}(W_2|\mathcal{E}_{1,2}) \leq$$

$$\int_0^{KD} 3K^2DL^2 \log(AL)^2 e^{-c\epsilon L} d\epsilon + O\left(\int_{KD}^{D+2L} \epsilon L^2 \log(AL)^2 e^{-\frac{4}{\pi}DL}\right) =$$

$$O(\log^2(AL)^2) + o(1).$$

Putting the above bounds together, we obtained

$$\mathbb{E}(W|\mathcal{E}_{1,2}) = (1 + o(1))\mathbb{E}(W).$$

As $\mathbb{E}(W^2) = \mathbb{E}(W\mathbb{E}(W|\mathcal{E}_{1,2}))$, we are done by Chebyshev's inequality. □

Theorem 2.9. *If $\log(AL^2) = o(L^2)$, then for any $\epsilon > 0$ we have $d_{A,L} \leq (1 + \epsilon)\frac{\pi}{4}\frac{\log(AL^2)}{L}$ w.h.p.*

Proof: The diameter of any cell is attained at two stick intersection points, thus it is enough to prove that any cell belonging to a stick intersection point P is contained in the circle C of radius $(1+\epsilon)\frac{\pi}{4}\frac{\log(AL^2))}{L}$ with center P. To do so we show that there is a stick that intersects C and all stick-circle intersection points are separated from P, where being separated means that there is a stick with both endpoints outside C that crosses the line segment between P and the stick-circle intersection point. The former statement is trivial as P is a stick intersection point, thus as $\log(AL^2) = o(L^2)$ implies $\frac{\log(AL^2)}{L} = o(L)$, the sticks that define P must intersect C. Let us choose $\epsilon' > 0$ such that $(1+\epsilon)(1-\epsilon') > 1$ holds. Then by Lemma 2.4 we obtain that the probability that one particular stick-circle intersection point is not separated from the center of the circle even with sticks of length $2(1-\epsilon')L$ is $(AL^2)^{-(1+\epsilon)(1-\epsilon')}$ and thus, by Lemma 2.6, the probability that this happens for at 1 stick-circle intersection point tends to 0. Finally, note that as $\frac{\log(AL^2)}{L} = o(L)$, the endpoints of these sticks lie outside the circle. □

By Theorem 2.7 and Theorem 2.9 we can determine $d_{A,L}$ asymptotically if $\log(AL^2) = o(L^2)$ holds.

Corollary 2.10. *If* $\log(AL^2) = o(L^2)$*, then with high probability we have*

$$d_{A,L} = (1+o(1))\frac{\pi}{4}\frac{\log(AL^2)}{L}.$$

Another immediate consequence of Theorem 2.9 is the following upper bound on $a_{A,L}$.

Corollary 2.11. *If* $\log(AL^2) = o(L^2)$*, then for any* $\epsilon > 0$ *we have* $a_{A,L} \leq (1+\epsilon)\frac{\pi^3}{64}\frac{\log^2(AL^2)}{L^2}$ *w.h.p.*

The following theorem is the analogue of Theorem 2.7. As its proof differs from that of Theorem 2.7 only by replacing Lemma 2.4 by a statement on the probability of no sticks meeting a circle, we omit the proof.

Theorem 2.12. *If* $A = \omega(L^{2+\delta})$ *for some* $\delta > 0$*, then with high probability there exists 2 stick intersection points at distance at least*

$$\frac{1}{2}\left(\frac{\log(AL^2)}{L} - 4\frac{\log\log(AL^2)}{L}\right)$$

such that the circle of which a diameter is the line segment between them is not intersected by any stick.

2.2.3 The area of the cell around the origin

In this subsection we derive lower and upper bounds on the expected area **A** of the cell around the origin, namely we prove the following theorem.

Theorem 2.13. *The expected area of the cell around the origin is*

$$\frac{\pi^3}{8L^2} + o(1/L^2).$$

Proof: To obtain the lower bound we compute the expected area of the set S containing the origin with the property that whenever $s \in S$, then the segment from s to the origin is not crossed by any stick. We will call S the *star-set* around the origin.

Let us fix a point Q and a half line starting from Q. By Lemma 2.4, the probability that no stick crosses the half line within distance R is

$$e^{-\frac{1}{\pi}\int_0^\pi 2RL|\cos\theta|d\theta} = e^{-\frac{4RL}{\pi}}. \tag{1}$$

Therefore the expected area of the star-set around the origin is

$$\pi \int_0^\infty e^{-\frac{4L}{\pi}\sqrt{R}} dR = \pi \int_0^\infty 2xe^{-\frac{4L}{\pi}x} dx = \frac{\pi^3}{8L^2}.$$

To obtain the upper bound let us define the following events: E_1 is the event that the cell around the origin is contained in the disc D of radius $L^{-2/3}$ centered at the origin, E_2 is the event that for any boundary point P of D there is a stick crossing D that separates P from the origin, E_3 is the event that D does not contain any endpoint of the sticks, E_4 is the event that the cell around the origin is the star-set around the origin. Clearly $E_2 \subset E_1$ and $E_2 \cap E_3 \subset E_4$, furthermore to ensure that all boundary points are separated from the origin, it is enough to prove that all the intersection points of the boundary and any stick are separated from the origin and that there exists at least one such intersection point.

It is enough to prove that

$$\mathbb{P}(E_1 \setminus E_4)L^{-4/3} = o(L^{-2}), \tag{2}$$

$$\mathbb{P}(\overline{E}_1)\mathbb{E}(\mathbf{A}|\overline{E}_1) = o(L^{-2}). \tag{3}$$

Let B be a disc of radius R and let Q be a point on its boundary. The probability that no stick separates Q from the center of B is at most

$$e^{-\frac{1}{\pi}\int_0^\pi (2L-2R)R|\cos\alpha|d\alpha} = e^{-\frac{4}{\pi}(L-R)R}. \tag{4}$$

The expected number of sticks that intersect a disc B of radius R is $4RL + R^2\pi$ (again, condition first on the angle of the stick, then see where the center must be). Since the number X_B of such sticks is a Poisson random variable, we obtain that **(a)** $\mathbb{P}(X_B = 0) = \exp(-(4RL+R^2\pi))$ and if $R \ll L$, then **(b)** $\mathbb{P}(X_B > 8RL) \leq \exp(-cRL)$ for some constant c.

Using **(a)**, **(b)** and 4 with $R = L^{-2/3}$ we obtain that for L large enough we have

$$\mathbb{P}(\overline{E}_2) \leq 8L^{1/3}e^{-L^{1/3}} + e^{-L^{1/3}} + e^{-cL^{1/3}} = e^{-\Omega(L^{1/3})}.$$

Furthermore $\mathbb{P}(E_3) = e^{-2\pi L^{-4/3}}$. Thus

$$\mathbb{P}(E_1 \setminus E_4)L^{-4/3} \leq \mathbb{P}(\overline{E}_4)L^{-4/3} \leq (\mathbb{P}(\overline{E}_2) + \mathbb{P}(\overline{E}_3))L^{-4/3} =$$

$$\left(e^{-\Omega(L^{1/3})} + O(L^{-4/3})\right) L^{-4/3} = O(L^{-8/3}),$$

which proves 2.

We still have to prove 3. Note that by the above we have $\mathbb{P}(\overline{E}_1) \leq \mathbb{P}(\overline{E}_2) = O(e^{-L^{1/3}})$. We will use 1-dependent percolation on $\mathbb{Z}^2$ defined in the previous section and apply the following lemma.

Lemma 2.14. *In any 1-dependent bond percolation measure on $\mathbb{Z}^2$ in which every bond is open with probability at least 0.95, there exists an open cycle surrounding the origin within distance 2^{k+1} with probability at least $1-10^{k+1}2^{-2^k}$ for any positive integer k.*

Proof: In the proof of Theorem 2.3, Balister, Bollobás and Walters define 10 pairs of non-adjacent bonds of the rectangle $T_u \cup T_v$ where $T_u = \{u, u+(0,1), u+(1,0), u+(1,1)\}$ and u and v are adjacent sites in $(2\mathbb{Z})^2$ and declare the bond (u, v) open in a new 1-dependent percolation on $(2\mathbb{Z})^2$ if in all pairs at least 1 of the bonds were open in the original percolation measure. Thus the probability that a bond is closed in the new percolation measure is at most $10q^2$, where q denotes the probability that a bond was closed in the original percolation measure. By repeating the procedure $k + 1$ times we obtain a measure on $(2^{k+1}\mathbb{Z})^2$ where the probability of a bond to be closed satisfies $10q' < 10^{k+1}2^{-2^k}$ (here we used the assumption that $10q < 0.5$ from the beginning) and in which if the origin is surrounded by an open cycle immediately (i.e. the 8 sites around form an open cycle), then there exists an open cycle surrounding the origin within distance 2^{k+1} in the original percolation measure. But this happens with probability at least $1 - 8q' > 1 - 10^{k+1}2^{-2^k}$. □

Since in the 1-dependent percolation measure defined in the previous section, S_u^t has side-length tL for some constant t, by Lemma 2.14 we obtain that if t is large enough to make the probability q of a bond being closed small enough to satisfy $10q < 0.5$, then the probability that the area of the cell around the origin is not contained in the surrounding square of side-length $2^{k+1}tL$ is at most $10^{k+1}2^{-2^k}$.

$$\mathbb{P}(\mathbf{A} \geq (2^{k+1}tL)^2 | \overline{E}_1) \leq 10^{k+1}2^{-2^k}$$

and thus

$$\mathbb{E}(\mathbf{A}|\overline{E}_1) \leq 1 + \sum_{k=2}^{\infty} (2^{k+1}tL)^4 \mathbb{P}\left(\mathbf{A} \geq (2^{k-1}tL)^2 | \overline{E}_1\right) \leq K \cdot L^4$$

for some constant K that does not depend on L. Therefore we have

$$\mathbb{P}(\overline{E}_1)\mathbb{E}(\mathbf{A}|\overline{E}_1) = O\left(e^{-L^{1/3}}L^4\right)$$

which proves 3. □

2.3 Stick length tending to 0

In this subsection we address the problem when L tends to 0 and show that in this case $\mu_{\lambda,L}$ behaves very much like the original independent bond percolation on $\mathbb{Z}^2$. To state this result precisely we introduce the following two quantities.

$$\lambda_c'(L) := \inf\{\lambda : \mu_{\lambda,L}(\text{the bond } ((0,0),(1,0)) \text{is open) }) > 1/2\},$$

$$\lambda_c''(L) := \inf\{\lambda : \mu_{\lambda,L}''(\text{the bond } ((0,0),(1,0)) \text{is open })) > 1/2\},$$

where $\mu_{\lambda,L}''$ is the bond percolation on $\mathbb{Z}^2$ in which a bond is declared to be open if the corresponding line segment is covered by sticks centered at points inside the line segment. Clearly we have $\lambda_c'(L) \leq \lambda_c''(L)$.

Proposition 2.15. *If L tends to 0, then we have*

$$(1 + o(1))\lambda_c'(L) = \lambda_c''(L) = (1 + o(1))\lambda_c(L) = (1 + o(1))\frac{|\log L|}{2L}.$$

Proof: In $\mu_{\lambda,L}''$ the bond $((0,0),(1,0))$ is open if and only if **(a)** there are points $(x_1, 0)$ and $(x_2, 0)$ with $x_1 \in [0, L]$ and $x_2 \in [1 - L, 1]$ and **(b)** any two consecutive points on the line segment $((0,0),(1,0))$ are within distance $2L$. These distances are independent of each other, and we know that as L tends to 0, the number of points on the line segment is $\lambda + O(\lambda^{2/3})$ with probability tending to 1. Therefore the probability that condition **(b)** is satisfied is $(1 - e^{-2\lambda L})^{\lambda + O(\lambda^{2/3})} + f(L)$, where $f(L)$ tends to 0 as L tends to 0. Using the fact that $1 + z \sim e^{-z}$ as z tends to 0, we obtain that condition **(b)** is satisfied with probability tending to 1/2 if $\lambda = (1 + o(1))\frac{|\log L|}{2L}$. The probability that condition **(a)** does not hold is $2e^{\lambda L} - e^{2\lambda L}$ which tends to 0 for $\lambda = (1 + o(1))\frac{|\log L|}{2L}$. This proves the equation $\lambda_c''(L) = (1 + o(1))\frac{|\log L|}{2L}$.

To see $(1 + o(1))\lambda_c'(L) = \lambda_c''(L)$ note that for $((0,0),(1,0))$ to be open in $\mu_{\lambda,L}$, condition **(b)** still must hold.

In $\mu''_{\lambda,L}$ the state of every bond is independent of that of the others, thus $\mu''_{\lambda,L}$ is simply the ordinary independent bond percolation model. The Harris-Kesten theorem [5],[6] states that percolation occurs if and only if bonds are open with probability higher than 1/2. Thus we have $\lambda_c(L) \leq \lambda_c''(L)$. The bond percolation $\mu_{\lambda,L}$ is 1-dependent (i.e. the state of any bond is independent of the set of all bonds that do not share a common site with). We know that the probability of an infinite component is 0 if the probability of an edge to be open is less than $1/c^2$, where c is the connective constant of $\mathbb{Z}^2$. The above computations show, that if $\lambda \leq (1 - \epsilon)\frac{|\log L|}{2L}$ for any fixed positive ϵ, then the probability of a bond to be open tends to 0. This proves $\lambda_c(L) = (1 + o(1))\lambda_c''(L)$. □

2.4 Stick length tending to ∞

In this subsection we consider the case of large stick length. We will compare the percolation process to another process that takes place in the plane and connect in this way Section 2 and 3. Given two independent Poisson processes in the plane Λ_h, Λ_v, each of density λ, for any $P = (p_1, p_2) \in \Lambda_h$ let us draw a horizontal stick of length $2L$ centered at P (i.e. the line segment from $(p_1 - L, p_2)$ to $(p_1 + L, p_2)$) and for any $P' \in \Lambda_v$ let us draw a vertical stick of length $2L$ centered at P'. Let $G_{hv}(\lambda, L)$ be the graph of which the vertex set is $\Lambda_u \cup \Lambda_v$ and two vertices are joined if the sticks centered at the vertices intersect and let us define $\lambda_{hv}(L) = \inf\{\lambda : \mathbb{P}(G_{hv}(\lambda, L)\text{contains an infinite component }) > 0\}$.

Proposition 2.16. *For any $L > 0$ we have $\lambda_c(L + 1) \leq \lambda_{hv}(L)$.*

Proof: For any $P = (p_1, p_2) \in \Lambda_h$ let $\pi(s) = (p_1, \lceil p_2 \rceil)$ and for any $p' = (p_1', p_2') \in \Lambda_v$ let $\pi(p') = (\lceil p_1' \rceil, p_2')$. For any $a, b \in \mathbb{N}$ the set of points $\{\pi(P) \in y = a\}$ and the set of points $\{\pi(P') \in x = b\}$ are distributed according to a Poisson process on these lines with density λ. Clearly, if (P, Q) is an edge in $G_{hv}(\lambda, L)$, then $(\pi(P), \pi(Q))$ is an edge in $G(\lambda, L + 1)$, which proves the statement of the proposition. □

Proposition 2.17. *For any $L > 0$ we have $\lambda_c(L) \geq \frac{1}{2L\lfloor 2L+1 \rfloor + 4L}$.*

Proof: The expected degree of any vertex v_P in $G(\lambda, L)$ is at most $\lambda(2L\lfloor 2L+1\rfloor + 4L)$ ($\lambda 4L$ for the vertices corresponding to points obtained in the Poisson process on the line of P, and at most $\lambda(2L\lfloor 2L+1\rfloor)$ for the vertices corresponding to points on perpendicular lines). Thus if λ is as in the condition of the proposition, then this expected degree is at most 1. The proposition follows as the lower bound in Theorem 2.1. □

Theorem 2.18. *For any $L > 0$ we have $\lambda_{hv}(L) \leq c/L^2$, where $c = 5.875$.*

Proof: We define a 1-dependent bond percolation on $\mathbb{Z}^2$ very similar to that of Section 2. Remember, for any vertex $(a, b) \in \mathbb{Z}^2$ and any $0 < t < 1$, we denote by $S^t_{a,b}$ the square of side length tL centered at (atL, btL), i.e $[(a - 1/2)tL, (a + 1/2)tL] \times [(b - 1/2)tL, (b + 1/2)tL]$. The horizontal bond $((a, b), (a, b+1))$ is open if an only if **(i)** there is a $P \in \Lambda_h \cap (S_{(a,b)} \cup S_{(a,b+1)})$ of which the stick meets both left and right side of $(S_{(a,b)} \cup S_{(a,b+1)})$ and **(ii)** a $P' \in \Lambda_v \cap S_{(a,b)}$, while the vertical bond $((c, d), (c+1, d))$ is open if an only if there is a $Q \in \Lambda_h \cap S_{(c,d)}$ and a $Q' \in \Lambda_v \cap (S_{(c,d)} \cup S_{(c+1,d)})$ of which the stick meets both upper and lower side of $(S_{(c,d)} \cup S_{(c+1,d)})$.

The probability that **(i)** does not hold $e^{-\lambda 2(1-t)tL^2} = e^{-2(1-t)tc}$, while the probability that **(ii)** does not hold is $e^{-\lambda t^2 L^2} = e^{-ct^2}$. Since Λ_h and Λ_v are independent, the two conditions are met (or not met) independently, thus for any bond e in $\mathbb{Z}^2$ we have

$$\mathbb{P}\,(e \text{ is closed}) = e^{-2(1-t)tc} + e^{-ct^2} - e^{-c(2t-t^2)}.$$

Plugging in $c = 5.875$ and $t = 0.707$ we obtain that the above quantity is less than 0.1361, thus by Theorem 2.3 the statement follows. □

Corollary 2.19. *For any $L > 0$ we have*

$$\frac{1}{2L\lfloor 2L+1\rfloor + 4L} \leq \lambda_{hv}(L) \leq c/L^2,$$

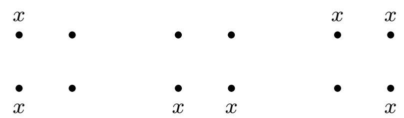
where $c = 5.875$.

2.5 Sticks of length 1

In this subsection we consider the case $2L = 1$, the maximum stick length when the bond percolation measure $\mu_{\lambda,1/2}$ is 1-dependent. In fact $\mu_{\lambda,1/2}$ has the stronger property that it is 1-dependent and any horizontal bond is mutually independent from any set of vertical bonds and vice versa. We will call such percolation measures *1-dependent and perpendicular independent.* We will determine lower and upper bounds on the critical probability of any 1-dependent perpendicular dependent percolation measure.

Theorem 2.20. *In any 1-dependent, perpendicular independent bond percolation on* $\mathbb{Z}^2$ *in which every bond is open with probability at least 0.7733, the origin is in an infinite open component with positive probability.*

Proof: We follow the lines of [2]. Consider the lattice $(2\mathbb{Z})^2$ of which the vertex u will correspond to the 2×2 square $S_u = \{u; u+(0,1); u+(1,0); u+(1,1)\}$ of the original lattice and the bond uv to the rectangle $S_u \cup S_v$. We declare this bond open if in the graph induced by $S_u \cup S_v$ there is an open component that contains one of the following 3 subsets of vertices both in S_u and S_v (vertices labeled with x form the required subsets):



Since any 2 of the above subsets intersect, therefore if there is an infinite component in $(2\mathbb{Z})^2$ then there is an infinite component in the original lattice. Unfortunately, the percolation obtained in this way will not be perpendicular independent, but it will still be 1-dependent, therefore we will be able to use Theorem 2.3. To do so we have to establish an upper bound on the probability q' that a fixed bond in $(2\mathbb{Z})^2$ is closed provided each bond in the original lattice is closed with probability at most q.

Drawing $S_u \cup S_v$ horizontally we define 12 sets E_i $(i = 1, ..., 12)$ of edges in $S_u \cup S_v$ and prove that if none of the E_i consist of only closed bonds then the bond uv is open.

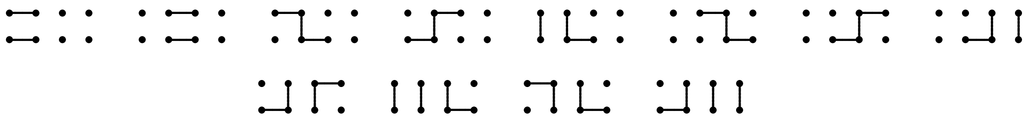


Figure: the subsets E_1-E_{12} of bonds

If no vertical bond is open then considering E_5, E_{10} and E_{12} gives that all lower horizontal bonds must be open, and their component contains the required subsets. The same argument holds if the only open vertical bond is either the leftmost or the rightmost. If the only open vertical bond is the left middle one, then E_1, E_6 and E_8 provide the component containing the required sets of vertices, while if the only open vertical bond is the right middle one, then we use E_1, E_4, E_5.

Let us assume now that there are exactly 2 open vertical bonds. If these are the 2 middle ones, then we are done by E_1 and E_2. If the 2 leftmost bonds are open, then we need E_1, E_2 and E_8, while if the 2 rightmost bonds are open, we obtain a good open component via E_1, E_2 and E_4. If the leftmost and the right middle vertical bonds are open, we need to use E_1, E_2, E_3 and E_4, while if the rightmost and left middle vertical bonds are open, we are done by E_1, E_2, E_6 and E_7. If the 2 outmost vertical bonds are open, then using $E_2, E_3, E_4, E_6, E_7, E_9$ and E_{11} we see that either all lower or all upper horizontal bonds are open forming a good component with the 2 open vertical bonds.

If there is at most 1 closed vertical bond, then we proceed as follows. Leftmost or rightmost bond closed: E_1 and E_2, left middle: E_1, E_2, E_3 and E_4, right middle: E_1, E_2, E_6 and E_7.

As among the above sets of bonds there are 2 of size 2, 6 of size 3 and 4 of size 4, we obtain that the probability that a bond is closed in the new lattice is at most $2q^2 + 6q^3 + 4q^4$. By Theorem 2.2 we know that we have an infinite component in $(2\mathbb{Z})^2$ if this expression is less than 0.1361, which holds if $q < 0.201$.

To improve on this bound, one might consider the lattice $(3\mathbb{Z})^2$ of which the vertex u will correspond to the 3×3 square $S_u = \{u; u + (0,1); u + (1,0); u + (1,1); u + (2,0); u + (2,1); u + (2,2); u + (1,2); u + (0,2)\}$ of the original lattice and the bond uv to the rectangle $S_u \cup S_v$, where we declare this bond open if in the graph induced by $S_u \cup S_v$ there is an open component that contains at least 5 of the 9 vertices both in S_u and S_v. One might find 3 configurations consisting of 3 independent bonds, 18 of 4 bonds, 54 of 5 bonds, 103 of 6 bonds, 181 of 7 bonds, 227 of 8 bonds and 82of 9 bonds such that if in all configurations at least one of the bonds is open in all configurations, then the there exists an open component containing at least 5 vertices both in S_u and S_v. (The list of all configurations and the programme that finds these can be downloaded from [1].) Therefore if in the original percolation the probability that a bond is closed is at most q, then in the auxiliary percolation defined as above the probability that a bond is closed is at most $3q^3 + 18q^4 + 54q^5 + 103q^6 + 181q^7 + 227q^8 + 82q^9$ which is less than 0.1361 if $q < 0.2267$. Thus we are done by Theorem 2.2

□

As lower bound on the critical probability of any 1-dependent bond percolation on $\mathbb{Z}^2$ one has $1/c^2$ where c is the connective constant of $\mathbb{Z}^2$. To see this note that for any self-avoiding walk the set of every other edge is independent, thus the standard first moment argument gives this result. We improve this lower bound a bit in the following theorem.

Theorem 2.21. *In any 1-dependent, perpendicular independent bond percolation on $\mathbb{Z}^2$ in which every bond is open with probability at most 0.2, the origin is in an infinite open component with zero probability.*

Proof: Let us say that $E \subset E(\mathbb{Z}^2)$ is perpendicular independent if any two edges in E that share a common vertex are perpendicular. Let us say that a self-avoiding walk W starting at the origin is in $\mathcal{W}_{n,m}$ if W contains n edges and the maximum size of a perpendicular independent subset of the edges of W is m. Clearly, if for some p we have $\sum_{m=\lceil n/2 \rceil}^{n} |\mathcal{W}_{n,m}| p^m \to 0$ as n tends to infinity, then the origin is in a finite open component with probability 1.

We would like to bound $|\mathcal{W}_{n,m}|$. In any self-avoiding walk (SAW), the maximum size of a perpendicular independent subset of the edges and one such set can be determined by a greedy algorithm. Let us denote by $a_{n,m}$ ($b_{n,m}$) the number of SAW's in $\mathcal{W}_{n,m}$ of which the maximum independent set of edges determined by this greedy algorithm contains (does not contain) the last edge of the SAW. Any SAW that starts at the origin can be continued by at most 3 edges, 2 perpendicular and a forward edge, thus we have

$$a_{n,m} \le 2a_{n-1,m-1} + 3b_{n-1,r-1},$$

$$b_{n,m} \le a_{n-1,m},$$

which gives

$$a_{n,m} \le 2a_{n-1,m-1} + 3a_{n-2,r-1}.$$

Let us write $f(x, y) = \sum_{n,m \ge 0} a'_{n,m} x^n y^m$, where $a'_{1,m} = a_{1,m}$ and for other values of n $a'_{n,m}$ is defined such that the above inequalities hold with equation. By the recurrence we obtain

$$f(x, y) = \frac{g(x, y)}{1 - 2xy - 3x^2 y},$$

where $g(x, y)$ is defined for all $x, y \in \mathbb{R}$. Plugging in $x = 1$ we obtain that the radius of convergence of $\sum_{n,m \ge 0} a'_{n,m} y^m$ is 0.2, thus it is at least that much for $\sum_{n,m \ge 0} a_{n,m} y^m$. $\square$

Proposition 2.22. *For any edge $e \in E(\mathbb{Z}^2)$ we have*

$$\mu_{\lambda,1/2}(e \text{ is open}) = 1 - (\lambda + 1)e^{-\lambda}$$

Proof: W.l.o.g let $e \in E(\mathbb{Z}^2)$ be a horizontal edge on the line l with endpoints A and B, let H be the center of e and for any $P \in \{A, B, H\}$ let l_{P+} (l_{P-}) denote the halfline of l starting from (ending at) P. If Λ_l is the set of the points of Poisson process on l, then e is open if and only if one of the following three pairwise disjoint events happen: **(i)** Λ_l contains points both from the segments AH and HB, **(ii)** $\Lambda_l \cap HB = \emptyset$ and there exist $Q_1 \in l_{H^-} \cap \Lambda_l, Q_2 \in l_{B^+} \cap \Lambda_l$ with $d(Q_1, H) + d(B, Q_2) \leq 1/2$, **(iii)** $\Lambda_l \cap AH = \emptyset$ and there exist $Q_1 \in l_{A^-} \cap \Lambda_l, Q_2 \in l_{H^+} \cap \Lambda_l$ with $d(Q_1, A) + d(H, Q_2) \leq 1/2$.

The probability of **(i)** is $(1 - e^{-\lambda/2})^2$, the probability of both **(ii)** and **(iii)** is $e^{-\lambda/2} \int_0^{1/2} \int_0^t \lambda e^{-\lambda s} \lambda e^- \lambda(t-s) ds dt = e^{-\lambda/2} \int_0^{1/2} \lambda^2 t e^{-\lambda t} dt$. Integrating by parts gives

$$\int_0^{1/2} \lambda t e^{-\lambda t} = \left[-te^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \right]_0^{1/2} = -\left(\frac{1}{2} + \frac{1}{\lambda} \right) e^{-\lambda/2} + \frac{1}{\lambda}.$$

Thus by the pairwise disjointness of **(i)**, **(ii)** and **(iii)**we obtain

$$\begin{aligned} \mu_{\lambda,1/2}(e \text{ is open}) &= (1 - e^{-\lambda/2})^2 + 2e^{-\lambda/2} \int_0^{1/2} \lambda^2 t e^{-\lambda t} dt \\ &= (1 - e^{-\lambda/2})^2 + 2e^{-\lambda/2} \lambda \left[-\left(\frac{1}{2} + \frac{1}{\lambda} \right) e^{-\lambda/2} + \frac{1}{\lambda} \right] \\ &= 1 - (\lambda + 1) e^{-\lambda}. \end{aligned}$$

□

Theorems 3.5 and 3.6 together with Proposition 3.7 give the following Corollary.

Corollary 2.23.

$$0.82438 \leq \lambda_c(1/2) \leq 2.826$$

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