On tight paths of fixed length

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Abstract

A k-uniform hypergraph $\mathcal{H}$ contains a tight path of length l if there exist vertices $x_1, x_2, \dots, x_{k+l-1}$ such that $\{x_i, x_{i+1}, \dots, x_{i+k-1}\}$ is an edge of $\mathcal{H}$ for all $i = 1, 2, \dots, l$. The maximum number of edges that a k-uniform hypergraph on n vertices can have without containing a tight path of length l is denoted by $ex_k(n, P_l^{tight})$. The currently best known upper bound for general values of k and l is $\frac{(l-1)(k-1)}{k}\binom{n}{k-1}$ due to Füredi, Jiang, Kostochka, Mubayi, Verstraëte. In this short note we *repeat* a lemma from a previous paper of ours that gives the bound $\sum_{j=2}^{l} \frac{j-1}{k-j+2}\binom{n}{k-1}$. This is better than the bound of Füredi et al if the uniformity k is somewhat larger than the length l of the forbidden path, say $l < 0.8k$.

The Turán number $ex_k(n, \mathcal{F})$ of a k-uniform hypergraph $\mathcal{F}$ is the maximum number of edges that a k-uniform hypergraph $\mathcal{H}$ on n vertices can have without containing $\mathcal{F}$. There are several extensions of the notion of path to hypergraphs, one of them is the so-called tight path: the k-uniorm tight path P_l^{tight} of length l has $k + l - 1$ vertices $x_1, x_2, \dots, x_{k+l-1}$ and its edge set is $\{\{x_i, x_{i+1}, \dots, x_{i+k-1}\} : i = 1, 2, \dots, l\}$. Kalai's tight tree conjecture [6] for the special case of tight paths is the following (stated first in print in [1] and more recently in [5]).

Conjecture 0.1. *For any pair k, l of integers we have*

$$ex_k(n, P_l^{tight}) \le \frac{l-1}{k}\binom{n}{k-1}.$$

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Győri, Katona, and Lemons proved [4] the bound $ex_k(n, P_l^{tight}) \leq (l-1)\binom{n}{k-1}$. This was improved by Füredi, Jiang, Kostochka, Mubayi, Verstraëte as follows.

Theorem 0.2 ([2])**.** *For any pair k, l of integers we have*

$$ex_k(n, P_l^{tight}) \leq \frac{(l-1)(k-1)}{k}\binom{n}{k-1}.$$

To this short note we include a lemma from a previous paper of ours [7] that did not attract too much attention possibly as its title and abstract do not refer to tight paths or Turán problems at all. The bound obtained in the paper is better than all known bounds if k is at least $2l$ (when writing [7] the results of Győri, Katona, and Lemons were already known [3]). Since in [7], this result was applied with $k = n/2 + o(n)$, it was stated as follows.

Lemma 0.3 ([7])**.** *For any positive integer l, if $\mathcal{H} \subseteq \binom{[n]}{k}$ does not contain a tight path of length $l+1$, then $|\mathcal{H}| = O_l(\frac{1}{k}\binom{n}{k-1})$ provided $k \geq 2l$.*

In Corollary 0.5 below we state the best constant coming from the proof of [7]. The inductive argument is based on the following lemma.

Lemma 0.4. *For any pair k, l of positive integers with $k > l$ we have*

$$ex_k(n, P_{l+1}^{tight}) \leq ex_k(n, P_l^{tight}) + \frac{l}{k-l+1}\binom{n}{k-1}.$$

Proof. Let $\mathcal{H}$ be a hypergraph on n vertices with $|\mathcal{H}| = ex_k(n, P_l^{tight}) + \frac{l}{k-l+1}\binom{n}{k-1} + 1$. By definition of the Turán number, $\mathcal{H}$ must contain a tight path of length l. Removing the last edge of this path we can still find another tight path of length l. In this way, we find $\frac{l}{k-l+1}\binom{n}{k-1} + 1$ different edges in $\mathcal{H}$ such that each of them is the last set in a certain tight path of length l.

Let $\mathcal{H}_1$ denote the subhypergraph of these edges and consider an edge $H \in E(\mathcal{H}_1)$. Let H' denote the first edge of (one of) the tight path(s) to which H belongs, i.e. if the vertices of the tight path are $x_1, x_2, ..., x_{k+l-1}$ and $H = \{x_l, x_{l+1}, ..., x_{k+l-1}\}$, then $H' = \{x_1, x_2, ..., x_k\}$. Let the *modified shadow* of H with respect to H' be $\{H \setminus \{x_j\} : l \leq j \leq k\}$. The size of the modified shadow determined by any tight path is $k - l + 1$. Therefore, there exists an $(k-1)$-set G that belongs to the modified shadows of at least $l+1$ edges $H^1, H^2, ..., H^{l+1}$ from $E(\mathcal{H}_1)$.

Let $P_1, P_2, ..., P_l = H^1$ be a tight path of length l on the vertices $\{y_1, y_2, ..., y_{k+l-1}\}$ with $\{y_j, y_{j+1}, ..., y_{j+k-1}\} = P_j \in E(\mathcal{H})$ for all $j = 1, 2, ..., l$ and let $G = H^1 \setminus \{y_t\}$ for some $l \leq t \leq k$. As the H^j's are all different containing G and have size k at least one of them, say H^2, is of the form $G \cup \{z\}$ such that $z \notin \{y_1, y_2, ..., y_{l-1}, y_t\}$. But then the sets $P_1, P_2, ..., P_l = H^1, H^2$ form a tight path of length $l+1$ on the vertices $\{y_1, y_2, ..., y_{l-1}, y_t, y_l, y_{l+1}, ... y_{t-1}, y_{t+1}, \ldots, y_{k+l-1}, z\}$. □

Corollary 0.5. *For any pair k, l of integers with $l \leq k$ we have*

$$ex_k(n, P_l^{tight}) \leq \sum_{j=2}^{l} \frac{j-1}{k-j+2}\binom{n}{k-1}.$$

Proof. For a k-set S its shadow $\Delta(S)$ is simply $\binom{S}{k-1}$. Observe that for $l = 2$, the statement follows from the fact that if a k-uniform hypergraph $\mathcal{H}$ on n vertices is P_2^{tight}-free, then the collection $\{\Delta(H) : H \in E(\mathcal{H})\}$ of shadows is pairwise disjoint, therefore $k \cdot |\mathcal{F}| \leq \binom{n}{k-1}$ must hold. The general statement then follows by induction on l using Lemma 0.4. $\square$

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