

# Families that remain $k$ -Sperner even after omitting an element of their ground set

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## Abstract

A family  $\mathcal{F} \subseteq 2^{[n]}$  of sets is said to be  $l$ -trace  $k$ -Sperner if for any  $l$ -subset  $L \subset [n]$  the family  $\mathcal{F}|_L = \{F|_L : F \in \mathcal{F}\} = \{F \cap L : F \in \mathcal{F}\}$  is  $k$ -Sperner, i.e. does not contain any chain of length  $k+1$ . The maximum size that an  $l$ -trace  $k$ -Sperner family  $\mathcal{F} \subseteq 2^{[n]}$  can have is denoted by  $f(n, k, l)$ . For pairs of integers  $l < k$ , if in a family  $\mathcal{G}$  every pair of sets satisfies  $||G_1| - |G_2|| < k - l$ , then  $\mathcal{G}$  possesses the  $(n - l)$ -trace  $k$ -Sperner property. Among such families, the largest one is  $\mathcal{F}_0 = \{F \in 2^{[n]} : \lfloor \frac{n-(k-l)}{2} \rfloor + 1 \leq |F| \leq \lfloor \frac{n-(k-l)}{2} \rfloor + k - l\}$  and also  $\mathcal{F}'_0 = \{F \in 2^{[n]} : \lfloor \frac{n-(k-l)}{2} \rfloor \leq |F| \leq \lfloor \frac{n-(k-l)}{2} \rfloor + k - l - 1\}$  if  $n - (k - l)$  is even. In an earlier paper, we proved that this is asymptotically optimal for all pair of integers  $l < k$ , i.e.  $f(n, k, n - l) = (1 + o(1))|\mathcal{F}_0|$ . In this paper we consider the case when  $l = 1$ ,  $k \geq 2$ , and prove that  $f(n, k, n - 1) = |\mathcal{F}_0|$  provided  $n$  is large enough. We also prove that the unique  $(n - 1)$ -trace  $k$ -Sperner family with size  $f(n, k, n - 1)$  is  $\mathcal{F}_0$  and also  $\mathcal{F}'_0$  when  $n + k$  is odd.

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## 1 Introduction

We use standard notation. The set of the first  $n$  positive integers is denoted by  $[n]$ . For a set  $X$  the family of all subsets of  $X$ , all  $i$ -subsets of  $X$ , all subsets of  $S$  of size at most  $i$ , all subsets of  $S$  of size at least  $i$  are denoted by  $2^X$ ,  $\binom{X}{i}$ ,  $\binom{X}{\leq i}$ ,  $\binom{X}{\geq i}$ , respectively. A chain of

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length  $k$  is a family of  $k$  sets satisfying  $F_1 \subset F_2 \subset \dots \subset F_k$ . A maximal chain  $\mathcal{C} \subseteq 2^{[n]}$  is a chain of length  $n + 1$ .

Let  $\Sigma(n, m) = \sum_{i=1}^m \binom{n}{\lfloor \frac{n-m}{2} \rfloor + i}$  denote the sum of the  $m$  largest binomial coefficients of order  $n$ .

A typical problem in extremal set system theory is to determine how many sets a family  $\mathcal{F} \subseteq 2^{[n]}$  may contain if it satisfies some prescribed property. As one of the first such result, Erdős [1] in 1945 proved that if a family  $\mathcal{F} \subseteq 2^{[n]}$  does not contain any chain of length  $k + 1$  (families with this property are called *k-Sperner families*), then the size of  $\mathcal{F}$  cannot exceed  $\Sigma(n, k)$  and the only  $k$ -Sperner family of this size is  $\{F \in 2^{[n]} : \lfloor \frac{n-k}{2} \rfloor + 1 \leq |F| \leq \lfloor \frac{n-k}{2} \rfloor + k\}$  if  $n + k$  is odd, and also  $\{F \in 2^{[n]} : \lfloor \frac{n-k}{2} \rfloor \leq |F| \leq \lfloor \frac{n-k}{2} \rfloor + k - 1\}$  if  $n + k$  is even. The case  $k = 1$  was proved by Sperner [6] in 1928.

The *trace* of a set  $F$  on another set  $X$  is  $F \cap X$  and is denoted by  $F|_X$ . The trace of a family  $\mathcal{F}$  on  $X$  is the family of traces  $\mathcal{F}_X = \{F|_X | F \in \mathcal{F}\}$ . The fundamental result about traces of families, known as Sauer-lemma, was proved in the early 70's independently by Sauer [4], Shelah [5], and Vapnik and Chervonenkis [7] and states that if for a family  $\mathcal{F} \subseteq 2^{[n]}$  there exists no set  $K$  of size  $k$  such that  $\mathcal{F}|_K = 2^K$  (i.e. all subsets of  $K$  appear as a trace of a set in  $\mathcal{F}$ ), then  $|\mathcal{F}| \leq \sum_{i=0}^{k-1} \binom{n}{i}$  holds. This bound is sharp as shown by the families  $\binom{[n]}{\leq k-1}$  and  $\binom{[n]}{\geq n-k+1}$ , but there are lots of other families of this size satisfying the condition of the Sauer-lemma.

One way to make sure that a family  $\mathcal{F}$  satisfies the condition of the Sauer-lemma is to prescribe not to contain any maximal chain as trace in any  $k$ -subset  $K$  of  $[n]$ . This observation leads to the following notion introduced in [2]: a family  $\mathcal{F}$  is said to be *l-trace k-Sperner* if for any  $l$ -set  $L$  the trace  $\mathcal{F}|_L$  is  $k$ -Sperner. In [2], it was proved that if  $k \leq l$  and  $n$  is large enough, then the maximum size  $f(n, k, l)$  that an  $l$ -trace  $k$ -Sperner family  $\mathcal{F} \subseteq 2^{[n]}$  can have is  $\sum_{i=0}^{k-1} \binom{n}{i}$  and the only  $l$ -trace  $k$ -Sperner families of this size are  $\binom{[n]}{\leq k-1}$  and  $\binom{[n]}{\geq n-k+1}$ , i.e. if the condition of the Sauer-lemma is strengthened to the  $l$ -trace  $k$ -Sperner property, then the uniqueness of the 'trivial' extremal families  $\binom{[n]}{\leq k-1}$  and  $\binom{[n]}{\geq n-k+1}$  holds.

The situation is entirely different if for fixed  $k$  and  $l$  with  $k > l$  and  $n$  large enough, we consider  $(n - l)$ -trace  $k$ -Sperner families, i.e. families of which the traces form a  $k$ -Sperner family no matter which  $l$  elements of the ground set we omit. For any  $l$ -set  $L$  and  $G \in 2^{[n]}$  we have  $|G| - l \leq |G|_{[n] \setminus L} \leq |G|$  and thus if in a family  $\mathcal{G}$  every pair of sets satisfies  $||G_1| - |G_2|| < k - l$ , then  $\mathcal{G}$  possesses the  $(n - l)$ -trace  $k$ -Sperner property. We obtain the largest such family if we take  $\mathcal{G}_0 = \{G \in 2^{[n]} : \lfloor \frac{n-(k-l)}{2} \rfloor + 1 \leq |G| \leq \lfloor \frac{n-(k-l)}{2} \rfloor + k - l\}$  and also  $\mathcal{G}'_0 = \{G \in 2^{[n]} : \lfloor \frac{n-(k-l)}{2} \rfloor \leq |G| \leq \lfloor \frac{n-(k-l)}{2} \rfloor + k - l - 1\}$  if  $n - (k - l)$  is even. The size of  $\mathcal{G}_0$  is  $\Sigma(n, k - l)$ . In [2] and [3] we conjectured that the families  $\mathcal{G}_0$  and  $\mathcal{G}'_0$  are optimal.

**Conjecture 1.1.** *Let  $k$  and  $l$  be positive integers with  $l < k$ . Then there exists  $n_0 = n_0(k, l)$  such that if  $n \geq n_0$ , then  $f(n, k, n - l) = \Sigma(n, k - l)$  holds.*

Conjecture 1.1 was proved asymptotically in [3] (the case  $l = 1, k = 2$  was already proved

in [2]).

**Theorem 1.2.** [2] *Let  $k$  and  $l$  be positive integers with  $l < k$ . Then  $f(n, k, n - l) = (1 + O(\frac{1}{n^{2/3}}))\Sigma(n, k - l)$  holds.*

Our main result verifies Conjecture 1.1 for  $l = 1, k \geq 2$  and it also describes the extremal family.

**Theorem 1.3.** *Let  $k \geq 2$  be an integer. Then there exists  $n_0 = n_0(k)$  such that if  $n \geq n_0$  and  $\mathcal{F} \subseteq 2^{[n]}$  is an  $(n - 1)$ -trace  $k$ -Sperner family, then  $|\mathcal{F}| \leq \Sigma(n, k - 1)$ . Furthermore, equality holds if and only if  $\mathcal{F}$  is the family  $\mathcal{G}_0$  and when  $n + k$  is odd, then also if  $\mathcal{F}$  is the family  $\mathcal{G}'_0$ .*

Note that Sperner's result follows from the case  $k = 2$  as if  $\mathcal{F} \subseteq 2^{[n]}$  is Sperner, then it is  $(n - 1)$ -trace 2-Sperner. Indeed, if  $x \in [n]$  and  $F_1, F_2, F_3 \in \mathcal{F}$  were such that  $F_1|_{[n]-x} \subsetneq F_2|_{[n]-x} \subsetneq F_3|_{[n]-x}$  would hold, then there would exist  $1 \leq i < j \leq 3$  with  $x \in F_i, F_j$  or  $x \notin F_i, F_j$  and thus  $F_i \subsetneq F_j$  would contradict the Sperner property of  $\mathcal{F}$ . In general it is not true that a  $k$ -Sperner family possesses the  $(n - 1)$ -trace  $(k - 1)$ -Sperner property, but the largest such family does. Theorem 1.3 states that no other  $(n - 1)$ -trace  $(k - 1)$ -Sperner family can have larger size.

In the proof of Theorem 1.3 we will need the next result which follows from the Corollary after Theorem 7 in [2].

**Theorem 1.4.**

$$f(n, n - 1, 1) = O\left(\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}\right) = O\left(\frac{1}{n} \Sigma(n - |F_0|, 1)\right).$$

**Notation.** The complement of a set  $F$  will be denoted by  $\overline{F}$ . For convenience, we will write  $F + z$  instead of  $F \cup \{z\}$  and  $F - z$  instead of  $F \setminus \{z\}$ . Also, if  $\sigma$  is a permutation of elements of an  $m$ -set  $M$ , then we will think of  $\sigma$  as an ordering of the elements of  $M$  and write  $\sigma_1, \sigma_2, \dots, \sigma_m$  for the first, second, ...,  $m$ th element in the ordering. The *index* of an element  $x$  in the permutation  $\sigma$  is the integer  $i$  for which  $\sigma_i = x$  and will be denoted by  $ind(x)$ . The set of permutations of  $[n]$  is denoted by  $S_n$ . Let  $\pi \in S_n$ ,  $a, b, c, d \in [n]$  and  $a, b, c, d \notin Y \subset [n]$ . Then we will write  $\pi$  as

$$\dots\dots\dots a \dots\dots\dots bcYd \dots\dots\dots$$

to denote the fact that  $ind(a) < ind(b) = ind(c) - 1 < ind(d) - 2$  and exactly those elements  $y$  belong to  $Y$  for which  $ind(c) < ind(y) < ind(d)$  holds. Furthermore, if we are interested in the relation of the indices of elements of two not necessarily disjoint subsets  $A = \{a, b, c, d\}$  and  $X = \{x, y, z\}$  of  $[n]$ , then the permutation  $\pi$  is written as

$$\dots\dots\dots \begin{matrix} & & x & y & z \\ & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & c & d \end{matrix} \dots\dots\dots$$

to denote  $ind(a) = ind(b) - 1 = ind(c) - 2 = ind(x) - 2 = ind(y) - 3 = ind(z) - 4 = ind(d) - 4$  and thus  $x = c, z = d$  and  $|A \cap X| = 2$  hold.

## 2 Proof of Theorem 1.3

Let  $\mathcal{F}$  be an  $(n - 1)$ -trace  $k$ -Sperner family and let  $\mathcal{C} \subset 2^{[n]}$  be a maximal chain. Note that if  $n > k$ , then  $|\mathcal{F} \cap \mathcal{C}| \leq k$  holds. Indeed, if  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_{k+1} \subseteq [n]$  holds, then either  $G_1$  or  $[n] \setminus G_{k+1}$  is non-empty or at least one of the sets  $G_i \setminus G_{i-1}$  contains two elements. When omitting an element from one such set, the traces of the  $G_i$ 's would still form a chain of length  $k + 1$ .

Let  $c^-, c, c^+$  denote the number of maximal chains  $\mathcal{C} \subseteq 2^{[n]}$  such that  $|\mathcal{F} \cap \mathcal{C}|$  is less than  $k - 1$ , exactly  $k - 1$ , exactly  $k$ , respectively. By the above observation, we know that  $c^- + c + c^+ = n!$  holds. Let  $\mathcal{F}^k = \{(F_1, F_2, \dots, F_k) : F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k, F_i \in \mathcal{F}\}$  denote the set of  $k$ -chains in  $\mathcal{F}$ . The main step of the proof of Theorem 1.3 is the following lemma which states that on average a maximal chain contains at most  $k - 1$  sets from an  $(n - 1)$ -trace  $k$ -Sperner family.

**Lemma 2.1.** *Let  $\mathcal{F}$  be an  $(n - 1)$ -trace  $k$ -Sperner family such that  $4 \leq |F| \leq n - 1$  holds for all  $F \in \mathcal{F}$ . Then the inequality  $c^- \geq c^+$  holds. Moreover, if there exists a  $k$ -chain  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  with  $5 \leq |F_1|$  and  $|F_{i+1} \setminus F_i| = 1$  for all  $i = 1, 2, \dots, k - 1$ , then  $c^- > c^+$  holds.*

*Proof.* Let  $\mathcal{F}$  be an  $(n - 1)$ -trace  $k$ -Sperner family. We will say that  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  is of type I if  $|F_{i+1} \setminus F_i| = 1$  holds for all  $1 \leq i \leq k - 1$  and  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  is of type II- $\ell$  if  $|F_{i+1} \setminus F_i| = 1$  holds for all  $1 \leq i \leq \ell - 1$  and  $|F_{\ell+1} \setminus F_\ell| \geq 2$ . Note that

$$c^+ = \sum_{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k} (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)!$$

where  $|F_0|$  is defined to be 0.

Let  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  be a  $k$ -chain of type I and  $x \in F_1, z \notin F_k$ . Then let  $\mathcal{C}(x, z, F_1, F_2, \dots, F_k)$  denote the set of those maximal chains that contain  $(F_i - x) + z$  for all  $1 \leq i \leq k$  and  $F_k + z$ . Note that if  $\{x, z\} \neq \{x', z'\}$ , then  $\mathcal{C}(x, z, F_1, F_2, \dots, F_k) \cap \mathcal{C}(x', z', F_1, F_2, \dots, F_k) = \emptyset$  as the sets of size  $|F_1|$  in  $\mathcal{C}(x, z, F_1, F_2, \dots, F_k)$  and  $\mathcal{C}(x', z', F_1, F_2, \dots, F_k)$  are  $(F_1 - x) + z$  and  $(F_1 - x') + z'$ , respectively. Therefore writing  $\mathcal{C}(F_1, F_2, \dots, F_k) = \bigcup_{x \in F_1, z \notin F_k} \mathcal{C}(x, z, F_1, F_2, \dots, F_k)$  we have

$$|\mathcal{C}(F_1, F_2, \dots, F_k)| = \sum_{x \in F_1, z \notin F_k} |\mathcal{C}(x, z, F_1, F_2, \dots, F_k)| = |F_1| \cdot (n - |F_k|) \cdot |F_1|! (n - |F_k| - 1)! = |F_1| \cdot (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)!$$

**Claim 2.2.** Let  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  be a  $k$ -chain of type I and  $x \in F_1, z \notin F_k$ . Then for any  $\mathcal{C} \in \mathcal{C}(x, z, F_1, F_2, \dots, F_k)$  we have  $|\mathcal{C} \cap \mathcal{F}| \leq k - 2$ .

*Proof of Claim.* Let  $\mathcal{C} \in \mathcal{C}(x, z, F_1, F_2, \dots, F_k)$  be a maximal chain. If  $C \in \mathcal{C}$  with  $C \subseteq (F_1 - x) + z$ , then  $C \notin \mathcal{F}$  as  $C|_{[n]-z} \subsetneq F_1|_{[n]-z} \subsetneq \dots \subsetneq F_k|_{[n]-z}$  would form a chain of length  $k+1$  in  $\mathcal{F}|_{[n]-z}$ . Also, if  $C \supseteq F_k + z$ , then  $C \notin \mathcal{F}$  as  $F_1, F_2, \dots, F_k, C$  would form a  $(k+1)$ -chain even without omitting any element of the ground set  $[n]$ .

Finally, it cannot happen that  $F'_i = (F_i - x) + z \in \mathcal{F}$  holds for all  $2 \leq i \leq k$  as then  $F_1|_{[n]-x} \subsetneq F_2|_{[n]-x} \subsetneq F'_2|_{[n]-x} \subsetneq \dots \subsetneq F'_k|_{[n]-x}$  would be a  $(k+1)$ -chain in  $\mathcal{F}|_{[n]-x}$ .  $\square$

Let  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  be a  $k$ -chain of type II- $\ell$  with  $1 \leq \ell, x \in F_1, z \notin F_k$  and  $\sigma$  be a permutation of  $F_{\ell+1} \setminus F_\ell$ . Then writing  $y_i$  for the unique element of  $F_{i+1} \setminus F_i$  for  $1 \leq i \leq \ell - 1$  and  $m = |F_{\ell+1} \setminus F_\ell|$ , let  $\mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$  denote the set of those maximal chains that contain all sets from

$$\begin{aligned} \mathcal{A}_1 &= \{F_i - x : 1 \leq i \leq \ell\}, \\ \mathcal{A}_2 &= \{(F_\ell - x) + \sigma_1, F_\ell + \sigma_1, (F_\ell + \sigma_1) + \sigma_2, \dots, F_{\ell+1} - \sigma_m, (F_{\ell+1} - \sigma_m) + z\}, \\ \mathcal{A}_3 &= \{F_j + z : \ell + 1 \leq j \leq k\}. \end{aligned}$$

Again, it is easy to see that for a fixed  $k$ -chain  $(F_1, F_2, \dots, F_k)$  of type II the sets of chains  $\mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$  are pairwise disjoint, therefore writing  $\mathcal{C}(F_1, F_2, \dots, F_k) = \bigcup_{x \in F_1, z \notin F_k, \sigma} \mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$  we have

$$\begin{aligned} |\mathcal{C}(F_1, F_2, \dots, F_k)| &= \sum_{x \in F_1, z \notin F_k, \sigma} |\mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)| = \\ &|F_1| \cdot (n - |F_k|) \cdot (|F_1| - 1)! (n - |F_k| - 1)! \prod_{i=2}^k (|F_i| - |F_{i-1}|)! = (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)! \end{aligned}$$

**Claim 2.3.** Let  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  be a  $k$ -chain of type II- $\ell$  with  $2 \leq \ell, x \in F_1, z \notin F_k$  and  $\sigma$  as above. Then for any  $\mathcal{C} \in \mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$  we have  $|\mathcal{C} \cap \mathcal{F}| \leq k - 2$ .

*Proof of Claim.* Let  $\mathcal{C} \in \mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$  be a maximal chain. If  $C \in \mathcal{C}$  with  $C \subseteq F_1 - x$ , then  $C \notin \mathcal{F}$ , as then  $C \subsetneq F_1$  would hold and  $\mathcal{F}$  would contain a  $(k+1)$ -chain.

Note also that  $\mathcal{A}_2 \cap \mathcal{C} = \emptyset$ . Indeed, for any  $1 \leq j < |F_{\ell+1} \setminus F_\ell|$  and writing  $A_j = (\dots((F_\ell + \sigma_1) + \sigma_2) + \dots) + \sigma_j$  we have  $F_\ell \subsetneq A_j \subsetneq F_{\ell+1}$  and thus  $A_j$  and the  $F_i$ 's would form a  $(k+1)$ -chain. Also, the traces of  $A_0 = (F_\ell - x) + \sigma_1$  and the  $F_i$ 's would form a  $(k+1)$ -chain in  $\mathcal{F}|_{[n]-x}$  and the traces of  $A_m = (F_{\ell+1} - \sigma_m) + z$  and the  $F_i$ 's would form a  $(k+1)$ -chain in  $\mathcal{F}|_{[n]-z}$ . (These two statements use the fact that  $m = |F_{\ell+1} \setminus F_\ell| \geq 2$ .)

We obtained that  $\mathcal{C} \cap \mathcal{F}$  might contain at most  $\ell - 1$  sets of  $\mathcal{A}_1$  and some sets  $C_1, \dots, C_t$  containing  $F_{\ell+1} + z$ . Observe that  $t \leq k - \ell - 1$  as otherwise  $F_1, \dots, F_{\ell+1}$  together with  $C_1, \dots, C_t$  would form a chain of length at least  $k + 1$ . Therefore  $|\mathcal{C} \cap \mathcal{F}| \leq \ell - 1 + k - \ell - 1 = k - 2$  as stated by the Claim.  $\square$

We introduce further notation. First note that maximal chains are in a one-to-one correspondence with permutations of the ground set  $[n]$  as with any maximal chain  $\mathcal{C} = \{F_0, F_1, \dots, F_n\}$  one can associate the permutation  $\pi = \pi(\mathcal{C})$  such that  $\pi_i = F_i \setminus F_{i-1}$ . The set of permutations corresponding to maximal chains in  $\mathcal{C}(x, z, \sigma, F_1, F_2, \dots, F_k)$ ,  $\mathcal{C}(x, z, F_1, F_2, \dots, F_k)$ ,  $\mathcal{C}(F_1, F_2, \dots, F_k)$  will be denoted  $\pi(x, z, \sigma, F_1, F_2, \dots, F_k)$ ,  $\pi(x, z, F_1, F_2, \dots, F_k)$ ,  $\pi(F_1, F_2, \dots, F_k)$ , respectively. Permutations  $\pi \in S_n$  belonging to  $\pi(x, z, F_1, F_2, \dots, F_k)$  for a  $k$ -chain of type I look like this

$$\dots\dots\dots z \dots\dots\dots y_1 y_2 \dots y_{k-1} x \dots\dots\dots .$$

Knowing  $\pi$ ,  $\text{ind}(x)$  and  $\text{ind}(z)$  we are able to recover the  $F_i$ 's by

$$F_i = (\{\pi_j : j \leq \text{ind}(x) - k + i - 1\} - \pi_{\text{ind}(z)}) + \pi_{\text{ind}(x)}.$$

Permutations  $\pi \in S_n$  belonging to  $\pi(x, z, \sigma, F_1, F_2, \dots, F_k)$  for a  $k$ -chain of type II- $\ell$  look like this

$$\dots\dots\dots y_1 y_2 \dots y_{\ell-1} \sigma_1 x \sigma_2 \dots \sigma_{m-1} z \sigma_m Y_{\ell+1} \dots Y_{k-1} \dots\dots\dots .$$

Just as for  $k$ -chains of type I, we are able to recover the  $F_i$ 's if we know  $\pi$ ,  $\text{ind}(y_i)$   $i = 1, \dots, \ell - 1$ ,  $\text{ind}(x)$ ,  $\text{ind}(z)$ ,  $\max\{\text{ind}(y) : y \in F_{j+1} \setminus F_j\}$   $j = \ell + 1, \dots, k - 1$ .

For every  $k$ -chain  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  we have defined a set of maximal chains that contain at most  $k - 2$  sets from  $\mathcal{F}$ . To show that the union of these sets is large we need to prove that there is not much of an overlap among them. We are not able to fully establish such a result, but we manage to prove such statements for subsets of the  $\mathcal{C}(F_1, F_2, \dots, F_k)$ 's. For every  $k$ -chain  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  of type I, let  $\pi^*(x, z, F_1, F_2, \dots, F_k) = \{\pi \in \pi(x, z, F_1, F_2, \dots, F_k) : \text{ind}(z) \leq \text{ind}(y_1) - 2\}$ , while for  $k$ -chains of type II- $\ell$ , let  $\pi^*(x, z, \sigma, F_1, F_2, \dots, F_k)$  equal  $\pi(x, z, \sigma, F_1, F_2, \dots, F_k)$ .  $\mathcal{C}^*(x, z, F_1, F_2, \dots, F_k)$  denotes the set of corresponding maximal chains and we obtain  $\pi^*(F_1, F_2, \dots, F_k)$  and  $\mathcal{C}^*(F_1, F_2, \dots, F_k)$  by taking union over all  $x \in F_1, z \notin F_k$  and  $\sigma$  being a permutation of  $F_{\ell+1} \setminus F_\ell$ . Clearly, for any  $k$ -chain of type I we have

$$|\pi^*(F_1, F_2, \dots, F_k)| = (|F_1| - 2) \cdot (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)$$

To make the reasoning in the previous paragraph more formal we need the following final notation. For any maximal chain  $\mathcal{C}$  let  $s^*(\mathcal{C}, \mathcal{F})$  denote the number of  $k$ -chains  $(F_1, F_2, \dots, F_k)$  in  $\mathcal{F}^k$  such that  $\mathcal{C} \in \mathcal{C}^*(F_1, F_2, \dots, F_k)$  and let  $s^*(F_1, F_2, \dots, F_k) = \max\{s^*(\mathcal{C}, \mathcal{F}) : \mathcal{C} \text{ is a maximal chain with } F_1, F_2, \dots, F_k \in \mathcal{C}\}$ . By Claim 2.2, and Claim 2.3, we have

$$c^- \geq \sum_{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k} \frac{|\mathcal{C}^*(F_1, F_2, \dots, F_k)|}{|s^*(F_1, F_2, \dots, F_k)|}.$$

The following two claims will allow us to establish good upper bounds on  $s^*(F_1, F_2, \dots, F_k)$ .









Then  $\sigma'_1 = y_{\ell-1}$ ,  $x' = y_\ell$  and  $\text{ind}(x') < \text{ind}(x)$ . Thus  $F'_1|_{[n]-x'} \subsetneq \dots \subsetneq F'_{\ell'}|_{[n]-x'} \subsetneq F_{\ell-1}|_{[n]-x'} \subsetneq F_{\ell'}|_{[n]-x'} \subsetneq F_{\ell'+2}|_{[n]-x'} \subsetneq \dots \subsetneq F_k|_{[n]-x'}$  would be a chain of length  $k+1$  contradicting the  $(n-1)$ -trace  $k$ -Sperner property of  $\mathcal{F}$ .  $\square$

By Claim 2.4 and Claim 2.5 we obtain  $s^*(F_1, F_2, \dots, F_k) \leq 2$  if  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  is of type I and  $s^*(F_1, F_2, \dots, F_k) = 1$  if  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  is of type II- $\ell$  for any  $1 \leq \ell \leq k-1$ . Therefore by Claim 2.2 and Claim 2.3 we have

$$c^- \geq \sum_{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k} \frac{|\mathcal{C}^*(F_1, F_2, \dots, F_k)|}{|s^*(F_1, F_2, \dots, F_k)|} \geq$$

$$\sum_{\substack{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k \\ \text{type I}}} \frac{(|F_1| - 2)(n - |F_k|)!}{2} \prod_{i=1}^k (|F_i| - |F_{i-1}|)! + \sum_{\substack{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k \\ \text{type II}}} (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)! \geq$$

$$\sum_{(F_1, F_2, \dots, F_k) \in \mathcal{F}^k} (n - |F_k|)! \prod_{i=1}^k (|F_i| - |F_{i-1}|)! = c^+$$

The moreover part of the statement follows as if  $|F_1| \geq 5$  for some  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  of type I, then for that particular summand we have at least a  $3/2$  fraction more than what we need.  $\square$

**Corollary 2.6.** *Let  $\mathcal{F}$  be an  $(n-1)$ -trace  $k$ -Sperner family such that  $4 \leq |F| \leq n-1$  holds for all  $F \in \mathcal{F}$ . Then the inequality*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq k-1$$

*holds. Moreover, if there exists a  $k$ -chain  $(F_1, F_2, \dots, F_k) \in \mathcal{F}^k$  of type I with  $5 \leq |F_1|$ , then the inequality is strict.*

*Proof.* Let us count the pairs  $(F, \mathcal{C})$  where  $\mathcal{C}$  is a maximal chain and  $F \in \mathcal{F} \cap \mathcal{C}$ . On the one hand the number of such pairs is  $\sum_{F \in \mathcal{F}} |F|!(n - |F|)!$ , on the other hand this is at most  $k \cdot c^+ + (k-1)c + (k-2)c^-$  which is, by Lemma 2.1, at most  $(k-1) \cdot n!$ . Dividing by  $n!$  gives the statement and the moreover part follows from the moreover part of Lemma 2.1.  $\square$

*Proof of Theorem 1.3.* Let  $\mathcal{F} \subseteq 2^{[n]}$  be an  $(n-1)$ -trace  $k$ -Sperner family and suppose first that there exists a set  $F \in \mathcal{F}$  with  $|F| = 0, 1, 2, 3$  or  $n$ . If  $F = \emptyset$  or  $[n]$ , then  $\mathcal{F}' = \{F \in \mathcal{F} : 2 \leq |F| \leq n-2\}$  is  $(n-1)$ -trace  $(k-1)$ -Sperner. Indeed, the trace  $[n]|_{[n]-x} = [n] - x$  always strictly contains the trace of any set of size at most  $n-2$  and the trace  $\emptyset|_{[n]-x} = \emptyset$  is always strictly contained in the trace of any set of size at least two. Therefore any  $k$ -chain

in  $\mathcal{F}'|_{[n]-x}$  could be extended to a  $(k+1)$ -chain in  $\mathcal{F}|_{[n]-x}$ . Using Theorem 1.2 if  $k \geq 3$  we obtain that  $|\mathcal{F}'| \leq (1 + O(\frac{1}{n^{1/3}}))\Sigma(n, k-2)$  and thus  $|\mathcal{F}| \leq 2 + 2n + (1 + O(\frac{1}{n^{1/3}}))\Sigma(n, k-2)$ , which is strictly less than  $\Sigma(n, k-1)$  if  $n$  is large enough, while if  $k = 2$  then Theorem 1.3 gives that  $|\mathcal{F}'| \leq O(\frac{1}{n}\Sigma(n, 1))$  and thus  $|\mathcal{F}| < \Sigma(n, 1)$  if  $n$  is large enough.

Suppose next that there exists  $F_0 \in \mathcal{F}$  with  $1 \leq |F| \leq 3$ . For every subset  $S$  of  $F_0$ , let us write  $\mathcal{F}_S = \{F \in \mathcal{F} : F \cap F_0 = S, 5 \leq |F| \leq n-1\}$ . Observe that for any  $S \subseteq F_0$  the family  $\mathcal{F}_S|_{[n]\setminus F_0}$  is  $(n - |F_0| - 1)$ -trace  $k$ -Sperner and  $|\mathcal{F}_S| = |\mathcal{F}_S|_{[n]\setminus F_0}|$ . Therefore, by Theorem 1.2, we obtain  $|\mathcal{F}_S| \leq (1 + o(1))\Sigma(n - |F_0|, k-1) = (1/2^{|F_0|} + o(1))\Sigma(n, k-1)$ . Furthermore, the family  $\mathcal{F}_{F_0}|_{[n]\setminus F_0}$  is  $(n - |F_0| - 1)$ -trace  $(k-1)$ -Sperner. Indeed, as  $|F_0| \leq 3$  and all sets in  $\mathcal{F}_{F_0}$  have size at least five, we have  $F_0|_{[n]-x} \subsetneq F|_{[n]-x}$  for any  $F \in \mathcal{F}_{F_0}$  and  $x \in [n]$  and thus adding  $F_0|_{[n]-x}$  to any  $k$ -chain in  $\mathcal{F}_{F_0}|_{[n]-x}$  would create a  $(k+1)$ -chain in  $\mathcal{F}|_{[n]-x}$ . If  $k \geq 3$ , then Theorem 1.2 yields  $|\mathcal{F}_{F_0}| \leq (1 + o(1))\Sigma(n - |F_0|, k-2) = (1/2^{|F_0|} + o(1))\Sigma(n, k-2) = (\frac{k-2}{(k-1)2^{|F_0|}} + o(1))\Sigma(n, k-1)$ , and thus

$$|\mathcal{F}| \leq \sum_{i=1}^4 \binom{n}{i} + \sum_{S \subseteq F_0} |\mathcal{F}_S| < \Sigma(n, k-1),$$

provided  $n$  is large enough. If  $k = 2$ , then Theorem 1.3 gives that  $|\mathcal{F}_{F_0}| \leq O(\frac{1}{n}\Sigma(n - |F_0|, 1))$  and thus

$$|\mathcal{F}| \leq \sum_{i=1}^4 \binom{n}{i} + \left( \frac{2^{|F_0|} - 1}{2^{|F_0|}} + o(1) \right) \Sigma(n, 1) < \Sigma(n, 1)$$

if  $n$  is large enough.

We are left with the case when  $\mathcal{F}$  does not contain any set of size 0, 1, 2, 3 or  $n$ . Corollary 2.6 yields the statement  $|\mathcal{F}| \leq \Sigma(n, k-1)$  and also the uniqueness of the extremal family if  $n+k$  is even. If  $n+k$  is odd, then  $\mathcal{F}$  must contain only sets of size between  $\lfloor \frac{n-k+1}{2} \rfloor$  and  $\lceil \frac{n+k-1}{2} \rceil$  and thus cannot contain sets of size four if  $\lfloor \frac{n-k+1}{2} \rfloor \geq 5$  holds. By the moreover part of Corollary 2.6,  $\mathcal{F}$  cannot contain a  $k$ -chain of type I with even the smallest set having size at least five. The sizes of the largest and smallest set of a  $k$ -chain of type II differ by at least  $k$  and thus at least one of them is outside the interval between  $\lfloor \frac{n-k+1}{2} \rfloor$  and  $\lceil \frac{n+k-1}{2} \rceil$ . We obtained that an  $(n-1)$ -trace  $k$ -Sperner family of size  $\Sigma(n, k-1)$  must be  $(k-1)$ -Sperner. Then the uniqueness follows from the uniqueness part of Erdős's result.  $\square$

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