A note on traces of set families

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Abstract

A family of sets $\mathcal{F} \subseteq 2^{[n]}$ is defined to be l-trace k-Sperner if for any l-subset L of $[n]$ the family of traces $\mathcal{F}|_L = \{F \cap L : F \in \mathcal{F}\}$ does not contain any chain of length $k+1$. In this paper we prove that for any positive integers l', k with $l' < k$ if $\mathcal{F}$ is $(n-l')$-trace k-Sperner, then $|\mathcal{F}| \leq (k - l' + o(1))\binom{n}{\lfloor n/2 \rfloor}$ and this bound is asymptotically tight.

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when you hear the instruction
TRACE, TRACE,
take up the following position

1 Introduction

We use standard notation. The set of the first n positive integers is denoted by $[n]$. For a set X the family of all subsets of X, all i-subsets of X, all subsets of S of size at most i, all subsets of S of size at least i are denoted by 2^X, $\binom{X}{i}$, $\binom{X}{\leq i}$, $\binom{X}{\geq i}$, respectively.

Probably the very first theorem in extremal finite set theory is Sperner's result [13] stating that if a family $\mathcal{F} \subseteq 2^{[n]}$ does not contain two sets F_1, F_2 with $F_1 \subset F_2$, then the size of $\mathcal{F}$ cannot exceed $\binom{n}{\lfloor n/2 \rfloor}$. Moreover, the only families attaining this size are $\binom{[n]}{\lfloor n/2 \rfloor}$ and, if n is odd, $\binom{[n]}{\lceil n/2 \rceil}$. This theorem was generalized by Erdős [3] in the following way: if a family $\mathcal{F} \subseteq 2^{[n]}$ does not contain any chain $F_1 \subset F_2 \subset ... \subset F_k \subset F_{k+1}$ of length $k+1$ (families with this property are called *k-Sperner families*), then the size of $\mathcal{F}$ cannot exceed $\sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}$.

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Another topic in extremal finite set theory deals with problems concerning *traces* of set families. The trace of a set F on another set X is $F|_X = F \cap X$, while the trace of a family $\mathcal{F}$ is $\mathcal{F}|_X = \{F|_X : F \in \mathcal{F}\}$. The fundamental theorem about traces is the so-called Sauer-lemma [11, 12, 15] that states that if $\mathcal{F} \subseteq 2^{[n]}$ contains more than $\sum_{i=0}^{l-1} \binom{n}{i}$ sets, then there exists an $L \in \binom{[n]}{l}$ such that $\mathcal{F}|_L = 2^L$. As opposed to the situation described in Erdős's theorem, there are lots of different extremal families (see e.g. [4]). In [10], the present author showed that $\binom{[n]}{\le l-1}$ and $\binom{[n]}{\ge n-l+1}$ are the only families $\mathcal{F}$ of size $\sum_{i=0}^{l-1} \binom{n}{i}$ such that for all $L \in \binom{[n]}{l}$ the trace $\mathcal{F}|_L$ does not contain any chain of length $l+1$ (i.e. maximal chains in 2^L). This result led to the following definition: a family $\mathcal{F}$ is said to be l-trace k-Sperner if for any l-set L the trace $\mathcal{F}|_L$ is k-Sperner. Let $f(n,k,l)$ denote the maximum size that an l-trace k-Sperner family $\mathcal{F} \subseteq 2^{[n]}$ can have. In [10], it was also shown that for any pair of integers k, l there exists $n_0(k,l)$ such that if $n \ge n_0$, then $f(n,k,l) = \sum_{i=0}^{k-1} \binom{n}{i}$.

The situation becomes totally different when one considers the problem of determining $f(n,k,n-l')$ with k, l' fixed and n large enough. Note that if $a \le |A| \le b$ holds, then for any l'-subset L the size of $A|_{[n]\setminus L}$ lies between $a - l'$ and b. Therefore, as a chain contains sets of different sizes, the family $\bigcup_{i=1}^{k-l'} \binom{[n]}{a+i}$ is $(n-l')$-trace k-Sperner for any values of a, k, l' and n. The following conjecture asserts that the largest $(n-l')$-trace k-Sperner family is of this sort if n is large enough.

Conjecture 1.1. *Let k and l' be positive integers with $l' < k$. Then there exists $n_0 = n_0(k,l')$ such that if $n \ge n_0$ and $\mathcal{F} \subseteq 2^{[n]}$ is an $(n-l')$-trace k-Sperner family, then* $|\mathcal{F}| \le \sum_{i=1}^{k-l'} \binom{n}{\lfloor \frac{n-(k-l')}{2} + i \rfloor}$.

Note that if true, the bound in Conjecture 1.1 is best possible as shown by the family $\bigcup_{i=1}^{k-l'} \binom{[n]}{\lfloor \frac{n-(k-l')}{2} + i \rfloor}$. In [10] it was shown that Conjecture 1.1 holds asymptotically when $l' = 1, k = 2$. The main result of this paper verifies Conjecture 1.1 asymptotically for all values of k and l'.

Theorem 1.2. *Let k and l' be positive integers with $l' < k$. Then if $\mathcal{F} \subseteq 2^{[n]}$ is an $(n-l')$-trace k-Sperner family, then* $|\mathcal{F}| \le (k - l' + o(1)) \binom{n}{\lfloor n/2 \rfloor}$.

The rest of the paper is organized as follows: in Section 2, we briefly summarize the problem of forbidden subposets in set families (for recent survey-like papers see [7, 8] and for the most recent results see [5]) and state a result of Bukh [2] that will be used in the proof of Theorem 1.2. In Section 3, we obtain a result about $f(n,l',n-l')$ and another one about the connection of $f(n,l',n-l')$ and $f(n,k,n-l')$. These two results will immediately imply Theorem 1.2. Section 4 contains some concluding remarks and open problems.

2 Families with forbidden subposets

The aim of this section is to describe the context of forbidden subposets, introduce some terminology and to state Theorem 2.2 that will serve as the main tool in proving Theorem 1.2.

We say that a family $\mathcal{F}$ of sets contains a poset P if there is an injective mapping $i : P \to \mathcal{F}$ such that whenever $p \leqslant_P q$ holds, then $i(p)$ is contained in $i(q)$. We say that $\mathcal{F}$ is *P-free* if it does not contain P. For any set $\mathcal{P}$ of posets $La(n, \mathcal{P})$ denotes the maximum size that a family $\mathcal{F} \subseteq 2^{[n]}$ can have such that $\mathcal{F}$ is P-free for all $P \in \mathcal{P}$. If $\mathcal{P}$ consists of a single poset P, we write $La(n, P)$ instead of $La(n, \{P\})$. With this notation Sperner's theorem determines $La(n, P_2)$ and Erdős's theorem determines $La(n, P_{k+1})$, where P_k denotes the poset consisting of a chain of length k. In these theorems, $La(n, P_k)$ is attained at a union of consecutive levels of $2^{[n]}$. It is natural to conjecture that something similar is true for all posets. For a poset P let $l(P)$ denote the largest integer l such that for any n, no l consecutive levels of $2^{[n]}$ contain P. The following conjecture is folklore.

Conjecture 2.1. *Let P be a finite poset. Then $La(n, P) = (l(P) + \frac{1}{n})\binom{n}{\lfloor n/2 \rfloor}$.*

The *Hasse graph* $H(P)$ of a poset P is a directed graph with vertex set P and (p, q) is an arc if and only if $p \prec_P q$ (i.e. $p <_P q$ and there does not exist $r \in P$ with $p <_P r <_P q$). The height $h(P)$ of a poset is the length of the longest chain in P. It is easy to verify that if $H(P)$ is a tree, then $l(P) = h(P) - 1$. Conjecture 2.1 was proved by Bukh for all posets P with $H(P)$ being a tree.

Theorem 2.2 (Bukh, [2])**.** *Let P be a finite poset such that $H(P)$ is a tree. Then $La(n, P) = (h(P) - 1 + O(\frac{1}{n}))\binom{n}{\lfloor n/2 \rfloor}$.*

3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. To be able to use Theorem 2.2, we need to define the following directed graph: $T_{h,c}$ is a tree with height h such that all arcs are directed towards the root and each vertex, with the exception of the leaves, has exactly c children. Let $P_{h,c}$ denote the poset with $H(P_{h,c}) = T_{h,c}$. The following two theorems immediately yield Theorem 1.2.

Theorem 3.1. *Let k, l' be positive integers with $l' < k$. Then the following inequality holds:*

$$f(n, k, n - l') \leq f(n, l', n - l') + La(n, P_{k-l'+1, 2^{l'}}).$$

Theorem 3.2. *For any positive integer l', the size of an $(n - l')$-trace l'-Sperner family $\mathcal{F} \subseteq 2^{[n]}$ is $O_{l'}(n^{-1/3}\binom{n}{\lfloor n/2 \rfloor})$.*

Proof of Theorem 3.1. Let $\mathcal{F} \subseteq 2^{[n]}$ be a set family of size $f(n, l', n-l') + La(n, P_{k-l'+1,2^{l'}}) + 1$. We will find an l'-subset $L \subset [n]$ and a chain of length $k+1$ in $\mathcal{F}|_{[n]\setminus L}$. By the size of $\mathcal{F}$, there exists a copy of $P_{k-l'+1,2^{l'}}$ in $\mathcal{F}$. We remove the set corresponding to the root of $T_{k-l'+1,2^{l'}}$ and repeat this procedure until there exists no more copy of $P_{k-l'+1,2^{l'}}$ in the remaining family. As $|\mathcal{F}| = f(n, l', n-l') + La(n, P_{k-l'+1,2^{l'}}) + 1$, we must have removed at least $f(n, l', n-l') + 1$ sets. Thus, there exists an l'-subset $L \subseteq [n]$ and $l' + 1$ removed sets $F_{k-l'+1}, F_{k-l'+2}, ..., F_k, F_{k+1}$ such that

$$F_{k-l'+1}|_{[n]\setminus L} \subsetneq F_{k-l'+2}|_{[n]\setminus L} \subsetneq ... \subsetneq F_k|_{[n]\setminus L} \subsetneq F_{k+1}|_{[n]\setminus L}$$

holds.

As $F_{k-l'+1}$ is a removed set, there exists a copy of $P_{k-l'+1,2^{l'}}$ such that $F_{k-l'+1}$ corresponds to its largest element. Therefore there are lots of chains of length $k - l'$ in $\mathcal{F}$ such that all of their elements are subsets of $F_{k-l'+1}$. Clearly, if $G \subseteq G'$, then $G|_{[n]\setminus L} \subseteq G'|_{[n]\setminus L}$, but we also require the sets of the chain not to coincide when considering their traces on $[n] - L$. Thus, we need a chain $F_1 \subsetneq F_2 \subsetneq ... \subsetneq F_{k-l'} \subsetneq F_{k-l'+1}$ such that $F_{i+1} \setminus F_i$ is not contained in L for all $i = 1, ..., k - l'$. Suppose we have already picked F_j from the jth level of the copy of $P_{k-l'+1,2^{l'}}$ for all $j = i + 1, ..., k - l' + 1$. Then F_{i+1} has $2^{l'}$ children in $P_{k-l'+1,2^{l'}}$. As for any F of these sets, we have $F_{i+1} \setminus F \neq \emptyset$, and L has $2^{l'} - 1$ non-empty subsets, at least one such F will satisfy $F|_{[n]\setminus L} \subsetneq F_{i+1}|_{[n]\setminus L}$. Letting this F be F_i we continue to define all F_j's and we get a chain of length $k + 1$ in $\mathcal{F}|_{[n]\setminus L}$. This shows that $\mathcal{F}$ cannot be $(n - l')$-trace k-Sperner. □

Proof of Theorem 3.2. Let $\mathcal{F} \subseteq 2^{[n]}$ be an $(n - l')$-trace l'-Sperner family and let $\mathcal{F}_i = \{F \in \mathcal{F} : |F| = i\}$ for all $i = 0, 1, ..., n$. Note that if $\mathcal{H} \subseteq \binom{[n]}{i}$ is $(n - l')$-trace l'-Sperner, then $\mathcal{H}$ does not contain sets $H_1, H_2, ..., H_{l'+1}$ such that for some $x_1, x_2, ..., x_{i+l'} \in [n]$ we have $H_j = \{x_j, x_{j+1}, ..., x_{j+i-1}\}$ for all $j = 1, 2, ..., l' + 1$ (sets satisfying these conditions are often said to form a *tight path of length* $l' + 1$). Indeed, if such sets exist, then the traces of the H_j's form a chain of length $l' + 1$ on the set $[n] \setminus \{x_1, x_2, ..., x_{l'}\}$ provided $i \geq l'$. The result we found in the literature concerning uniform families not containing tight paths of given length [6] is not strong enough for our purposes, thus we prove the following lemma.

Lemma 3.3. *For any positive integer l', if $\mathcal{H} \subseteq \binom{[n]}{i}$ does not contain a tight path of length $l' + 1$, then $|\mathcal{H}| = O_{l'}(\frac{1}{i}\binom{n}{i-1})$ provided $i \geq 2l'$.*

Proof. We proceed by induction on l'. If $l' = 1$, then the above requirement is equivalent to the fact that for any $H, H' \in \mathcal{H}$ the shadows $\{G \subset H : |G| = |H| - 1\}$ and $\{G' \subset H' : |G'| = |H'| - 1\}$ are disjoint. Therefore $|\mathcal{H}| \leq \frac{1}{i}\binom{n}{i-1}$.

Let us assume that we have already proved the existence of a constant $c_{l'}$ such that any family $\mathcal{H} \subseteq \binom{[n]}{i}$ without a tight path of length l' has size at most $\frac{c_{l'}}{i}\binom{n}{i-1}$. Let us define $c_{l'+1} = c_{l'} + 2(l' + 1)$ and consider a family $\mathcal{H} \subseteq \binom{n}{i}$ with $|\mathcal{H}| \geq \frac{c_{l'+1}}{i}\binom{n}{i-1}$. By the induction hypothesis we find a tight path of length l'. Removing the last set of this path we can still find another tight path of length l'. In this way, we find $\frac{c_{l'+1}-c_{l'}}{i}\binom{n}{i-1} = \frac{2(l'+1)}{i}\binom{n}{i-1}$ different sets in $\mathcal{H}$ such that each of them is the last set in a certain tight path of length l'.

Let $\mathcal{H}_1$ denote the subfamily of these sets and consider a set $H \in \mathcal{H}_1$. Let H' denote the first set of (one of) the tight path(s) to which H belongs, i.e. if the vertices of the tight path are $x_1, x_2, ..., x_{i+l'-1}$ and $H = \{x_{l'}, x_{l'+1}, ..., x_{i+l'-1}\}$, then $H' = \{x_1, x_2, ..., x_i\}$. Let the *modified shadow* of H with respect to H' be $\{H \setminus \{x_j\} : l' \leq j \leq i\}$. Clearly, the size of the modified shadow determined by all tight paths is $i - l' + 1 \geq i/2$ by the assumption $i \geq 2l'$. Therefore, there exists an $(i-1)$-set G that belongs to the modified shadows of at least $l'+1$ sets $H^1, H^2, ..., H^{l'+1}$ from $\mathcal{H}_1$.

Let $P_1, P_2, ..., P_{l'} = H^1$ be a tight path of length l' on the vertices $\{y_1, y_2, ..., y_{i+l'-1}\}$ with $\{y_j, y_{j+1}, ..., y_{j+i-1}\} = P_j \in \mathcal{H}$ for all $j = 1, 2, ..., l'$ and let $G = H^1 \setminus \{y_t\}$ for some $l' \leq t \leq i$. As the H^j's are all different containing G and have size i at least one of them, say H^2, is of the form $G \cup \{z\}$ such that $z \notin \{y_1, y_2, ..., y_{l'-1}, y_t\}$. But then the sets $P_1, P_2, ..., P_{l'} = H^1, H^2$ form a tight path of length $l' + 1$ on the vertices $\{y_1, y_2, ..., y_{l'-1}, y_t, y_{l'}, y_{l'+1}, ..., y_{i+l'-1}, z\}$. This finishes the proof of the induction step. □

It is well known that $|\{X \subseteq [n] : ||X| - n/2| \geq n^{2/3}\}| = o(\frac{1}{n}\binom{n}{\lfloor n/2 \rfloor})$. Therefore by Lemma 3.3 we have

$$|\mathcal{F}| = o\left(\frac{1}{n}\binom{n}{\lfloor n/2 \rfloor}\right) + \sum_{i=n/2-n^{2/3}}^{n/2+n^{2/3}} |\mathcal{F}_i| = 2n^{2/3} O_{l'}\left(\frac{1}{n}\binom{n}{\lfloor n/2 \rfloor}\right) = O_{l'}\left(n^{-1/3}\binom{n}{\lfloor n/2 \rfloor}\right).$$

□

4 Concluding remarks

Let us first remark that we do not need the full strength of Bukh's theorem. An almost identical proof to that of Theorem 3.1 shows that the inequality $f(n, k+1, n-l') \leq f(n, k, n-l') + La(n, P_{2,l'})$ holds. Thanh showed $La(n, P_{2,l'}) = (1 + O_{l'}(\frac{1}{n}))\binom{n}{\lfloor n/2 \rfloor}$ in an earlier paper [14] and with a much easier proof than that of Theorem 2.2. (Later, De Bonis and Katona improved the error term [1].) However, as it is very rare that the extremal family for a forbidden subposet problem consists only of full levels, it seems unlikely that Conjecture 1.1 could be proved using only results from that area.

Theorem 1.2 and Conjecture 1.1 do not consider the case $k \leq l'$. In [10] it was proved that $f(n, 1, n - l') = \Theta_{l'}(\frac{1}{n^{l'}}\binom{n}{\lfloor n/2 \rfloor})$. Theorem 3.2 states that $f(n, l', n - l') = O_{l'}(\frac{1}{n^{1/3}}\binom{n}{\lfloor n/2 \rfloor})$ and it is natural to conjecture that bound $O_{l'}(\frac{1}{n}\binom{n}{\lfloor n/2 \rfloor})$ holds in general, not only for uniform families as proved by Lemma 3.3. We would like to propose the following conjecture that, if true, would generalize all results and conjectures above.

Conjecture 4.1. *For any pair of integers $k \leq l'$, the following holds*

$$f(n, k, n - l') = \Theta_{k,l'} \left(\frac{1}{n^{l'-k+1}} \binom{n}{\lfloor n/2 \rfloor} \right).$$

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