l-trace k-Sperner families of sets

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Abstract

A family of sets $\mathcal{F} \subseteq 2^X$ is defined to be l-trace k-Sperner if for any subset Y of X with size l the trace of $\mathcal{F}$ on Y (the restriction of $\mathcal{F}$ to Y) does not contain any chain of length $k+1$. In this paper we investigate the maximum size that an l-trace k-Sperner family (with underlying set $[n] = \{1, 2, ..., n\}$) can have for various values of k, l and n.

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1 Introduction

The trace of a set F on another set X is $F \cap X$ and is denoted by $F|_X$. The trace of a family $\mathcal{F}$ of sets is just the family of traces, i.e. $\mathcal{F}|_X = \{F|_X : F \in \mathcal{F}\}$. The fundamental result concerning traces of families was proved in the early 1970s independently by Sauer [12], Shelah [13] and Vapnik and Chernovenkis [15].

In order to state this result and some others that we will use in the proofs of the present paper, we have to introduce some notation and definitions:

Notation: $[n]$ denotes the set of the first n integers $\{1, 2, ..., n\}$. The power set of a set X is denoted by 2^X. The complement of a set $F \subseteq [n]$ is written $\overline{F}$. The system consisting of all subsets of X of size k (all *k-subsets* for short) is denoted by $\binom{X}{k}$ and will be sometimes referred as the k^{th} *level*. We define $\binom{X}{\le k}$ and $\binom{X}{\ge k}$ similarly. Given $\mathcal{F} \subseteq 2^{[n]}$ and $Y \subseteq [n]$ we write $Y + \mathcal{F}$ for $\{Y \cup F : F \in \mathcal{F}\}$.

Definition: A set system $\mathcal{F} \subseteq 2^{[n]}$ *traces* a set $X \subseteq [n]$ if for any subset Y of X there exists $F \in \mathcal{F}$ such that $F|_X = Y$.
The set system $\mathcal{F} \subseteq 2^{[n]}$ *strongly traces* a set $X \subseteq [n]$ if there exists a set $B \subseteq \overline{X}$ such that for any subset Y of X, we have $B \cup Y \in \mathcal{F}$. This B is called the *support* of X by $\mathcal{F}$. (Note that the support is not necessarily unique!) The set of supports of X is denoted by $\mathbf{S}_{\mathcal{F}}(X)$ and we will denote an element of $\mathbf{S}_{\mathcal{F}}(X)$ by $S_{\mathcal{F}}(X)$.

We use the following notations:

$$tr(\mathcal{F}) = \{X : \mathcal{F} \text{ traces } X\} \qquad str(\mathcal{F}) = \{X : \mathcal{F} \text{ strongly traces } X\}.$$

With these definitions and notations we are able to state the above-mentioned result. This formulation is due to Pajor [11].

Theorem A [12], [13], [15] *For any set system $\mathcal{F} \subseteq 2^{[n]}$ we have*

$$|tr(\mathcal{F})| \ge |\mathcal{F}|.$$

In particular, if $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$, then $\mathcal{F}$ traces a subset X of $[n]$ with $|X| = k$ (and this is sharp as $\binom{[n]}{\le k-1}$ shows).

The following results will turn out to be very useful in our proofs.

Theorem B [3] *For any set system $\mathcal{F} \subseteq 2^{[n]}$ we have*

$$|str(\mathcal{F})| \le |\mathcal{F}|.$$

Theorem C [4] *For any set system $\mathcal{F} \subseteq 2^{[n]}$ the following two properties are equivalent*

(a) $|\mathcal{F}| = |tr(\mathcal{F})|$,
(b) $|\mathcal{F}| = |str(\mathcal{F})|$.

Theorem A leads in several directions (besides the direction of Theorem B and C); a very good, but not very recent survey is the paper of Füredi and Pach [6]. We will focus on the direction that seems somewhat similar to the area known as Turán type problems (in such problems, given a 'small' family of sets $\mathcal{F}$ we are interested in the maximum size that a 'big' family $\mathcal{H} \subseteq 2^{[n]}$ can have that does not contain any copy of $\mathcal{F}$ as a subfamily). In this paper we will be interested in the maximum size that a family $\mathcal{H} \subseteq 2^{[n]}$ can have that does not contain any copy of $\mathcal{F}$ *as trace*. This type of results can be found in (among others) [7], [1], [2]. Theorem A can be interpreted in this context as well: the forbidden configuration should be $2^{[k]}$.

The most important definition for the present paper is the following:

Definition: A family $\mathcal{F} \subseteq 2^X$ of sets is said to be *l-trace k-Sperner* if for any subset Y of X with size l the trace of $\mathcal{F}$ on Y does not contain any chain of length $k+1$ (a chain of length $k+1$ is a family of $k+1$ sets $C_1, C_2, ..., C_{k+1}$ with $C_1 \subset C_2 \subset ... \subset C_{k+1}$).

The l-trace k-Sperner property can be formalized through forbidden traces, too. One has to exclude $\binom{l+1}{k+1}$ families as trace (all possibilities how we can choose $k+1$ levels out of the $l+1$ that $2^{[l]}$ possesses).

We will be interested in the function $f(n,k,l)$ which stands for the maximum size that an l-trace k-Sperner family $\mathcal{F} \subseteq 2^{[n]}$ can have. The rest of the paper is organized as follows. In Section 2, we consider the case $k = l$. In Section 3, we address the problem when n is large compared to both k and l. In Section 4 we prove some results about the $k = 1$ case. In Section 5 we address the problem when $l = n-1$, while in Section 6 we make some concluding remarks and gather some of the open problems.

2 The case $k = l$

In this section we consider the case $k = l$. The value of $f(n,k,k)$ is a trivial consequence of Theorem A. The really interesting part of the main theorem of this section states that forbidding the existence of a full chain as trace is strong enough to ensure the uniqueness of the optimal families $\binom{[n]}{\leq k-1}$ and $\binom{[n]}{\geq n-k+1}$. Note that there is no uniqueness when the forbidden configuration is the whole power set $2^{[k]}$ as shown by (among others) the examples of [7].

Theorem 1
(a) $f(n,k,k) = \sum_{i=0}^{k-1} \binom{n}{i}$.
(b) *If* $\mathcal{F} \subseteq 2^{[n]}$ *is k-trace k-Sperner with* $|\mathcal{F}| = \sum_{i=0}^{k-1} \binom{n}{i}$, *then either* $\mathcal{F} = \binom{[n]}{\leq k-1}$ *or* $\mathcal{F} = \binom{[n]}{\geq n-k+1}$.

Proof: The statement about $f(n,k,k)$ is straightforward from Theorem A.

To prove **(b)**, let us consider a k-trace k-Sperner set system $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| = \sum_{i=0}^{k-1} \binom{n}{i}$. By Theorem A we have $|tr(\mathcal{F})| \geq \sum_{i=0}^{k-1} \binom{n}{i}$. But if $\mathcal{F}$ traces a k-subset of $[n]$, then it is not k-trace k-Sperner, so $tr(\mathcal{F}) = \binom{[n]}{\leq k-1}$ and in particular, $|\mathcal{F}| = |tr(\mathcal{F})|$. From Theorem C, it follows that $|\mathcal{F}| = |str(\mathcal{F})|$, and using the k-trace k-Sperner property of $\mathcal{F}$, we have that $str(\mathcal{F}) = tr(\mathcal{F}) = \binom{[n]}{\leq k-1}$.

Now let us consider a set $F \in \mathcal{F}$ with minimum size. If $|F| > n - k + 1$, then $|\mathcal{F}| < f(n, k, k)$ - a contradiction. Therefore $|F| \leq n-k+1$, so there exists $X \subseteq [n] \setminus F$ with $|X| = k - 1$. By the paragraph above, we have $X \in str(\mathcal{F})$. Let us take an arbitrary $S(X) \in \mathbf{S}_X$. We claim that there is no element $s \in S(X) \setminus F$. Indeed, if there is, then let us consider $\mathcal{F}|_{X \cup \{s\}}$. Since $F \in \mathcal{F}$ and $s \notin F$, we have $\emptyset \in \mathcal{F}|_{X \cup \{s\}}$. Since $X \in str(\mathcal{F})$ and $s \in S(X)$, there is a chain in $\mathcal{F}|_{X \cup \{s\}}$ of length k with set sizes $1, 2, ..., k$, which together with the empty set form a chain of length $k+1$ - a contradiction. Thus $S(X) \subseteq F$, but since by the definition of support, $S(X) \cup \emptyset = S(X) \in \mathcal{F}$ and F is of minimum size, we must have $S(X) = F$ and thus $F + 2^X \subseteq \mathcal{F}$. As X was chosen arbitrarily, we obtain that for any Y with $Y \cap F = \emptyset$ and $|Y| = k - 1$, we have $F + 2^Y \subseteq \mathcal{F}$.

We claim that for any such Y, the set $F \cup Y$ is maximal in $\mathcal{F}$. Indeed, if not, then $F \cup Y \cup A \in \mathcal{F}$ for some non empty A. Therefore, for some $a \in A$, the trace $\mathcal{F}|_{Y \cup \{a\}}$ contains a chain of length $k+1$ (the trace of $F \cup Y \cup A$ is $Y \cup \{a\}$ and from the trace of $F + 2^Y$ we can pick the other k sets) - a contradiction.

In the argument proving that $F + 2^Y \subseteq \mathcal{F}$, we needed only the minimality of F (and not the fact that F is of minimum size), and with an almost identical argument we get that for any $Y' \subseteq F \cup Y$ with $|Y'| = k - 1$ we have $F \cup Y - 2^{Y'} \subseteq \mathcal{F}$ (and $F \cup Y \setminus Y'$ is minimal in $\mathcal{F}$). Repeating this for several times, we get that for any $G \subseteq [n]$ with $|F| \leq |G| \leq |F| + k - 1$, we have $G \in \mathcal{F}$. But because of **(a)**, it is possible if and only if $\mathcal{F} = \binom{[n]}{\leq k-1}$ or $\mathcal{F} = \binom{[n]}{\geq n-k+1}$. □

3 The case of fixed k and l

In this section we consider cases when n is large compared to both k and l. Our main result is the following theorem.

Theorem 2 *For every pair of integers k and l ($1 \leq k \leq l$) there exist $N(k,l)$ such that if $n \geq N(k,l)$, then $f(n,k,l) = \sum_{i=0}^{k-1} \binom{n}{i}$. Furthermore, if $2 \leq k \leq l$, then the only optimal l-trace k-Sperner families are $\binom{[n]}{\leq k-1}$ and $\binom{[n]}{\geq n-k+1}$.*

Proof: If $k = 1$, then $N(1,l) = 2l - 1$ is a good choice. Indeed, if $A \subset B \subseteq [n]$, then picking any l-subset L of $[n]$ which contains an element from $B \setminus A$ and considering the L-trace would yield to a contradiction. If none of the sets contains the other, then (since $n \geq 2l - 1$) either $\overline{B} \cup (A \cap B)$ or $\overline{A} \cup (B \cap A)$ is of size at least l, so we can find an l-subset, where the traces of the sets are in inclusion. (In fact, $2l - 1$ is sharp as shown by any pair $A, \overline{A} \subset [2l - 2]$, $|A| = l - 1$.)

Since in the case $k = 1$ there is no uniqueness for the extremal family, we still have to establish the base case $k = 2$, but as this case and the inductive step is very similar, we describe them simultanously. Suppose that for some fixed k and l, we have already proved the statement of the theorem for every k', l' with $k' \leq k$, $l' \leq l$ and with at least one of k' and l' strictly smaller then k or l. Let M denote the maximum of $N(k', l')$, where k', l' are as above and put $N = M + k + \sum_{i=0}^{l-1} \binom{M+k}{i}$. We will prove that the statement about $f(n, k, l)$ is true if $n \geq N$, and the statement about the optimal families holds provided $n \geq N + 1$.

Before we proceed to the actual proof, we need to introduce some notation. For any family $\mathcal{F} \subseteq 2^X$ and $x \in X$, we put $\mathcal{F}_x^0 := \{F \in \mathcal{F} : x \notin F, F \cup \{x\} \in \mathcal{F}\}$, $\mathcal{F}_x^1 := \{F \in \mathcal{F} : x \in F, F \setminus \{x\} \in \mathcal{F}\}$ and $\mathcal{F}_{\overline{x}} := \mathcal{F} \setminus (\mathcal{F}_x^0 \cup \mathcal{F}_x^1)$. Trivially $|\mathcal{F}_x^0| = |\mathcal{F}_x^1|$, $|\mathcal{F}| = |\mathcal{F}_x^0| + |\mathcal{F}_x^1| + |\mathcal{F}_{\overline{x}}|$ and $|\mathcal{F}|_{X \setminus \{x\}}| = |\mathcal{F}_x^0| + |\mathcal{F}_{\overline{x}}|$.

Lemma 3 *If $\mathcal{F}$ is l-trace k-Sperner on the underlying set X, then for any $x \in X$, $\mathcal{F}_x^0$ is $(l-1)$-trace $(k-1)$-Sperner on $X \setminus \{x\}$.*

Proof of Lemma: Suppose not. Then there exist an $l - 1$-set $L' \subseteq X \setminus \{x\}$ and $F_1, F_2, ..., F_k \in \mathcal{F}_x^0$ such that $F_1|_{L'} \subset F_2|_{L'} \subset ... \subset F_k|_{L'}$. But then, putting $F_{k+1} = F_k \cup \{x\} \in \mathcal{F}$ and $L = L' \cup \{x\}$, we would have $F_1|_L \subset F_2|_L \subset ... \subset F_k|_L \subset F_{k+1}|_L$ - a contradiction. $\square$

Suppose there exists an l-trace k-Sperner family $\mathcal{F} \subseteq 2^{[n]}$ $(n \geq N)$ with $|\mathcal{F}| = \sum_{i=0}^{k-1} \binom{n}{i} + C$ (where C is positive). We claim that there is a subset $X \subseteq [n]$ with $|X| \geq M + k$ such that for any element $x \in X$ we have $|\mathcal{F}|_X| \geq \sum_{i=0}^{k-1} \binom{|X|}{i} + C$ and $|(\mathcal{F}|_X)_x^0| = \sum_{i=0}^{k-2} \binom{|X|-1}{i}$.

We know that for any $x \in X \subseteq [n]$ with $|X| \geq M + 1$ we have $|(\mathcal{F}|_X)_x^0| \leq \sum_{i=0}^{k-2} \binom{|X|-1}{i}$, because of the lemma above and the inductive hypothesis on $f(n, k-1, l-1)$. Therefore if $|(\mathcal{F}|_X)_x^0| \neq \sum_{i=0}^{k-2} \binom{|X|-1}{i}$ then we must have $|(\mathcal{F}|_X)_x^0| < \sum_{i=0}^{k-2} \binom{|X|-1}{i}$ and consequently $|\mathcal{F}|_{X \setminus \{x\}}| \geq \sum_{i=0}^{k-1} \binom{|X|-1}{i} + C + 1$. If $X = [n]$ is not a good choice for our claim, then there is an $x_1 \in [n]$ which shows this fact. If $X = [n] \setminus \{x_1\}$ is not good either, then some $x_2 \in [n] \setminus \{x_1\}$ shows this and we have that $|\mathcal{F}|_{[n] \setminus \{x_1, x_2\}}| \geq \sum_{i=0}^{k-1} \binom{n-2}{i} + C + 2$. Continuing in this way we get that if there is no good set, then there is a subset $Y \subset [n]$ with $|Y| = M + k$ such that we have $|\mathcal{F}|_Y| > \sum_{i=0}^{l-1} \binom{M+k}{i}$. But then, by Theorem A, $\mathcal{F}|_Y$ (and so $\mathcal{F}$ as well) traces a set of size l contradicting the l-trace k-Sperner property.

So we established that for some $X \subseteq [n]$ with $|X| \geq M + k$ and any of its elements $x \in X$ we have $|\mathcal{F}|_X| \geq \sum_{i=0}^{k-1} \binom{|X|}{i} + C$ and $|(\mathcal{F}|_X)_x^0| = \sum_{i=0}^{k-2} \binom{|X|-1}{i}$. If $(\mathcal{F}|_X)_x^0 = \binom{X \setminus \{x\}}{\leq k-2}$ or $(\mathcal{F}|_X)_x^0 = \binom{X \setminus \{x\}}{\geq |X|-k+1}$, then $\mathcal{F}|_X$ contains $\binom{X}{\leq k-1}$ or $\binom{X}{\geq |X|-k+1}$ and at least one additional set which contradicts the l-trace k-Sperner property. Why is it true that $(\mathcal{F}|_X)_x^0) = \binom{X \setminus \{x\}}{\leq k-2}$ or $(\mathcal{F}|_X)_x^0 = \binom{X \setminus \{x\}}{\geq |X|-k+1}$? If $k \geq 3$, this is simply the inductive hypothesis for the uniqueness of the extremal systems. If $k = 2$ we need to work a bit more.

In this case, what we have already proved is that for the above set X and for any $x \in X$ we have $(\mathcal{F}|_X)^0_x \neq \emptyset$, i.e. the singleton $\{x\}$ is strongly traced by $\mathcal{F}|_X$. Since $\mathcal{F}|_X$ is l-trace k-Sperner, we need the following lemma.

Lemma 4 *If for some l with $2l \leq n$, the family $\mathcal{F} \subseteq 2^{[n]}$ is l-trace 2-Sperner and $\mathcal{F}$ strongly traces all singletons, then $\mathcal{F} = \binom{[n]}{\leq 1}$ or $\mathcal{F} = \binom{[n]}{\geq n-1}$.*

Proof: If for all singletons, one of the supports with respect to $\mathcal{F}$ is the empty set, then $\mathcal{F} \supseteq \binom{[n]}{\leq k-1}$, and since $\binom{[n]}{\leq k-1}$ is maximal l-trace k-Sperner, we must have $\mathcal{F} = \binom{[n]}{\leq k-1}$. Likewise, if for all singletons, one of the supports with respect to $\mathcal{F}$ is the complement set, then $\mathcal{F} = \binom{[n]}{\geq n-k+1}$.

If there exist $x, y \in [n]$, such that a support of $\{x\}$ is the empty set and a support of $\{y\}$ is $[n] \setminus \{y\}$, then $\mathcal{F}$ is not l-trace k-Sperner. Indeed, since the support of $\{y\}$ is $[n] \setminus \{y\}$, we have $[n], ([n] \setminus \{y\}) \in \mathcal{F}$ and since $\emptyset \in \mathbf{S}(\{x\})$ we have $\emptyset \in \mathcal{F}$. But then for any l-subset L containing y the sets $\emptyset|_L, ([n] \setminus \{y\})|_L, [n]|_L$ form a 3-chain.

So we may assume, that there is a singleton x such that all of its support with respect to $\mathcal{F}$ is not empty and not $[n] \setminus \{x\}$. Let us pick x such that (one of) its support $S(x) \in \mathbf{S}(x)$ is of minimum size.

Claim I *For any singleton $x' \subset [n] \setminus S(x)$ we have $S(x) \in \mathbf{S}(x')$.*

Proof of Claim: Let us consider an arbitrary $S(x') \in \mathbf{S}(x')$ and suppose there is an element $s \notin S(x)$ belonging to $S(x')$. Let us put $L = \{x'\} \cup \{s\} \cup L'$, where $L' \subseteq ([n] \setminus S(x)) \cup S(x')$ with $|L'| = l - 2$ (the existence of such a set follows from the assumption $2l \leq n$ and the minimality of $S(x)$). But then $\mathcal{F}|_L$ would contain a chain of length 3 as shown by $(S(x') \cup \{x'\})|_L$, $S(x')|_L$ and $S(x)|_L$. We get that $S(x') \subseteq S(x)$, so by the minimality of $S(x)$, we have $S(x') = S(x)$. $\square$

Claim II *For every $y \in S(x)$ and $S(y) \in \mathbf{S}(y)$, we have $|S(x)| = |S(y)|$.*

Proof of Claim: If $S(y)$ contained two elements $x_1, x_2 \notin S(x)$, then putting $L = \{x_1, x_2\} \cup L'$, where $L' \subseteq ([n] \setminus S(x_1)) \cup S(y)$ with $|L'| = l-2$ (the existence of such L' follows from the assumption $2l \leq n$ and the minimality of $S(x) = S(x_1)$, which holds by the previous claim), $\mathcal{F}|_L$ would contain the 3-chain: $S(x_1)|_L \subset S(x) \cup \{x_1\}|_L \subset S(y)|_L$. $\square$

Because of Claim II, Claim I could be applied to y and an arbitrary $x' \notin S(x) \cup S(y)$ (note, that by the above, $|S(x) \cup S(y)| = S(x) + 1$, so there is such x'), giving $S(x) = S(x') = S(y)$ - a contradiction as $y \in S(x), y \notin S(y)$.

We obtained that the support of any singleton is either the empty set or the complement of the singleton, so the proof the lemma is complete by the paragraph preceeding the claims. $\square$

We still have to show, that if $n \geq N + 1$, then the only optimal families are $\binom{[n]}{\leq k-1}$ and $\binom{[n]}{\geq n-k+1}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an l-trace k-Sperner family with $n \geq N + 1$. If for any

$x \in [n]$, we had $|\mathcal{F}^0_x| < \sum_{i=0}^{k-2}\binom{n-1}{i}$, then $|\mathcal{F}|_{[n]\setminus\{x\}}| > \sum_{i=0}^{k-1}\binom{n-1}{i}$ would hold, but this cannot happen, since $n-1 \geq N$ and we have already proved that for any $n' \geq N$ we have $f(n',k,l) = \sum_{i=0}^{k-1}\binom{n'}{i}$. So if $k=2$ we can apply Lemma 4 to obtain that $\mathcal{F}$ is either $\binom{[n]}{\leq k-1}$ or $\binom{[n]}{\geq n-k+1}$, while if $k>2$ by the induction hypothesis for any $x \in [n]$ we have that $\mathcal{F}^0_x$ is $\binom{[n]\setminus\{x\}}{\leq k-2}$ or $\binom{[n]\setminus\{x\}}{\geq n-k+2}$. □

We finish this section with mentioning that Lemma 4 holds for $k>2$ as well, but as the proof is very similar (although not identical) to the case $k=2$, we omit its proof.

Lemma 5 *If for some $2 \leq k \leq l$, $2l \leq n$, the family $\mathcal{F} \subseteq 2^{[n]}$ is l-trace k-Sperner and $\mathcal{F}$ strongly traces all $G \subseteq [n]$, $|G| = k-1$, then $\mathcal{F} = \binom{[n]}{\leq k-1}$ or $\mathcal{F} = \binom{[n]}{\geq n-k+1}$.*

4 The case $k=1$

In this section we consider the case $k=1$. It will be convenient to write the parameter l in the form $l = n - l'$. With this notation we have the following observation:

Lemma 6 *A family $\mathcal{F} \subseteq 2^{[n]}$ is l-trace Sperner if and only if for any $F, G \in \mathcal{F}$ we have $|F \setminus G| > l'$.*

Proof: Let $\mathcal{F}$ be an l-trace Sperner family. Therefore $G \not\subseteq F$ for all pairs of sets $F, G \in \mathcal{F}$. Suppose we have $|F \setminus G| \leq l'$ for at least one pair of sets. Then $|[n] \setminus (F \setminus G)| \geq n - l' = l$ holds. Let us pick $x \in G \setminus F$ and $L \subseteq F \setminus G$ with $x \notin L, |L| = l-1$. Then $F|_{\{x\}\cup L} \subset G|_{\{x\}\cup L}$ - a contradiction.

Now suppose that the family $\mathcal{F}$ is such that for any $F, G \in \mathcal{F}$ we have $|F \setminus G| > l'$ and (contradicting the Lemma) does not satisfy the l-trace Sperner property, i.e. there are sets $F, G \in \mathcal{F}$ and a subset $L \subseteq [n]$ with $|L| = l$ such that $F|_L \subset G|_L$. But then $(F \setminus G) \subseteq [n] \setminus L$ and therefore $|F \setminus G| \leq n - l = l'$ -a contradiction. □

From Lemma 6 it follows that for any two sets F, G in an l-trace Sperner family $\mathcal{F}$, the set of l'-shadows $\Delta^{l'}(F) = \{F' : F' \subset F, |F'| = |F| - l'\}$, $\Delta^{l'}(G) = \{G' : G' \subset G, |G'| = |G| - l'\}$ are disjoint and the l'-shadow of $\mathcal{F}$ ($\Delta^{l'}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} \Delta^{l'}(F)$) is an antichain. So we can apply the famous LYM-inequality [9], [16], [10] to $\Delta^{l'}(\mathcal{F})$, which gives

$$\sum_{F\in\mathcal{F}} \frac{\binom{|F|}{|F|-l'}}{\binom{n}{|F|-l'}} \leq 1.$$

The same argument (using Lemma 6) can be applied to the l'-shade of $\mathcal{F}$ ($\nabla^{l'}(F) = \{F' : F \subset F', |F'| = |F| + l'\}$ and $\nabla^{l'}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} \nabla^{l'}(F)$). This gives

$$\sum_{F\in\mathcal{F}} \frac{\binom{n-|F|}{l'}}{\binom{n}{|F|+l'}} \leq 1.$$

Summing the two inequalities above, we obtain the following theorem.

Theorem 7 *For any l-trace Sperner family $\mathcal{F} \subseteq 2^{[n]}$ the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \frac{\binom{|F|}{|F|-l'}}{\binom{n}{|F|-l'}} + \frac{\binom{n-|F|}{l'}}{\binom{n}{|F|+l'}} \leq 2.$$

Since $h(m) := \frac{\binom{m}{m-l'}}{\binom{n}{m-l'}} + \frac{\binom{n-m}{l'}}{\binom{n}{m+l'}} = \frac{1}{l'!n!}[m!(n-m+l')! + (n-m)!(m+l')!]$ it is easy to see that $h(m) \leq h(m+1)$ if and only if $m \leq \frac{n-1}{2}$, so $\frac{\binom{|F|}{|F|-l'}}{\binom{n}{|F|-l'}} + \frac{\binom{n-|F|}{l'}}{\binom{n}{|F|+l'}}$ is minimized when $|F| = \lfloor n/2 \rfloor$. Thus we get have:

Corollary:

$$f(n,1,l) \leq \frac{2}{\frac{\binom{\lfloor n/2 \rfloor}{l'}}{\binom{n}{\lfloor n/2 \rfloor - l'}} + \frac{\binom{\lceil n/2 \rceil}{l'}}{\binom{n}{\lfloor n/2 \rfloor + l'}}}$$

in particular, if n is even, then

$$f(n,1,l) \leq \frac{\binom{n}{n/2-l'}}{\binom{n/2}{l'}}.$$

To show that this upper bound is tight (or gives the right order of magnitude), we need a construction. The following construction is well-known but gives only the right order of magnitude.

For the sake of simplicity let $l' = 1$. Then for any $m \in [n]$ the family $\binom{[n]}{\lfloor n/2 \rfloor} \supset \mathcal{F}_m = \{F : F \in \binom{[n]}{\lfloor n/2 \rfloor}, \sum_{i \in F} i \equiv m \pmod{n}\}$ is clearly $(n-1)$-trace Sperner. So for at least one $m \in [n]$ we have $\mathcal{F}_m \geq \frac{\binom{n}{\lfloor n/2 \rfloor}}{n}$ which is half as large as the upper bound given by the corollary.

For larger but fixed l' (while n tends to infinity), one can construct families with the same order of magnitude as given by the upper bound (though the constants become worse as l' grows), in a very similar way using the elementary symmetric polynomials and the fact that prime numbers (and thus prime powers) are dense among integers.

5 The case $l = n-1$

In this section, we consider the case $l = n-1$. For $k \geq 2$, there is a natural construction that we conjecture to be optimal (at least when n is large enough): the $k-1$ largest levels (or any $k-1$ consecutive levels) form an $(n-1)$-trace k-Sperner family, since the traces are from k consecutive levels. The aim of this section is to prove that if $k = 2$, then this construction is asymptotically optimal.

Theorem 8

$$f(n, 2, n-1) \leq \left(1 + \frac{8}{n} + o\left(\frac{1}{n}\right)\right) \binom{n}{\lfloor n/2 \rfloor}.$$

Proof: Let $\mathcal{F}$ be an $(n-1)$-trace 2-Sperner family. Let us divide $\mathcal{F}$ into two: $\mathcal{F} = \mathcal{U} \cup \mathcal{D}$, where $\mathcal{U} = \{F \in \mathcal{F} : \exists G \in \mathcal{F} \text{ such that } G \subset F\}$ and $\mathcal{D} = \mathcal{F} \setminus \mathcal{U}$. (Note that since $\mathcal{F}$ cannot contain a 3-chain, the set G in the definition of $\mathcal{U}$ is from $\mathcal{D}$.) Furthermore we divide $\mathcal{U}$ into four parts: $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$, where

$$\mathcal{U}_0 = \{U \in \mathcal{U} : \forall U' \in \mathcal{U} : \quad |U \setminus U'| \geq 2, |U' \setminus U| \geq 2\},$$

$$\mathcal{U}_1 = \{U \in \mathcal{U} : \exists U' \in \mathcal{U} : \quad |U| = |U'|, |U \setminus U'| = 1\},$$

$$\mathcal{U}_2 = \{U \in \mathcal{U} \setminus \mathcal{U}_1 : \exists U' \in \mathcal{U} : \quad |U| < |U'|, |U \setminus U'| = 1\},$$

$$\mathcal{U}_3 = \{U \in \mathcal{U} \setminus \mathcal{U}_1 : \exists U' \in \mathcal{U} : \quad |U| > |U'|, |U' \setminus U| = 1\}.$$

By the results of the previous section (Lemma 6 and Theorem 7), we have $|\mathcal{U}_0| \leq \left(\frac{2}{n} + o\left(\frac{1}{n}\right)\right) \binom{n}{\lfloor n/2 \rfloor}$, so we may assume from now on that $\mathcal{U}_0$ is empty (and then we have to prove, that $|\mathcal{F}| \leq \left(1 + \frac{6}{n} + o\left(\frac{1}{n}\right)\right) \binom{n}{\lfloor n/2 \rfloor}$.

The following easy observation will be used frequently during the proof:

Lemma 9 *If $U \in \mathcal{U}_1 \cup \mathcal{U}_2$, then there is exactly 1 set $D \in \mathcal{D}$ with $D \subset U$, furthermore $|D| = |U| - 1$.*

Proof of Lemma: Let $U \in \mathcal{U}_1 \cup \mathcal{U}_2$. Then by definition there exists a set $U' \in \mathcal{U}$ with $|U'| \geq |U|$ and $|U \setminus U'| = 1$. Let u be the single element of $U \setminus U'$. Since $U \in \mathcal{U}$, we know that there is a $D \in \mathcal{D}$ with $D \subset U$. If $D \neq U \cap U' = U \setminus \{u\}$, then $D|_{[n]\setminus\{u\}} \subset U|_{[n]\setminus\{u\}} \subset U'|_{[n]\setminus\{u\}}$ contradicting the fact that $\mathcal{F}$ is $(n-1)$-trace 2-Sperner. $\square$

Lemma 10 *For every $U_1, U_2 \in \mathcal{U}_3$ we have $|U_1 \setminus U_2| \geq 2$ and $|U_2 \setminus U_1| \geq 2$.*

Proof of Lemma: Suppose to the contrary that there are sets U_1, U_2 with $|U_1 \setminus U_2| = 1$. Since by definition $U_1, U_2 \notin \mathcal{U}_1$, we have $|U_1| < |U_2|$. Therefore we have $U_1 \in \mathcal{U}_2$, and so by Lemma 8, $D = U_1 \cap U_2$ is the unique set in $\mathcal{D}$ with $D \subset U_1$. Since by assumption of the Lemma, $U_1 \in \mathcal{U}_3$, there is a set $U_3 \in \mathcal{U}$ with $|U_3 \setminus U_1| = 1$. Thus we have $U_3 \in \mathcal{U}_1 \cup \mathcal{U}_2$ and applying Lemma 8 again, we obtain that $D' = U_1 \cap U_3$ is the only set in $\mathcal{D}$ contained in U_3. But by definition $D' \subset U_1$ and $D' \neq D$ (they are not of the same size) - a contradiction. $\square$

With an almost identical proof one can obtain the following statement.

Lemma 11: *For every $U_1, U_2 \in \mathcal{U}_2$ we have $|U_1 \setminus U_2| \geq 2$ and $|U_2 \setminus U_1| \geq 2$.* $\square$

Lemma 10 and 11 together with Lemma 6 and Theorem 7 give that both $|\mathcal{U}_2|, |\mathcal{U}_3|$ have size at most $\left(\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2\rfloor}$, so just as with $\mathcal{U}_0$ we may suppose that $\mathcal{U}_2$ and $\mathcal{U}_3$ are empty (and we have to show that the size of the remaining family is at most $\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2\rfloor}$).

In order to prove that the inequality $|\mathcal{D}\cup\mathcal{U}_1| \leq \left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2\rfloor}$ holds, let us consider $\mathcal{D}\cup\mathcal{U}_1$ as a subposet of the Boolean poset, i.e for $F, G \in 2^{[n]}$ we have $F \leq G \Leftrightarrow F \subseteq G$. A poset P is said to be *connected* if for any $p_1, p_2 \in P$ there is a sequence $r_1, r_2, ..., r_k$ such that $r_i < r_{i+1}$ or $r_i > r_{i+1}$ for every $i = 1, 2, ..., k-1$ and $p_1 < r_1$ or $p_1 > r_1$ and $r_k < p_2$ or $r_k > p_2$. Maximal connected subposets of a poset are called the *connected components* of the poset.

How do the connected components of $\mathcal{D}\cup\mathcal{U}_1$ look like? Clearly, components that do not contain sets from $\mathcal{U}_1$ consist of a single set from $\mathcal{D}$. We claim, that components with sets from $\mathcal{U}_1$ are r-*forks* for some $r \geq 2$, i.e. each consist of sets $F, G_1, G_2, ..., G_3$ such that $F \subset G_i$, $i = 1, 2, ..., r$ and there is no containment between the G_is. Indeed, by Lemma 9 we know that sets from $\mathcal{U}_1$ contain exactly one set from $\mathcal{D}$ (and no sets from $\mathcal{U}_1$ since there are no 3-chains in the family), so sets from $\mathcal{U}_1$ can 'get connected' only through this one set. As a consequence we have that the poset $\mathcal{D}\cup\mathcal{U}_1$ cannot have four sets A, B, C, D with $A \subset B$, $C \subset B$, $C \subset D$. A theorem of Griggs and Katona [8] states that families with this property are of size at most $\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2\rfloor}$. This proves the theorem. □

6 Concluding remarks and open problems

There are lots of values of k, l and n for which $f(n, k, l)$ is yet to be determined. We enumerate here some of them and some related questions:

- Theorem 2 states that for every k and l there exists a number $N(k, l)$ such that if $n \geq N(k, l)$, then $f(n, k, l) = \sum_{i=0}^{k-1}\binom{n}{i}$. It would be interesting to determine the smallest such $N(k, l)$. Theorem 1 states that $N(k, k) = k$, and at the beginning of the proof of Theorem 2, it is shown that $N(1, l) = 2l - 1$. Furthermore it is not difficult to prove (though we omit the details) that $N(2, l) \leq 6l$.

- In the $k = 1$ case it is natural to conjecture that the upper bound given by Theorem 7 is asymptotically tight and even that the optimal families are $\lfloor n/2\rfloor$-uniform (or $\lceil n/2\rceil$-uniform). To prove the upper bound we reduced the problem to antichains and applied the LYM-inequality. The original proof of Sperner's theorem [14] on the maximum possible size of an antichain uses a different idea, namely if the minimum size m of a set in a maximum size antichain $\mathcal{A}$ would be strictly less than $\lfloor n/2\rfloor$, then one can remove $\mathcal{A}' = \{A \in \mathcal{A} : |A| = m\}$ from $\mathcal{A}$ and add all sets of $\nabla^1(\mathcal{A}')$ and the resulting family is an antichain containing more sets than $\mathcal{A}$. Can a similar method work in our case? Let $\mathcal{F}$ be an $(n-1)$-trace Sperner family with $m = \min\{|F| : F \in \mathcal{F}\} < \lfloor n/2\rfloor$, and let us put $\mathcal{F}' = \{F \in \mathcal{F} : |F| = m\}$ (for smaller values of l our reasoning is similar). As we saw in Section 4, being $(n-1)$-trace Sperner is equivalent to the property that for all $F, G \in \mathcal{F}$ we have $|F \setminus G| > 1$. Therefore we would need to prove the existence of a function $f : F \mapsto f(F) \in \nabla^1(F)$ such that for all $F, G \in \mathcal{F}'$ we have $|f(F) \setminus f(G)| > 1$. An easy application of the Lovász local lemma [5] shows that there is such a function if $m \le cn^{1/2}$ for some constant c (and the exponent gets smaller if we choose l to be smaller). Can one prove this for larger values of m?

- In Section 5, we conjectured that for the case $k \ge 2$, $l = n-1$ the family consisting of the $k-1$ largest levels is optimal and proved that for $k = 2$ this construction is asymptotically optimal. We think that to prove the exact conjecture for $k = 2$ may not necessarily require a much more complicated argument, but we think that to obtain results for larger values of k one would need to apply deeper techniques.

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