Majority and Plurality Problems

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A ball is said to be k-majority if its color class contains at least k balls. By definition a ball is majority if and only if it is k-majority for $k = \lfloor n/2 \rfloor + 1$. It is natural to consider only the case when $k > n/2$ since then k-majority is a strengthening of majority.

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1) adaptive: the ith query might depend on the answers to the first $i-1$ queries,

2) non-adaptive: all queries should be asked in advance.

Any non-adaptive strategy is adaptive, therefore the best non-adaptive strategy always uses at least as many queries as the best non-adaptive strategy.

History:

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Theorem (Saks and Werman)

If the number of colors is two, then the minimum number of queries needed in an adaptive search to find a majority ball is $n - b(n)$, *where* $b(n)$ *is the number of 1's in the binary representation of* n.

For a set Q of queries let the query graph G_Q be the graph where the vertices correspond to balls and two vertices are joined by an edge if and only if there exists a query in Q that asks for the comparison of the two corresponding balls.

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Theorem (Aigner, De Marco, and Montangero)

For any fixed positive integer c, if the number of possible colors is at most c, then the minimum number of queries needed in an adaptive search for a plurality ball is $O_c(n)$.

Let $M_c^*(n,k)$ ($M_c(n,k)$) denote the minimum number of queries needed to find a k-majority ball in a non-adaptive (adaptive) search if the number of colors is at most c and the number of balls is n.

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Theorem (Aigner)

For $n \geq 3$

$$M^*_2(n,k) = \begin{cases} n-1 & \textit{if } n < 2k-1 \\ n-2 & \textit{if } n = 2k-1. \end{cases}$$

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Thus the theorem of Saks and Werman compared to this theorem of Aigner tells us that if n is not a power of 2, then there exists a strictly better adaptive algorithm than the best non-adaptive one, but the gain is at most logarithmic (in case $n = 2^l - 1$ for some integer l).

Theorem (GKPP)

Suppose $c > 2$, $n > k > n/2$ and $n > 1$. Then a query graph G_Q solves the k-majority problem in the non-adaptive case if and only if G_Q is $(n - k + 1)$-connected.

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Corollary (GKPP)

Let $M^*_c(n)$ denote the minimum number of queries needed to find a majority ball in a non-adaptive search if the number of colors is at most c and the number of balls is n, and suppose $n \geq c > 2$. Then $M^*_c(n) = \lceil \lceil n/2 \rceil n/2 \rceil$.

Let $\mu(n)$ denote the largest integer l such that 2^l divides n, i.e. the exponent of two in the prime factorization of n.

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For any pair of integers $n \geq k > n/2$, the inequality $M_2(n,k) \geq n - 1 - \mu(\binom{n-1}{k-1})$ holds.

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Note that this result is a generalization of the theorem of Saks and Werman as if n is even, then for $k = \lfloor n/2 + 1 \rfloor$ we have $\mu(\binom{n-1}{n/2}) = b(n) - 1$, where $b(n)$ is the number of 1's in the binary representation of n.

Proposition (GKPP)

Let $k > n/2$. Then

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A little calculation involving the μ-function shows that the above propositions yield the theorem of Aigner.

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Theorem (GKPP)

For any pair of integers n and c, the following holds:

$$\left\lceil \frac{1}{2}\left(n-1-\frac{n-1}{c-1}\right)n \right\rceil \le P_c^*(n) \le \frac{c-2}{2(c-1)}n^2+n.$$

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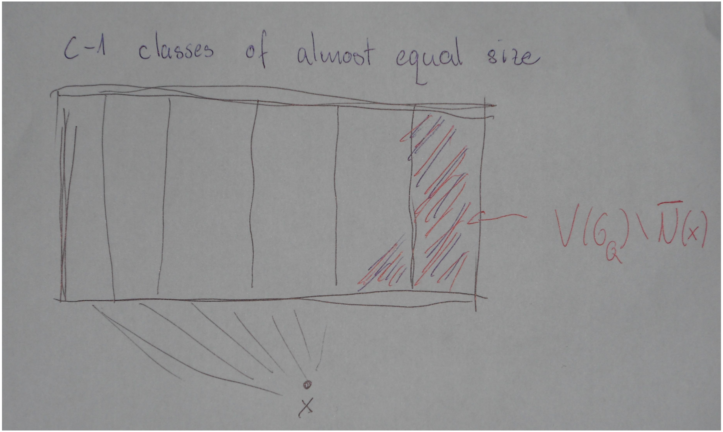
Lemma

If Q is a set of queries that solve the problem, then the minimum degree in G_Q is larger than $n-1-\lceil\frac{n-1}{c-1}\rceil$. Furthermore, if $n-1\equiv 1 \mod c-1$, then the minimum degree in G_Q is larger than $n-1-\lfloor\frac{n-1}{c-1}\rfloor$.


c-1 classes of almost equal size
x

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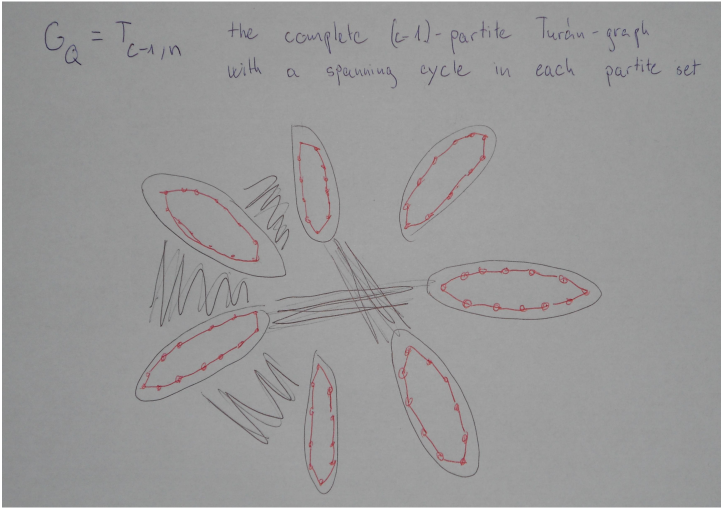
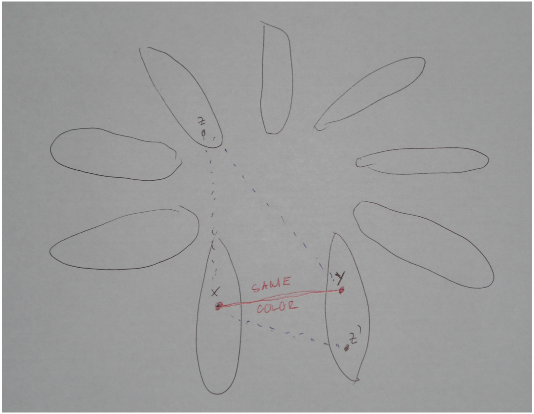


Proposition

If for some $x \in V_i, y \in V_j$ with $i \neq j$ the answer to the query (x, y) is SAME COLOR, then after receiving answers to all queries we are able to determine the whole color class of x and y.



Let k denote the number of color classes $C_1, ..., C_k$ that intersect at least two of the V_i's and let l denote the number of partite sets that are not contained in $K := \cup_{i=1}^{k} C_i$.

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Moreover, if $l = c - k$, then to cover all partite sets, all $c - k$ colors need to be used in different partite sets and thus they should be of the form $V_j \setminus K$. As we are able to determine all color classes, the proof is finished in this case.

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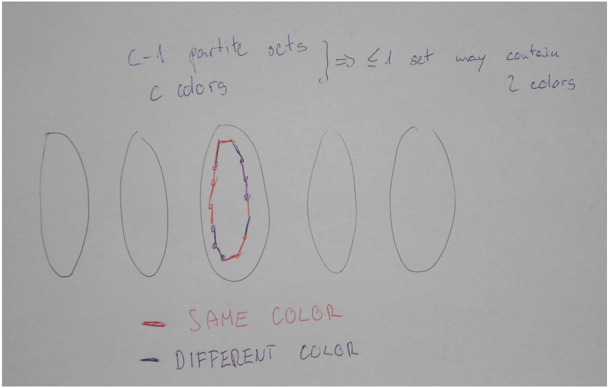

Then as all color classes are included in one of the partite sets we have that all but at most one partite sets form one color class each and the last partite set is the union of at most two color classes which we can identify due to the additional spanning cycle.

Thus we may suppose that $l \leq c - k - 1$ and $k \geq 1$ hold. This means that the number of already covered partite sets is $c - 1 - l \geq k$, thus $|K| \geq k\lfloor \frac{n}{c-1} \rfloor$ and therefore at least one of $C_1, ..., C_k$ has size at least $\lfloor \frac{n}{c-1} \rfloor$.

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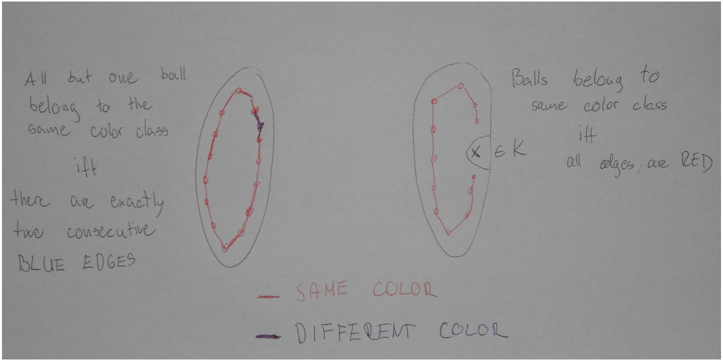
As all remaining color classes are contained in one of the V_i's, their size is at most $\lceil \frac{n}{c-1} \rceil$. Thus if a V_j contains at least two points from K, then it cannot contain a color class of size at least $\lfloor \frac{n}{c-1} \rfloor$.

On the other hand if a V_j contains at most one point of $\cup_{i=1}^k C_i$, then, due to the spanning cycle, we are able to tell whether it contains a color class of size $\lceil \frac{n}{c-1} \rceil$ or $\lfloor \frac{n}{c-1} \rfloor$.



Theorem (GKPP)

(i) $P_3^*(2k) = k(k+1)$,
(ii) $\frac{1}{2}(k+1)(2k+1) \leq P_3^*(2k+1) \leq \frac{1}{2}(k+1)(2k+1) + k - 1$.

Thank you for your attention!