Some Best Possible Inequalities Concerning Cross-Intersecting Families

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Communicated by the Managing Editors
Received July 30, 1990

Let \( \mathcal{A} \) be a non-empty family of \( a \)-subsets of an \( n \)-element set and \( \mathcal{B} \) a non-empty family of \( b \)-subsets satisfying \( A \cap B \neq \emptyset \) for all \( A \in \mathcal{A}, B \in \mathcal{B} \). Suppose that \( n > a + b \), \( b \geq a \). It is proved that in this case \( |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{a} - \binom{n-a}{a} + 1 \) holds. Various extensions of this result are proved. Two new proofs of the Hilton-Milner theorem on non-trivial intersection families are given as well.

1. INTRODUCTION

Let \( X := \{1, 2, \ldots, n\} \) be an \( n \)-element set. For an integer \( k \), \( 0 \leq k \leq n \), we denote by \( \binom{X}{k} \) the set of all \( k \)-element subsets of \( X \). A family \( \mathcal{F} \subseteq \binom{X}{k} \) is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in \mathcal{F} \). One of the best known results in extremal set theory is the following.

**THEOREM [EKR].** Let \( \mathcal{F} \subseteq \binom{X}{k} \) be an intersecting family with \( n = |X| \geq 2k \). Then, \( |\mathcal{F}| \leq \binom{n-1}{k-1} \).

Two families \( \mathcal{A} \subseteq \binom{X}{a} \) and \( \mathcal{B} \subseteq \binom{X}{b} \) are said to be cross-intersecting if and only if \( A \cap B \neq \emptyset \) holds for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). Recall the following result of Hilton and Milner.

**THEOREM A [HM].** Let \( \mathcal{A} \subseteq \binom{X}{a} \) and \( \mathcal{B} \subseteq \binom{X}{b} \) be non-empty cross-intersecting families with \( n = |X| \geq 2a \). Then, \( |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{a} - \binom{n-a}{a} + 1 \).
Recently, Simpson [S] rediscovered this theorem. In this paper, we generalize the above result in various ways. Probably the following is the most natural extension of Theorem A.

**Theorem 1.** Let \( \mathcal{A} \subset \binom{X}{a} \) and \( \mathcal{B} \subset \binom{X}{b} \) be non-empty cross-intersecting families with \( n = |X| \geq a + b \), \( a \leq b \). Then the following hold:

1. \[ |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{a} - \binom{n-a}{a} + 1. \]
2. If \( |\mathcal{A}| \geq \binom{n-1}{n-a} \), then
   \[ |\mathcal{A}| + |\mathcal{B}| \leq \begin{cases} \binom{n}{a} - \binom{n-a}{a} + 1 & \text{if } a = b \geq 2 \\ \binom{n-1}{a-1} + \binom{n-1}{b-1} & \text{otherwise.} \end{cases} \]

Putting restrictions on the size of \( \mathcal{A} \) we can obtain stronger bounds.

**Theorem 2.** Let \( \mathcal{A} \subset \binom{X}{a} \) and \( \mathcal{B} \subset \binom{X}{b} \) be non-empty cross-intersecting families with \( n = |X| \geq a + b \), \( a \leq b \). Suppose that \( \mathcal{A} \mathcal{B} \mathcal{A}^c = \mathcal{B} \mathcal{B}^c \mathcal{A} \) holds for some real number \( \alpha \) with \( n-a \leq \alpha \leq n-1 \). Then the following holds:

\[ |\mathcal{A}| + |\mathcal{B}| \leq \begin{cases} \binom{n}{b} - \binom{n}{b} + \binom{n}{n-a} & \text{if } a < b \text{ or } \alpha \leq n-2 \\ 2 \binom{n-1}{a-1} & \text{if } a = b \text{ and } \alpha \geq n-2. \end{cases} \]

The next result is of similar flavor, and it will be used for one of the new proofs for the Hilton–Milner theorem (see Section 4).

**Theorem 3.** Let \( \mathcal{A} \subset \binom{Y}{a} \), \( \mathcal{B} \subset \binom{Y}{b} \) be non-empty cross-intersecting families with \( m = |Y| \geq 2a - 1 \). Suppose that \( |\mathcal{A}| < \binom{m-1}{a-1} \), then \( |\mathcal{A}| + |\mathcal{B}| \leq \binom{m-1}{a-1} - \binom{m-a}{a-1} + 1. \)

2. **Proof of Theorem 1**

To prove the theorem, we start with an easy inequality.

**Lemma 1.** Let \( a, b, \) and \( n \) be integers. Suppose that \( n \geq a + b \) and \( a \leq b \). Then, it follows that

\[ \binom{n-1}{a-1} + \binom{n-1}{b-1} \leq \binom{n}{b} - \binom{n-a}{b} + 1. \]
or equivalently,
\[
\binom{n-1}{n-a} - \binom{n-1}{b} \leq \binom{n-a}{n-a} - \binom{n-a}{b}.
\]

**Proof.** To prove the above inequality, it suffices to show that
\[
\binom{x-1}{n-a-1} \leq \binom{x-1}{b-1}
\]
holds for all real numbers \( x, n-a+1 \leq x \leq n-1 \). This is equivalent to
\[
(x-b) \cdots (x-n+a+1) \leq (n-a-1) \cdots b
\]
\[
\iff x-b \leq n-a-1 \iff x \leq n-1 + (b-a).
\]
The above inequality follows from \( x \leq n-1 \) and \( a \leq b \).

**Proof of Theorem 1.** We prove the theorem by induction on \( n \). Since the theorem clearly holds for \( n = a+b \), we assume that \( n > a+b \). Further, by the Kruskal–Katona theorem [Kr, Kal], we may assume that \( \mathcal{A} := \{ X - A : A \in \mathcal{A} \} \) is the collection of the smallest \( |\mathcal{A}| \) sets in \( \binom{X}{n-a} \) with respect to the colex order (see Appendix). Let us define
\[
\mathcal{A}(n) := \{ A - \{ n \} : n \in A \in \mathcal{A} \} \subset \binom{X - \{ n \}}{a-1}.
\]
\[
\mathcal{A}(\bar{n}) := \{ A : n \notin A \in \mathcal{A} \} \subset \binom{X - \{ n \}}{a}.
\]
We also define \( \mathcal{B}(n) \) and \( \mathcal{B}(\bar{n}) \) in the same way.

**Proof of (i).** Since the RHS of the inequality in (ii) does not exceed the RHS of that of (i) we may suppose that \( \mathcal{A} < \binom{X}{n-a} \) and therefore \( \mathcal{A}(\bar{n}) = \emptyset \).

**Case 1.** \( \mathcal{B}(\bar{n}) \neq \emptyset \). By the induction hypothesis, we have
\[
|\mathcal{A}(n)| + |\mathcal{B}(\bar{n})| \leq \binom{n-1}{b} - \left( \binom{n-1}{a} - \binom{n-a}{b} \right) + 1.
\]
This, together with \( |\mathcal{B}(n)| \leq \binom{n}{b} \), gives
\[
|\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}(n)| + |\mathcal{B}(\bar{n})| + |\mathcal{B}(n)| \leq \binom{n}{b} - \binom{n-a}{b} + 1.
\]
Case 2. \( \mathcal{B}(\bar{n}) = \emptyset \). In this case, we have
\[
|\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}(n)| + |\mathcal{B}(n)| \leq \binom{n-1}{a-1} + \binom{n-1}{b-1}.
\]
Using Lemma 1, we obtain the desired inequality.

Proof of (ii). Since the theorem holds if \( |\mathcal{A}| = \binom{n-1}{a-1} \), we assume that \( |\mathcal{A}(n)| = \binom{n-1}{a-1} \) and \( |\mathcal{A}(\bar{n})| > 0 \). Note that \( |\mathcal{A}(n)| = \binom{n-1}{a-1} \) implies \( |\mathcal{B}(\bar{n})| = 0 \).

Case 1. \( a < b \). By the induction hypothesis, we have
\[
|\mathcal{A}(\bar{n})| + |\mathcal{B}(n)| \leq \binom{n-1}{b-1} - \binom{(n-1)-a}{b-1} + 1.
\]
So we obtain
\[
|\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}(n)| + |\mathcal{A}(\bar{n})| + |\mathcal{B}(n)|
\leq \left\{ \binom{n-1}{a-1} + \binom{n-1}{b-1} \right\} + \left\{ 1 - \binom{n-a-1}{b-1} \right\}
\leq \binom{n-1}{a-1} + \binom{n-1}{b-1}.
\]
Using Lemma 1, we obtain the desired inequality.

Case 2. \( a = b \). By the induction hypothesis, we have
\[
|\mathcal{A}(\bar{n})| + |\mathcal{B}(n)| \leq \binom{n-1}{a} - \binom{(n-1)-(a-1)}{a} + 1.
\]
This, together with \( |\mathcal{A}(n)| = \binom{n-1}{a-1} \), gives
\[
|\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}(n)| + |\mathcal{A}(\bar{n})| + |\mathcal{B}(n)| \leq \binom{n}{a} \binom{n-a}{a} + 1.
\]
This completes the proof of (ii).

3. Proofs of Theorem 2 and Theorem 3

In this section, we use Lovász' version of the Kruskal–Katona theorem, and so we need the following technical lemma.

Lemma 2. Let \( s, t, \) and \( n \) be integers with \( n > s + t \). Define a real valued function \( f(x) := \binom{n}{x} - \binom{s}{x} + \binom{x}{t} \). Then, the following hold:
(i) Suppose that \( 1 + (n - s - t) v/s(v - n + t + 1) < (r^v)/(n - v) \), then \( f'(x) < 0 \) holds for all real numbers \( x \leq v \).

(ii) Let \( u, v \) be real numbers with \( u < v \), \( u < n - t + s \). Suppose that \( f'(u) < 0 \) and \( f(u) \geq f(v) \), then \( f(u) \geq f(x) \) holds for all real numbers \( x \), \( u \leq x \leq v \).

Proof. Proof of (i). Since
\[
f'(x) = -\left(\frac{x}{s}\right)^{s-1} \sum_{j=0}^{n-t-1} \frac{1}{x-j} + \left(\frac{x}{n-t}\right)^{n-t-1} \sum_{j=0}^{s-1} \frac{1}{x-j},
\]
the inequality \( f'(x) < 0 \) is equivalent to
\[
\left(\sum_{j=0}^{n-t-1} \frac{1}{x-j}\right) / \left(\sum_{j=0}^{s-1} \frac{1}{x-j}\right) < \left(\frac{x}{s}\right)^{s-1} \left(\frac{n}{n-t}\right)^{n-t-1}.
\]
By simple estimation, we have
\[
\text{LHS} = 1 + \left(\sum_{j=s}^{n-t-1} \frac{1}{x-j}\right) / \left(\sum_{j=0}^{s-1} \frac{1}{x-j}\right) \leq 1 + \frac{n-t-s}{x-n+t+1} \cdot \frac{x}{s}.
\]
Thus, to prove (1), it suffices to show that
\[
(x-s) \cdot \ldots \cdot (x-n+t+1) \left(1 + \frac{n-t-s}{x-n+t+1} \cdot \frac{x}{s}\right) < (n-t) \cdot \ldots \cdot (s+1).
\]
Since the LHS of (2) is increasing with \( x \), it suffices to show (2) for \( x = v \), that is,
\[
1 + \frac{n-t-s}{v-n+t+1} \cdot \frac{v}{s} < \left(\frac{v}{s}\right)^{s-1} \left(\frac{v}{n-t}\right)^{n-t-1}.
\]
This was exactly our assumption.

Proof of (ii). Suppose on the contrary that \( f(u) < f(x) \) holds for some \( x, x > u \). Then, we may assume that there exist \( p, q \) which satisfy
\[
u < p < q \leq v, f'(p) = f'(q) = 0, f(p) < f(u) < f(q).
\]
If $f'(x) = 0$, it follows that

$$\binom{n}{s} = \binom{x}{n-t} \left\{ 1 + \left( \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right) \right\} \left\{ \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right\}.$$ 

Substituting this into $f(x)$, we define a new function:

$$g(x) := \binom{n}{s} - \binom{x}{n-t} \left\{ \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right\} \left\{ \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right\}.$$ 

Note that $g(x) = f(x)$ holds if $f'(x) = 0$. Thus, $f(u) < g(q)$ must hold. We derive a contradiction by showing that $f(u) \geq g(x)$, or equivalently,

$$\left\{ \binom{x}{s} - \binom{u}{n-t} \right\} \sum_{j=0}^{n-t-1} \frac{1}{x-j} \leq \binom{x}{n-t} \sum_{j=0}^{n-t-1} \frac{1}{x-j}$$

holds for all $x \geq p$. Since $u < n - t + s$, $\binom{x}{s} - \binom{u}{n-t}$ is positive, and so the LHS is decreasing with $x$. On the other hand, the RHS is increasing with $x$. Therefore, it suffices to check the inequality for $x = p$, that is $f(u) \geq g(p) = f(p)$. This was our assumption. 

Using the above lemma, we prove Theorem 2, which contains Theorem 1 (i).

**Proof of Theorem 2.** Since the theorem holds for $n = a + b$, we assume that $n > a + b$. Let $|\mathcal{A}| = \binom{n}{x}$, $n - a \leq x \leq n - 1$. Then, by the Kruskal–Katona theorem we have $|\mathcal{B}| \leq \binom{n}{b} - \binom{x}{b}$. Define $f(x) := \binom{x}{b} - \binom{x}{a}$. Define $f(x) := \binom{x}{b} - \binom{x}{a}$.

Case 1. $a < b$. In this case, we prove that $f'(x) < 0$ holds for $n - a \leq x \leq n - 1$. By Lemma 2 (i), it suffices to show that

$$1 + \frac{(n-a-b)(n-1)}{ba} < \binom{n-1}{b}/\binom{n-1}{n-a}. \quad (1)$$

This holds for $n = a + b + 1$. So we may assume that $n \geq a + b + 2$. Then,

$$\text{RHS} = \frac{(n-a) \cdot \ldots \cdot (n-b)}{b \cdot \ldots \cdot a}$$

$$= \frac{(n-a)(n-a-1)}{a(a+1)} \cdot \frac{(n-a-2) \cdot \ldots \cdot (n-b)}{b \cdot \ldots \cdot (a+2)}$$

$$\geq \frac{(n-a)(n-a-1)}{a(a+1)},$$

$$\text{LHS} = 1 + \frac{n-a-(a+1)}{a+1} \cdot \frac{n-1}{a}.$$
To prove (1), it suffices to show that
\[
1 + \frac{(n-2a-1)(n-1)}{a(a+1)} < \frac{(n-a)(n-a-1)}{a(a+1)},
\]
or equivalently, \(n > 2a + 1\), and this was our assumption.

**Case 2.** \(a = b\).

**Subcase 2.1.** \(x \leq n - 2\). In this case, we prove that \(f'(x) < 0\) holds for \(n - a \leq x \leq n - 2\). By Lemma 2 (i), it suffices to show that
\[
1 + \frac{(n-2a)(n-2)}{a(a-1)} < \binom{n-2}{a}/\binom{n-a}{n-a}.
\]
This holds for \(n = 2a + 1\). So we assume that \(n > 2a + 2\). Then,
\[
(2) \iff 1 + \frac{(n-2a)(n-2)}{a(a-1)} < \frac{(n-a)(n-a-1)}{a(a-1)} \iff n > 2a.
\]
This was our assumption.

**Subcase 2.2.** \(x \geq n - 2\). Note that \(f(n-2) = f(n-1) = 2\binom{n-1}{a-1}\). So by Lemma 2 (ii), \(f(x) < 2\binom{n-1}{a-1}\) holds for \(n-2 < x \leq n-1\).

Next we prove Theorem 3, which will be used to prove the Hilton–Milner theorem.

**Proof of Theorem 3.** Since the theorem clearly holds for \(m = 2a - 1\), we assume that \(m \geq 2a\). We distinguish two cases according to the size of \(\mathcal{A}\).

**Case 1.** \(1 \leq |\mathcal{A}| \leq \binom{m-2}{m-a}\). Let \(\mathcal{A} = (\binom{x}{m-a})\), \(m-a \leq x \leq m-2\). Then, by the Kruskal–Katona theorem we have \(|\mathcal{B}| \leq (\binom{m}{a-1}) - (\binom{x}{a-1})\). Define \(f(x) := (\binom{m}{a-1}) - (\binom{x}{a-1}) + (\binom{x}{m-a})\). First we prove that \(f'(x) < 0\) holds for all \(x\), \(m-a \leq x \leq m-3\). By Lemma 2(i), it suffices to show that
\[
1 + \frac{(m-2a+1)(m-3)}{(a-1)(a-2)} < \binom{m-3}{a-1}/\binom{m-3}{m-a}.
\]
This holds for \(m = 2a\), and if \(m > 2a\) this is equivalent to \(m > 2a - 1\) which is our assumption.

Next, we prove that \(f(m-2) \geq f(x)\) holds for all \(x\), \(m-3 \leq x \leq m-2\). By Lemma 2 (i), it suffices to show that \(f(m-a) \geq f(m-2)\), or equivalently,
This follows from Lemma 1.

**Case 2.** $|\mathcal{A}| > \binom{m-2}{m-a}$. In this case, we have $|\mathcal{B}| < \binom{m-1}{a-1} - \binom{m-a-1}{a-1} = \binom{m-a}{m-a-1} + \binom{m-a-1}{m-a-1}$. Let $|\mathcal{B}| = \binom{m-1}{m-(a-1)} + \binom{m-a-1}{m-(a-1)-1}$, $m-a \leq x < m-2$. Then, by the Kruskal–Katona theorem, we have $|\mathcal{A}| \leq \binom{m}{a} - \binom{m-a}{a-x} - \binom{m-a}{x}$. Define $f(x) := \binom{m-1}{m-(a-1)} + \binom{m-a-1}{m-(a-1)-1} + \binom{m}{a} - \binom{m-a}{a-x} - \binom{m-a}{x}$. By arguments in Case 1, $f(x) \leq f(m-a)$ holds for all $x$, $m-a \leq x \leq m-2$.

**4. Application**

Using results of earlier sections, we give two new proofs of the Hilton–Milner theorem. Let us mention that other short proofs were given in [FF, M]. Recall that an intersecting family $\mathcal{F}$ is called non-trivial if $\bigcap_{F \in \mathcal{F}} F = \emptyset$ holds.

**Theorem [HM].** Let $\mathcal{F} \subset \binom{X}{k}$ be a non-trivial intersecting family with $n = |X| \geq 2k$, Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

**Proof 1.** Suppose that $|\mathcal{F}|$ is maximal with respect to the conditions. First we deal with an important special case. Suppose that there exists $A := \{a, b\} \in \binom{X}{2}$ such that $A \cap F \neq \emptyset$ holds for all $F \in \mathcal{F}$. By the maximality of $|\mathcal{F}|$, $\{G : A \subset G \in \binom{X}{2}\} \subset \mathcal{F}$ holds. Define

$$\mathcal{A} := \{F - \{a\} : F \in \mathcal{F}, F \cap A = \{a\}\},$$

$$\mathcal{B} := \{F - \{b\} : F \in \mathcal{F}, F \cap A = \{b\}\}.$$ 

Then $\mathcal{A}, \mathcal{B}$ are cross-intersecting families on $X - A$. By Theorem A,

$$|\mathcal{A}| + |\mathcal{B}| \leq 1 + \binom{n-2}{k-1} - \binom{n-k-1}{k-1}.$$ 

Consequently,

$$|\mathcal{F}| \leq 1 + \binom{n-2}{k-1} - \binom{n-k-1}{k-1} + \binom{n-2}{k-2} = 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1},$$

as desired.
Next consider the case when $9$ is shifted (see Appendix). Note that
\{2, 3, \ldots, k + 1\} \in \mathcal{F}$. Now define
\[
\mathcal{A} := \{ F - \{1\} : F \cap \{1, 2\} = \{1\}, F \in \mathcal{F}\},
\]
\[
\mathcal{B} := \{ F - \{2\} : F \cap \{1, 2\} = \{2\}, F \in \mathcal{F}\},
\]
\[
\mathcal{C} := \{ F - \{1, 2\} : \{1, 2\} \subseteq F \in \mathcal{F}\},
\]
\[
\mathcal{D} := \{ F - \{1, 2\} : \{1, 2\} \cap F = \emptyset\}.
\]

Then by Theorem A and \{3, 4, \ldots, k + 1\} \in \mathcal{A} \cap \mathcal{B}, |\mathcal{A}| + |\mathcal{B}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1} holds.

On the other hand, \mathcal{C}, \mathcal{D} are cross-intersecting and \mathcal{D} is 2-intersecting. Thus, \mathcal{D}' := \{ X - D : D \in \mathcal{D}, (n-2) - (2k-2) = (n-2k)-\text{-intersecting. By the Intersecting Kruskal–Katona theorem (cf. [Ka2]),}\]
\[
|\mathcal{D} : \sigma_{k-2}(\mathcal{D}')| \geq |\mathcal{D}^*| = |\mathcal{D}| \text{ (see Appendix) and by the cross-intersecting property } \mathcal{S} \cap \mathcal{C} = \emptyset. \text{ Therefore,}\]
\[
|\mathcal{C}| + |\mathcal{D}| \leq |\mathcal{S}| + |\mathcal{C}| \leq \binom{n-2}{k-2} = \binom{n-2}{k-2}.
\]

Again, we obtain $|\mathcal{S}| = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$.

Now to the general case. Apply repeatedly to \mathcal{S} the shift operator (see Appendix) \mathcal{S}_{ij}, 1 \leq i < j \leq n. Either we obtain a shifted non-trivial intersecting family of the same size (and we are done by the second case) or at some point the family stops to be non-trivial. That is for some \mathcal{G} \subseteq \binom{X}{k}, \mathcal{G} \text{ non-trivial intersecting, } |\mathcal{S}| = |\mathcal{G}| \text{ we have that } \bigcap_{H \in \mathcal{S}_{ij}(\mathcal{G})} H \neq \emptyset. \text{ In this case, clearly } \{i\} = \bigcap_{H \in \mathcal{S}_{ij}(\mathcal{G})} H \text{ and consequently } \{i, j\} \cap G \neq \emptyset \text{ for all } G \in \mathcal{G}. \text{ Thus we are done by the first special case.} \]

Proof II. Since the theorem clearly holds for $n = 2k$, we assume that $n \geq 2k + 1$. We may assume that $n \in F \in \mathcal{F}$ holds for some $F$. Let us define $Y := X - \{n\}, m := |Y|, a := k$,
\[
\mathcal{A} := \{ F : n \notin F \in \mathcal{F}\} \subseteq \binom{Y}{a},
\]
\[
\mathcal{B} := \{ F - \{n\} : n \in F \in \mathcal{F}\} \subseteq \binom{Y}{a-1}.
\]

Then \mathcal{A} and \mathcal{B} are non-empty cross-intersecting families. Since \mathcal{A} is intersecting itself, $|\mathcal{A}| \leq \binom{m}{a-1}$ holds. First suppose that $|\mathcal{A}| = \binom{m}{a-1}$. If $m = 2a$, then $|\mathcal{A}| = \frac{1}{2}\binom{m}{a}$. Hence for all $B \in \mathcal{B}$ and for all $y \in Y - B$, $B \cup \{y\} \in \mathcal{A}$.
holds. Therefore, $B$ is also intersecting, and so we may that $m \in B$ holds for all $B \in B$. Since $\mathcal{F}$ is non-trivial, there exists $A \in \mathcal{A}$ such that $m \notin A$. So,

$$|\mathcal{B}| \leq \left| \left( \frac{Y-\{m\}}{a-2} \right) - \left( \frac{Y-A}{a-1} \right) \right| = \left( \frac{m-1}{a-2} \right) - \left( \frac{m-a}{a-1} \right).$$

This implies that $|\mathcal{F}| = |\mathcal{A}| + |\mathcal{B}| \leq \left( \binom{n}{a-1} - \binom{m-a}{a-1} \right) = \left( \binom{n}{k-1} - \binom{n-k-1}{a-1} \right)$. If $m > 2a$, then we may assume that $m \in A$ holds for all $A \in \mathcal{A}$, that is, $\mathcal{A} = \{ A \in \binom{\mathcal{Y}}{a} : m \in A \}$. Since $\mathcal{F}$ is non-trivial, there exists $B \in B$ such that $m \notin B$. But, for all $A \in \binom{\mathcal{Y}-(B \cup \{m\})}{a-1}$, $A := A_0 \cup \{m\} \in \mathcal{A}$ must hold, a contradiction.

Next suppose that $|\mathcal{A}| < \binom{n}{a-1}$. Then by Theorem 3, we have

$$|\mathcal{F}| = |\mathcal{A}| + |\mathcal{B}| \leq \left( \frac{m}{a-1} \right) - \left( \frac{m-a}{a-1} \right) + 1 = \left( \frac{n-1}{k-1} \right) - \left( \frac{n-k-1}{a-1} \right) + 1,$$

as desired. 

**APPENDIX**

Let $n$, $k$ be integers and let $X$ be an $n$-element set. We define the colex order $<$ on $\binom{\mathcal{X}}{k}$ by setting $A < B$ if $\max\{i : i \in A-B\} < \max\{i : i \in B-A\}$. The shift operator $S_{ij}$, $1 \leq i < j \leq n$, on $\binom{\mathcal{X}}{k}$ is defined as follows: Let $\mathcal{F} \subseteq \binom{\mathcal{X}}{k}$. For $F \in \mathcal{F}$, define

$$s_{ij}(F) := \begin{cases} (F-(j)) \cup \{i\} & \text{if } i \notin F, j \in F, \text{ and } (F-\{j\}) \cup \{i\} \notin \mathcal{F} \\ F & \text{otherwise}, \end{cases}$$

and $S_{ij}(\mathcal{F}) := \{ s_{ij}(F) : F \in \mathcal{F} \}$. It is easily checked that (i) $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$ and (ii) $S_{ij}(\mathcal{F})$ is intersecting if $\mathcal{F}$ is intersecting. A family $\mathcal{F} \subseteq \binom{\mathcal{X}}{k}$ is called shifted if $S_{ij}(\mathcal{F}) = \mathcal{F}$ holds for all $1 \leq i < j \leq n$. For a family $\mathcal{F} \subseteq \binom{\mathcal{X}}{k}$ and an integer $l \leq k$, we define the $l$th shadow of $\mathcal{F}$ by $\sigma_l(\mathcal{F}) := \{ G \in \binom{\mathcal{X}}{k} : G \subseteq \exists F \in \mathcal{F} \}$. 

**ACKNOWLEDGMENT**

The authors are indebted to J. E. Simpson for calling their attention to Theorem A, which initiated this research.
REFERENCES


