Some Inequalities Concerning Cross-Intersecting Families

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Some Inequalities Concerning Cross-Intersecting Families

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Let \(a, b\) and \(n\) be integers with \(2 \leq a \leq b\) and \(n \geq a + b\). Suppose that \(A \subset \left[\begin{array}{l} n \\ a \end{array}\right]\) and \(B \subset \left[\begin{array}{l} n \\ b \end{array}\right]\) are nontrivial cross-intersecting families. Then
\[
|A| + |B| \leq 2 + \left(\frac{n}{b}\right) - 2 \left(\frac{n-a}{b}\right) + \left(\frac{n-2a}{b}\right).
\]
This result is best possible.

1. Introduction

Let \([n] := \{1, 2, \ldots, n\}\) be an \(n\)-element set. For an integer \(k, 0 \leq k \leq n\), we denote by \(\left[\begin{array}{l} n \\ k \end{array}\right]\) the set of all \(k\)-element subsets of \([n]\). A family \(\mathcal{F} \subset \left[\begin{array}{l} n \\ k \end{array}\right]\) is called nontrivial if \(\bigcap_{F \in \mathcal{F}} F = \emptyset\). Two families, \(\mathcal{A} \subset \left[\begin{array}{l} n \\ a \end{array}\right]\) and \(\mathcal{B} \subset \left[\begin{array}{l} n \\ b \end{array}\right]\), are said to be cross-intersecting if \(A \cap B \neq \emptyset\) holds for all \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\). A family \(\mathcal{F} \subset \left[\begin{array}{l} n \\ k \end{array}\right]\) is called intersecting if \(\mathcal{A} \text{ and } \mathcal{B}\) are cross-intersecting.

Let us recall the following two fundamental results.

**Theorem A (Erdős, Ko and Rado [1])**. Let \(k\) and \(n\) be integers with \(n \geq 2k\). If \(\mathcal{F} \subset \left[\begin{array}{l} n \\ k \end{array}\right]\) is intersecting, then \(|\mathcal{F}| \leq \left(\binom{n}{k-1}\right)\).

**Theorem B (Hilton and Milner [6])**. Let \(k\) and \(n\) be integers with \(n \geq 2k\). If \(\mathcal{F} \subset \left[\begin{array}{l} n \\ k \end{array}\right]\) is nontrivial intersecting, then \(|\mathcal{F}| \leq \left(\binom{n-1}{k-1}\right) - \left(\binom{n-k-1}{k-1}\right) + 1\).

In [4], Füredi proposed the following conjectures.

**Conjecture 1**. Let \(a, b\) and \(n\) be integers with \(n > a + b\). Suppose that \(\mathcal{A} \subset \left[\begin{array}{l} n \\ a \end{array}\right]\) and \(\mathcal{B} \subset \left[\begin{array}{l} n \\ b \end{array}\right]\) are cross-intersecting families. Then \(|\mathcal{A}| |\mathcal{B}| \leq \left(\binom{n-1}{a-1}\right) \left(\binom{n-1}{b-1}\right)\).
Conjecture 2. Let $a, b$ and $n$ be integers with $a \leq b$ and $n \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting families. If $|\mathcal{A}| \geq \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ and $\mathcal{A}$ is nontrivial, then $|\mathcal{B}| \leq \binom{n}{b-1} - \binom{n-a-1}{b-1} + \binom{n-a}{b-a}$.

Conjecture 3. Let $a, b$ and $n$ be integers with $a \leq b$ and $n \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are nontrivial cross-intersecting families. Then

$$|\mathcal{A}| + |\mathcal{B}| \leq \left| \binom{[a+1]}{a} \right| + \left| \left\{ B \in \binom{[n]}{b} : |[a+1] \cap B| \geq 2 \right\} \right|.$$ 

Conjecture 1 was known to be true if $n \geq \max \{2a, 2b\}$ (see [10], [13]). But if $n < \max \{2a, 2b\}$ then the conjecture is not true in general. A simple counterexample is given in Section 2.

In Section 3, we show that Conjecture 2 is a direct consequence of a theorem of Mőrs.

Conjecture 3 is false even if we fix $|\mathcal{A}| = a + 1$. In this case, the best construction is the following. Let

$$A_i := \{1, \ldots, a - 1\} \cup \{a + i\} \quad \text{for} \quad 0 \leq i < a,$$

and set

$$\mathcal{A} := \{A_0, \ldots, A_{a-1}\} \cup \{\{a, \ldots, 2a - 1\}\},$$
$$\mathcal{B} := \left\{ B \in \binom{[n]}{b} : A \cap B \neq \emptyset \right\} \text{for all } A \in \mathcal{A}.$$ 

If we do not restrict $|\mathcal{A}|$, the following construction is much better.

Example. Choose disjoint $A_0, A_1 \in \binom{[n]}{a}$, and set $\mathcal{A}_0 := \{A_0, A_1\}$,

$$\mathcal{B}_0 := \left\{ B \in \binom{[n]}{b} : B \cap A_0 \neq \emptyset, \ B \cap A_1 \neq \emptyset \right\}.$$ 

Then $\mathcal{A}_0$ and $\mathcal{B}_0$ are nontrivial cross-intersecting families. ($\mathcal{A}_0$ has size 2.)

Actually, if $b \geq a + 2$ then we have the following result.

**Theorem 1.** Let $a, b$ and $n$ be integers with $2 \leq a \leq b - 2$ and $n \geq a + b$. Suppose that two families $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting, and the family $\mathcal{A}$ is nontrivial. Then, $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0|$ holds. For $n > a + b$, equality holds if and only if $\mathcal{A} \cong \mathcal{A}_0$ and $\mathcal{B} \cong \mathcal{B}_0$.

Note that in the above theorem it is not assumed that $\mathcal{B}$ is nontrivial. We prove Theorem 1 in Section 5. If $|\mathcal{A}|$ is relatively small then the same inequality holds for the cases $b = a$ or $b = a + 1$ as well.

**Theorem 2.** Let $a, b$ and $n$ be integers with $2 \leq a \leq b$ and $n \geq a + b$. Suppose that two families $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting, and the family $\mathcal{A}$ is nontrivial. Then the following statements hold.
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(i) If \( b = a + 1 \) and \( |\mathcal{A}| \leq \binom{n-1}{a-1} + \frac{n-2}{a-1} \), then \( |\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0| \). For \( n > a + b \), equality holds if and only if \( \mathcal{A} \cong \mathcal{A}_0 \) and \( \mathcal{B} \cong \mathcal{B}_0 \).

(ii) If \( b = a \) and \( |\mathcal{A}| \leq \binom{n-1}{a-1} - \binom{n-2}{a-1} + 1 \) then \( |\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0| \). For \( n > a + b \) and \( a \geq 3 \), equality holds if and only if \( \mathcal{A} \cong \mathcal{A}_0 \) and \( \mathcal{B} \cong \mathcal{B}_0 \).

Using Theorems 1 and 2, we have the following.

**Theorem 3.** Let \( a, b \) and \( n \) be integers with \( 2 \leq a \leq b \) and \( n \geq a + b \). Suppose that \( \mathcal{A} \subset \binom{[n]}{a} \) and \( \mathcal{B} \subset \binom{[n]}{b} \) are nontrivial cross-intersecting families. Then \( |\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0| \). For \( n > a + b \) and \( b \geq 3 \), equality holds if and only if \( \mathcal{A} \cong \mathcal{A}_0 \) and \( \mathcal{B} \cong \mathcal{B}_0 \).

Since Theorems 1, 2, 3 are trivial if \( n = a + b \), throughout this paper we consider the case \( n > a + b \).

2. Counterexample to Conjecture 1

Define

\[
\mathcal{A} := \{ A \in \binom{[n]}{a} : \{1, 2\} \subseteq A \neq \emptyset \}, \\
\mathcal{B} := \{ B \in \binom{[n]}{b} : \{1, 2\} \subseteq B \}.
\]

These two families are cross-intersecting, and

\[
|\mathcal{A}| = \binom{n-1}{a-1} + \binom{n-2}{a-1}, \quad |\mathcal{B}| = \binom{n-1}{b-1} - \binom{n-2}{b-1}.
\]

Set \( \delta := |\mathcal{A}| + |\mathcal{B}| - \binom{n-1}{a-1} \). Then \( \delta > 0 \) is equivalent to

\[
\frac{(n-1)(b-a)}{(n-b)(n-a)} > 1. \tag{2.1}
\]

Let \( n = (2 - x)b, a = (1 - \beta)b \), where

\[
0 < x < \beta < 1. \tag{2.2}
\]

Then \( n > a + b \) holds and condition (2.1) is equivalent to

\[
(1 - 1/b)\beta > (1 - x)^2. \tag{2.3}
\]

If we choose \( x, \beta \) and \( b \) so that (2.2) and (2.3) hold, then \( n > a + b \), but \( \delta > 0 \). For example, choose an integer \( c > 5 \) and set \( n = 17c, a = 5c \) and \( b = 10c \): then the pair of \( \mathcal{A} \) and \( \mathcal{B} \) is a counterexample to Conjecture 1.

3. The Mörs theorem

Let \( \mathcal{F} \subset \binom{[n]}{k} \) and \( 0 < l < k \). The \( l \)th shadow \( \Delta_l(\mathcal{F}) \) of \( \mathcal{F} \) is defined by

\[
\Delta_l(\mathcal{F}) := \{ G : |G| = l, G \subset F \text{ for some } F \in \mathcal{F} \}.
\]
Let us define the colex order on $\binom{n}{k}$ by

$$A < B \quad \text{if and only if} \quad \max\{A - B\} < \max\{B - A\}.$$ 

Define $\text{Colex}(k, j)$ to be the first $j$ sets in $\binom{n}{k}$ with respect to the colex order. Let us define $\|\mathcal{F}\| := \bigcup_{F \in \mathcal{F}} F$. For given integers $n, k, i, l$, what is the minimum of $|\Delta_l(\mathcal{F})|$ if $\mathcal{F} \subset \binom{n}{k}$? The Mörs theorem (Theorem C below) gives the complete answer to this question.

Let $n, k, i$ be integers with $n/k \leq i \leq \binom{n}{k}$. Let us construct a family $\mathcal{F}_0 \subset \binom{n}{k}$ with $\|\mathcal{F}_0\| = n, |\mathcal{F}| = i$ as follows. Define $g := \max\{j : n - \|\text{Colex}(k, j)\| \leq (i - j)k\}$, $h := \|\text{Colex}(k, g)\|$. For $1 \leq j < i - g$, define $F_j := \{(j - 1)k + h + 1, \ldots, jk + h\}$. Further, define $G := \{(i - g - 1)k + h + 1, \ldots, n/k, 1, 2, \ldots, k - (n - (i - g - 1)k - h)\}$. Finally, define $\mathcal{F}_0 := \text{Colex}(k, g) \cup \{F_1, \ldots, F_{i-g-1}, G\}$.

**Theorem C (Mörs [12]).** Let $n, k, i, l$ be integers with $1 \leq l < k \leq n$, $n/k \leq i \leq \binom{n}{k}$. Suppose that $\mathcal{F} \subset \binom{n}{k}$, $\|\mathcal{F}\| = n, |\mathcal{F}| = i$. Then $|\Delta_l(\mathcal{F})| \geq |\Delta_l(\mathcal{F}_0)|$.

If $n \leq 2k$, the situation is much simpler. In this case, the optimal family $\mathcal{F}_0$ is given by $\mathcal{F}_0 := \text{Colex}(k, i - 1) \cup \{h + 1, \ldots, n/k, 1, 2, \ldots, k + h - n\}$.

Let us show how Conjecture 2 follows from Theorem C (see also [5]). Note that

$$|\mathcal{F}| \geq \binom{n - 1}{a - 1} - \binom{n - a - 1}{a - 1} + 1 = \binom{n - 2}{n - a} + \binom{n - 3}{n - a - 1} + \cdots + \binom{n - a - 1}{n - 2a + 1} + \binom{n - a - 1}{n - a - 1}.$$ 

By the Mörs theorem, we have

$$|\mathcal{F}| \leq \binom{n}{b} - \binom{n - 2}{b} - \binom{n - 3}{b - 1} - \cdots - \binom{n - a - 1}{b + a} - \binom{n - a - 1}{b - 1} = \binom{n - 1}{b - 1} + \binom{n - a - 1}{b - a} - \binom{n - a - 1}{b - 1}.$$ 

4. Tools for proofs

In this section, we list several inequalities concerning binomial coefficients (see [2], [3], [10], [11]). These inequalities will be used in later sections.

**Lemma 1.** Let $b \geq a, a \geq e + 3$ and $n \geq a + b$. Then inequality $P(j, n)$ holds for $0 \leq e \leq a - 3$ and $0 \leq j \leq e + 1$.

$$P(j, n) : \binom{n - a + e}{b - 1 - j} - \binom{n - 2a + e}{b - 1 - j} > \binom{n - a + e}{e + 1 - j}.$$ 

**Proof.** We prove $P(j, n)$ by double induction on $j$ and $n$. Fix $a, b, e$. 

If \( j = e + 1 \), then the desired inequality is
\[
P(e + 1, n) : \quad \left( \frac{n - a + e}{b - j} \right) - \left( \frac{n - 2a + e}{b - j} \right) > 1.
\]
Since \( b \geq a \geq e + 3 \), we have \( b - 2 - e \geq 1 \). Thus \( P(e + 1, n) \) holds for all \( n \geq a + b \).

Now fix \( 0 < j \leq e \) and assume that \( P(j, n) \) holds for all \( n \geq a + b \). We prove
\[
P(j - 1, n) : \quad \left( \frac{n - a + e}{b - j} \right) - \left( \frac{n - 2a + e}{b - j} \right) > \left( \frac{n - a + e}{e + 2 - j} \right)
\]
using induction on \( n \).

First we check the case \( n = a + b \), that is,
\[
P(j - 1, a + b) : \quad \left( \frac{b + e}{b - j} \right) - \left( \frac{b - a + e}{b - j} \right) > \left( \frac{b + e}{e + 2 - j} \right).
\]
The above inequality is trivial if \( b - a + e \leq b - j \). So assume \( a < e + j \). By the induction hypothesis \( P(j, a + b) \), it follows that
\[
\left( \frac{b + e}{b - 1 - j} \right) - \left( \frac{b - a + e}{b - 1 - j} \right) > \left( \frac{b + e}{e + 1 - j} \right) = \frac{e + 2 - j}{b - 1 + j} \left( \frac{b + e}{e + 2 - j} \right).
\]
Thus, to prove \( P(j - 1, a + b) \), it suffices to show
\[
\left( \frac{b + e}{b - j} \right) \left( \frac{1 - b - 1 + j}{b - j} \right) > \left( \frac{b - a + e}{b - j} \right) \left( 1 - \frac{b - 1 + j}{b - j} \right.
\]
or, equivalently,
\[
\left( \frac{b + e}{b - j} \right) \left( \frac{1 - b - 1 + j}{b - j} \right) > \left( \frac{b - a + e}{b - j} \right) \left( 1 - \frac{b - 1 + j}{b - j} \right.
\]
The above inequality clearly holds.

Next we fix \( n \) and assume \( P(j - 1, n) \). We prove \( P(j - 1, n + 1) \). Using the induction hypotheses \( P(j - 1, n) \) and \( P(j, n) \), we have
\[
\left( \frac{n + 1 - a + e}{b - j} \right) - \left( \frac{n + 1 - 2a + e}{b - j} \right)
\]
\[
= \left\{ \left( \frac{n - a + e}{b - j} \right) - \left( \frac{n - 2a + e}{b - j} \right) \right\} + \left\{ \left( \frac{n - a + e}{b - j} \right) - \left( \frac{n - 2a + e}{b - j} \right) \right\}
\]
\[
> \left( \frac{n - a + e}{e + 2 - j} \right) + \left( \frac{n - a + e}{e + 1 - j} \right) = \left( \frac{n - a + e + 1}{e + 2 - j} \right)
\]
This proves \( P(j - 1, n + 1) \), and by induction \( P(j - 1, n) \) holds for all \( n \geq a + b \).

Lemma 2. Let \( n \) and \( a \) be integers with \( n \geq 2a \), \( a > 1 \). Define \( f(n, a) := \binom{n-1}{a} + \binom{n-2a}{a} - 2\binom{n-a-1}{a-1} - \binom{n-1}{a-1} \). Then we have \( f(n, a) > 0 \).

Proof. We prove \( f(n, a) > 0 \) by double induction on \( n \) and \( a \). It is easily checked that \( f(n, 1) = 0 \) and \( f(2a, a) = 0 \). Fix \( n \) and \( a \), and assume \( f(n, a) \geq 0 \) and \( f(n, a - 1) \geq 0 \). Using
these assumptions, let us prove \( f(n, a + 1) > 0 \). Since
\[
\begin{align*}
f(n + 1, a) &= \left\{ \binom{n}{a + 1} - \binom{n}{a} - 2\binom{n - a - 1}{a + 1} + \binom{n - 2a}{a + 1} - \binom{n - 2a - 2}{a} \right\} + \\
& \quad \left\{ \binom{n}{a} - \binom{n}{a - 1} - 2\binom{n - a - 1}{a} + \binom{n - 2a}{a} - \binom{n - 2a - 2}{a - 1} \right\} \\
&= f(n, a) + f(n, a - 1) + 2\binom{n - a - 1}{a - 2} - 2\binom{n - 2a}{a - 2} \left( 1 + \frac{a - 2}{2(n - 3a + 3)} \right),
\end{align*}
\]
it suffices to show that
\[
\frac{(n - a - 1) \cdots (n - 2a + 2)}{(n - 2a) \cdots (n - 3a + 3)} \geq 1 + \frac{a - 2}{2(n - 3a + 3)}.
\]
Let us check the above inequality:
\[
\begin{align*}
\text{LHS} &= \left( 1 + \frac{a - 1}{n - 2a} \right) \cdots \left( 1 + \frac{a - 1}{n - 3a + 3} \right) \\
&> 1 + \frac{a - 1}{n - 2a} + \cdots + \frac{a - 1}{n - 3a + 3} > 1 + \frac{a - 2}{2(n - 3a + 3)} = \text{RHS}.
\end{align*}
\]
This proves \( f(n + 1, a) > 0 \).

**Lemma 3.** Let \( n \) and \( a \) be integers with \( n > 2a + 1, a > 0 \). Define \( f(n, a) := \binom{n}{a + 1} - \binom{n}{a} - 2\binom{n - a - 1}{a + 1} + \binom{n - 2a}{a + 1} - \binom{n - 2a - 2}{a} \). Then \( f(n, a) > 0 \).

**Proof.** We prove \( f(n, a) > 0 \) by double induction on \( n \) and \( a \). One can easily check that \( f(n, 0) = 0 \) and \( f(2a + 1, a) = 0 \). Fix \( n \) and \( a \), and assume that \( f(n, a) > 0 \) and \( f(n, a - 1) > 0 \). Using these assumptions, let us prove \( f(n + 1, a) > 0 \). In fact,
\[
\begin{align*}
f(n + 1, a) &= \left\{ \binom{n}{a + 1} - \binom{n}{a} - 2\binom{n - a - 1}{a + 1} + \binom{n - 2a}{a + 1} - \binom{n - 2a - 2}{a} \right\} + \\
& \quad \left\{ \binom{n}{a} - \binom{n}{a - 1} - 2\binom{n - a - 1}{a} + \binom{n - 2a}{a} - \binom{n - 2a - 2}{a - 1} \right\} \\
&= f(n, a) + f(n, a - 1) + 2\binom{n - a - 1}{a - 2} - 2\binom{n - 2a}{a - 2} \left( 1 + \frac{a - 2}{2(n - 3a + 3)} \right) \\
&> 0.
\end{align*}
\]

For an integer \( k \) and a real \( x \geq k \), define \( \binom{x}{k} := \prod_{i=0}^{k-1}(x - i)!/k! \).

**Lemma 4.** Let \( s, t \) and \( n \) be integers with \( n > s + t \). Define a real valued function \( f(x) := -\binom{x}{s} + \binom{x}{t} \). Then the following statements hold.

(i) Suppose that \( 1 + \frac{(n - s - t)x}{(n - s - t + 1)} < \binom{n}{s}/\binom{n}{t} \). Then \( f'(x) < 0 \) holds for all real numbers \( x \leq n \).
(ii) Let \( u, v \) be real numbers with \( u < v \), and let \( w \in [u, v] \). Suppose that \( f'(u) < 0 \) and \( f(w) = \max(f(u), f(v)) \). Then \( f(w) \geq f(x) \) holds for all real numbers \( x, u \leq x \leq v \).

**Proof.** (i) Since \( f'(x) = -\binom{n}{t} \sum_{j=0}^{t-1} \frac{1}{x-j} + \frac{x}{(n-t)} \sum_{j=0}^{n-t-1} \frac{1}{x-j} \), \( f'(x) < 0 \) is equivalent to

\[
\left( \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right) / \left( \sum_{j=0}^{x-t} \frac{1}{x-j} \right) < \left( \frac{x}{s} \right) / \left( \frac{n-t}{s} \right) = \frac{(n-t) \cdots (s+1)}{(x-s) \cdots (x-n+t+1)}. \tag{4.1}
\]

By simple estimation, we have

\[
\text{LHS} = 1 + \left( \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right) / \left( \sum_{j=0}^{x-t} \frac{1}{x-j} \right) \leq 1 + \frac{n-t-s}{x-n+t+1} \cdot \frac{x}{s}.
\]

Thus, to prove (4.1), it suffices to show that

\[
(x-s) \cdots (x-n+t+1) \left( 1 + \frac{n-t-s}{x-n+t+1} \cdot \frac{x}{s} \right) < (n-t) \cdots (s+1). \tag{4.2}
\]

Since the LHS of (4.2) is increasing with \( x \), it suffices to show (4.2) for \( x = v \), that is,

\[
1 + \frac{n-t-s}{v-n+t+1} \cdot \frac{v}{s} < \left( \frac{v}{s} \right) / \left( \frac{n-t}{s} \right).
\]

But this was our assumption.

(ii) Suppose on the contrary that \( f(w) < f(x) \) holds for some \( x, x > u \). Then, we may assume that there exist \( p, q \) which satisfy

\[
u < p < q \leq v,
\]

\[
f'(p) = f'(q) = 0,
\]

\[
f(p) < f(w) < f(q).
\]

If \( f'(x) = 0 \), it follows that

\[
\left( \frac{x}{s} \right) = \left( \frac{x}{n-t} \right) \left\{ 1 + \left( \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right) / \left( \sum_{j=0}^{x-t} \frac{1}{x-j} \right) \right\}.
\]

Substituting this into \( f(x) \), we define a new function:

\[
g(x) := -\left( \frac{x}{n-t} \right) \left( \sum_{j=0}^{n-t-1} \frac{1}{x-j} \right) / \left( \sum_{j=0}^{x-t} \frac{1}{x-j} \right).
\]

Note that \( g(x) = f(x) \) holds if \( f'(x) = 0 \). Thus, \( f(w) < g(q) \) must hold. We derive a contradiction by showing that \( f(w) \geq g(x) \) or, equivalently,

\[
\left\{ \left( \frac{w}{s} \right) - \left( \frac{w}{n-t} \right) \right\} \sum_{j=0}^{n-t-1} \frac{1}{x-j} \leq \left( \frac{x}{n-t} \right) \sum_{j=0}^{x-t} \frac{1}{x-j}.
\]
Our goal is to show $P$ in order to maximize $|A|$. Section 3 that the $u$

Next, define $f$ using Lemma 4(i), we have $x$ decreasing with $x$. On the other hand, the RHS is increasing with $x$. Therefore, it suffices to check the inequality for $x = p$, that is, $f(w) = g(p) = f(p)$. But this was our assumption.

**Lemma 5.** Let $a, b$ and $n$ be integers with $n > a + b$. Define a real valued function $f(y) := -\binom{y}{b} - \binom{y}{n-a-1}$. Then, the following hold.

(i) If $b \geq a + 3$ then $f(y) < f(n - a - 1)$ holds for $n - a - 1 < y \leq n - 1$.

(ii) If $b = a + e$ then $f(y) < f(n - a - 1)$ holds for $n - a - 1 < y \leq n - 3 + e$, $e = 0, 1, 2$.

**Proof.** Set $s := b - 1$ and $t := a + 1$.

(i) Set $v := n - 1$. Then, we have

$$1 + \frac{(n - s - t)v}{s(n + t + 1)} = \frac{(n - a - 1)(n - b + 1) - (n - a - b)}{(b - 1)(a + 1)}.$$

Using Lemma 4(i), we have $f'(y) < 0$ for $y \leq n - 1$.

(ii) Set $v := n - 4 + e$. Using Lemma 4(i), one can check $f'(y) < 0$ for $y \leq n - 4 + e$.

Next, define $u := n - 4 + e$, $v := n - 3 + e$, $w := u$. Using Lemma 4(ii), one can check $f(y) \leq f(n - 4 + e) = f(n - 3 + e)$ for $n - 4 + e \leq y \leq n - 3 + e$.

5. **Proof of Theorem 1**

Let $n > a + b$ and consider cross-intersecting families $\mathcal{A} \subset [n]_a$ and $\mathcal{B} \subset \binom{[n]}{b}$. Define $P(t) := \max\{|\mathcal{A}| + |\mathcal{B}| : \mathcal{A}, \mathcal{B} \text{ are cross-intersecting and } \mathcal{A} \text{ is nontrivial}|t\}$. Our goal is to show $P(|\mathcal{A}|) \leq P(2)$ for $2 \leq |\mathcal{A}| \leq \binom{n}{a}$.

Define the complement of $\mathcal{A}$ by $\mathcal{A}^c := \{[n] - A : A \in \mathcal{A}\} \subset \binom{[n]}{a - 1}$, and recall from Section 3 that the $b$th shadow of $\mathcal{A}^c$ is

$$\Delta_b(\mathcal{A}^c) := \left\{F \in \binom{[n]}{b} : F \cap A = \emptyset \text{ for some } A \in \mathcal{A}\right\}.$$

Since $\mathcal{A}$ is nontrivial, we have $\bigcup_{F \in \Delta_b} F = [n]$. The cross-intersecting property implies $\Delta_b(\mathcal{A}^c) \cap \mathcal{B} = \emptyset$.

**Case 1.** $|\mathcal{A}| \leq \binom{n - 1}{a - 1}$.

In this case, we assume $b \geq a + 1$ instead of $b \geq a + 2$. (We will use this part of the proof for a proof of Theorems 2 and 3 later.) Suppose that $|\mathcal{A}| = |\mathcal{A}^c| \leq \binom{n - 1}{a - 1}$ is fixed. Then, in order to maximize $|\mathcal{A}| + |\mathcal{B}|$, we have to choose $\mathcal{A}$ so that $|\Delta_b(\mathcal{A}^c)|$ is minimal. (Then
Therefore, it follows that

\[ f = \Delta_b(\mathcal{A}) \text{ has the maximal size.} \]

By the Mörs theorem, the optimal family is the following. Let \( \mathcal{F} \subset \binom{[b]}{[a]} \) be the first \( |\mathcal{A}| - 1 \) sets with respect to the colex order. Let \( \bigcup_{E \in \mathcal{F}} E = \{1, 2, \ldots, x\} \) and define \( F := \{1, \ldots, x - a\} \cup \{x + 1, \ldots, n\} \). Finally, the optimal family \( \mathcal{A} \) is given by \( \mathcal{A} = \mathcal{F} \cup \{F\} \). Then we have

\[ P(|\mathcal{A}|) = P(|\mathcal{F}| + 1) = |\mathcal{F}| + 1 + \binom{n}{b} - |\Delta_b(\mathcal{F} \cup \{F\})|. \]

**Lemma 6.** Let \( b \geq a \). For any integer \( x, n - a < x \leq n - 2 \), we have \( P(2) > P(\binom{x}{n-a} + 1) \).

**Proof.** Let \( \mathcal{A}^c = \mathcal{F} \cup \{F\} \) and \( |\mathcal{A}^c| = \binom{x}{n-a} \). In this case, \( \mathcal{F} = \binom{[b]}{[a]} \) and \( F = \{1, \ldots, x - a\} \cup \{x + 1, \ldots, n\} \) hold. Thus, \( |\Delta_b(\mathcal{A}^c)| = \binom{x}{b} + \binom{n-a}{b} - \binom{x-a}{b} \). Therefore, we have

\[ P \left( \binom{x}{n-a} + 1 \right) = \binom{x}{n-a} + 1 + \binom{n}{b} - \binom{x}{b} - \binom{n-a-1}{b-1} - \binom{x-a}{b-1}. \]

Let \( f(x) := \binom{x}{n-a} - \binom{x}{b} + \binom{x-a}{b} \). We want to show \( f(x) < f(n-a) \) for \( n-a < x \leq n-2 \).

Let us define \( g(x) := f(x) - f(x+1) \). It suffices to show \( g(n-a+e) > 0 \) for \( 0 \leq e \leq a-3 \). This follows from Lemma 1 by setting \( j = 0 \).

**Lemma 7.** Let \( b \geq a \). For any integer \( x, n - a < x \leq n - 2 \), we have \( P(\binom{x}{n-a} + 1) \geq P(\binom{x}{n-a} + 2) \).

**Proof.** We calculated \( P(\binom{x}{n-a} + 1) \) in the proof of Lemma 6. Now we consider the case \( |\mathcal{A}^c| = \binom{x}{n-a} + 2 = \binom{x}{n-a} + \binom{n-a-1}{b} + 1 \). This time, we have \( \mathcal{F} = \binom{[b]}{[a]} \cup \{1, \ldots, n-a-1, x+1\} \) and \( F = \{1, \ldots, x - a + 1\} \cup \{x + 2, \ldots, n\} \). Thus,

\[ P \left( \binom{x}{n-a} + 2 \right) = \binom{x}{n-a} + 2 + \binom{n}{b} - \binom{x}{b} - \binom{n-a-1}{b-1} - \binom{x-a}{b-1}. \]

Therefore,

\[ P \left( \binom{x}{n-a} + 1 \right) - P \left( \binom{x}{n-a} + 2 \right) = \binom{n-a-1}{b-1} - \binom{x-a}{b-1} - 1 \geq 0. \]

**Lemma 8.** Let \( b \geq a + 1 \), and let \( x \) be an integer with \( n - a \leq x \leq n - 2 \). If \( \binom{x}{n-a} + 2 \leq |\mathcal{A}| \leq \binom{x+1}{n-a} + 1 \) then \( P(\binom{x}{n-a} + 2) \geq P(|\mathcal{A}|) \).

**Proof.** Choose a real \( y, n - a - 1 \leq y < x \), so that \( |\mathcal{A}| = \binom{x}{n-a} + \binom{y}{n-a-1} + 1 \). In this case, it follows that \( \mathcal{A}^c = \mathcal{F} \cup \{F\} \),

\[ \mathcal{F} \subset \left[ \frac{[x]}{n-a} \right] \cup \left\{ G \cup \{x+1\} : G \in \left( \left[ \frac{[y]}{n-a-1} \right] \right) \right\}, \]

\[ F = \{1, \ldots, x-a+1\} \cup \{x+2, \ldots, n\}. \]
Using the Kruskal–Katona theorem ([7], [8], [9]), we have

\[ P(|\mathcal{A}|) = |\mathcal{A}| + \binom{n}{b} - |\Delta_b(\mathcal{A}^c)| \]

\[ \leq \left( \frac{x}{n-a} \right) + \left( \frac{y}{n-a-1} \right) + 1 + \left( \frac{n}{b} - \frac{x}{b} \right) - \left( \frac{y}{b-1} \right) - \left( \frac{n-a}{b} \right) + \left( \frac{x-a+1}{b} \right). \]

Now define a real valued function \( f(y) := -(y_{b-1} + (y_{a-1}) \) for \( n-a-1 \leq y \leq n-1 \). By Lemma 5, we have \( f(y) \leq f(n-a-1) \), that is, \( P(|\mathcal{A}|) \leq P(\left( \frac{n-a}{b} + 2 \right) \). Equality holds if and only if \( x = n-a-1 \), that is, \( |\mathcal{A}| = \left( \frac{x}{b} \right) + 2 \).

By Lemmas 6, 7 and 8, we have

\[ P(2) > P(|\mathcal{A}|) \text{ for } 2 < |\mathcal{A}| \leq \binom{n-1}{a-1}. \]

Equality holds only if \( a = b = 2 \). Since we have assumed \( b \geq a + 1 \), we obtain \( P(2) > P(|\mathcal{A}|) \).

**Case 2.** \( |\mathcal{A}| > \binom{n-1}{a-1} \).

By the Erdős–Ko–Rado theorem ([1]), \( \mathcal{A} \) is nontrivial no matter how we choose \( \mathcal{A} \). Suppose that \( |\mathcal{A}| = |\mathcal{A}^c| > \binom{n-1}{a-1} \) is fixed. Then, to maximize \( |\mathcal{A}| + |\mathcal{A}| \), we have to choose \( \mathcal{A} \) so that \( |\Delta_b(\mathcal{A}^c)| \) is minimal. By the Kruskal–Katona theorem, we may assume that \( \mathcal{A}^c \) is the first \( |\mathcal{A}| \) sets with respect to the colex order. Choose a real \( y, n-a-1 \leq y \leq n-1 \), so that \( |\mathcal{A}^c| = \left( \frac{n-1}{a-1} \right) + \left( \frac{y}{a-1} \right) \). Then we have

\[ P(|\mathcal{A}|) = |\mathcal{A}| + \binom{n}{b} - |\Delta_b(\mathcal{A}^c)| \]

\[ \leq \left( \frac{n-1}{n-a} \right) + \left( \frac{y}{n-a-1} \right) + \left( \frac{n}{b} - \frac{n-1}{b} \right) - \left( \frac{y}{b-1} \right). \]

Let us define a real valued function \( f(y) := -(y_{b-1} + (y_{a-1}) \) for \( n-a-1 \leq y \leq n-1 \). Then, by our assumption \( b \geq a + 2 \) and Lemma 5, we have \( f(y) \leq f(n-a-1) \). Thus,

\[ P(|\mathcal{A}|) \leq P \left( \frac{n-1}{a-1} + 1 \right) = P \left( \frac{n-1}{a-1} \right) + 1 - \left( \frac{n-a-1}{b-1} \right) < P(2). \]

This completes the proof of Theorem 1. \( \square \)

**6. Proof of Theorem 2**

The proof is similar to the proof of Theorem 1. We leave some of the computations in the proof of Theorem 2 to the reader. We use the same definitions and notation as in the proof of Theorem 1.

**Proof of Theorem 2 (i)**

**Case 1.** \( |\mathcal{A}| \leq \binom{n-1}{a-1} \).

The proof of this case is exactly same as the proof of Theorem 1.
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Note that if $h$ is odd, then $\Delta_0(\mathcal{A}) = \mathcal{A}$, so that $|\mathcal{A}| = (n-1) + (y-a)$. Then we have

$$P(|\mathcal{A}|) = |\mathcal{A}| + \binom{n}{b} - |\Delta_0(\mathcal{A})| \leq \left( \frac{n-1}{n-a} \right) + \left( \binom{y}{n-a-1} + \binom{n}{b} - \binom{n-1}{b} - \binom{y}{b-1} \right).$$

Let us define a real valued function $f(y) := -\binom{y}{b-1} + \binom{y-a}{n-a-1}$ for $n-a-1 \leq y \leq n-2$. Then, by Lemma 5, we have $f(y) \leq f(n-a-1)$. Thus,

$$P(|\mathcal{A}|) \leq P\left( \left( \frac{n-1}{a-1} + 1 \right) = P\left( \left( \frac{n-1}{a-1} \right) + 1 - \left( \frac{n-a-1}{b-1} \right) < P(2). \right)$$

Proof of Theorem 2 (ii)

Let us settle the case $a = b = 2$ first. In this case, it is not difficult to check that $|\mathcal{A}| + |\mathcal{B}| \leq 6 = |\mathcal{A}_0| + |\mathcal{B}_0|$ by hand. Equality holds if and only if $\mathcal{A}, \mathcal{B} \cong \{\mathcal{A}_0, \mathcal{B}_0\}$ or $\mathcal{A} = \mathcal{B} = \{12, 13, 23\}$ or $\mathcal{A} = \mathcal{B} = \{12, 23, 34\}, \{13, 23, 24\}$.

From now on, we assume $a = b \geq 3$.

Case 1. $|\mathcal{A}| \leq \binom{n-2}{a-2} + \binom{n-2}{a-2}$.

We follow the proof of Theorem 1. This time, Lemmas 6 and 7 are still valid. Instead of Lemma 8, we use the following.

**Lemma 9.** Let $x$ be any integer with $n-a \leq x \leq n-3$. If $\binom{x}{n-a} + 2 \leq |\mathcal{A}| \leq \binom{x+1}{n-a}$ then

$$P\left( \binom{x}{n-a} + 2 \right) \geq P(|\mathcal{A}|) \quad \square$$

We can prove the above lemma in exactly the same way as in the proof of Lemma 8. Now using Lemmas 6, 7, 9, it follows that $P(2) < P(|\mathcal{A}|)$ for $2 \leq |\mathcal{A}| \leq \binom{n-2}{a-2} + \binom{n-2}{a-2}$.

Case 2. $\binom{n-2}{a-2} + \binom{n-2}{a-2} < |\mathcal{A}| \leq \binom{n-1}{a-1} + \binom{n-1}{a-2} + 1 = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$.

For an integer $x$, $2 \leq x \leq a + 1$, let us define

$$g(x) := \binom{n-2}{a-2} + \cdots + \binom{n-x}{a-2} + 1,$$

$$h(x) := \binom{n}{a} - \binom{n-2}{a-2} - \cdots - \binom{n-x}{a-2} - \binom{n-a-1}{a-1}.$$

Note that if $|\mathcal{A}| = g(x)$ then, by the Kruskal–Katona theorem, we have $|\mathcal{B}| \leq h(x)$. Note also that $h(a + 1) = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1 = g(a + 1)$. Thus, if $|\mathcal{A}| \geq g(a + 1)$ then $|\mathcal{B}| \leq g(a + 1)$.

**Lemma 10.** For any integer $x$, $2 \leq x \leq a + 1$, we have $P(2) > P(g(x))$. 

The above inequality follows from Lemma 2.

We have

\[ P(g(2)) \geq P(g(3)) = P(g(4)) \leq P(g(5)) \leq \cdots \leq P(g(a+1)). \]

Thus, it suffices to show \( P(2) > P(g(a+1)) \). Note that

\[ g(a+1) = h(a+1) = \binom{n-1}{a+1} - \binom{n-a-1}{a-1} + 1, \]

and

\[ P(a+1) = 2g(a+1) = 2\binom{n-1}{a+1} - 2\binom{n-a-1}{a-1} + 2. \]

Therefore, the desired inequality \( P(2) > P(g(a+1)) \) is equivalent to

\[ \binom{n-1}{a} + \binom{n-2a}{a} - 2\binom{n-a-1}{a} - \binom{n-1}{a-1} > 0. \]

The above inequality follows from Lemma 2. \( \square \)

**Lemma 11.** For any integer \( x, 2 \leq x \leq a \), we have \( P(g(x)) > P(g(x)+1) \).

**Proof.** If \( |\mathcal{A}| = g(x) + 1 = g(x) + \binom{n-x-a+1}{n-x-a+1} \), then by the Mörs theorem, we have

\[ |\mathcal{A}| \leq h(x) - \binom{n-x-a+1}{n-x-a+1}. \]

Thus, \( P(g(x)) > P(g(x)+1) \) is equivalent to \( \binom{n-x-a+1}{n-x-a+1} > 1 \). This follows from our assumption \( n > 2a \). \( \square \)

**Lemma 12.** Let \( x \) be an integer with \( 2 \leq x \leq a \). If \( g(x) + 1 \leq |\mathcal{A}| \leq g(x+1) \) then

\[ P(|\mathcal{A}|) \leq \max\{P(g(x)+1), P(g(x+1))\} \]

**Proof.** Choose a real \( y, n-x-a+1 \leq y \leq n-x-1 \), so that \( |\mathcal{A}| = g(x) + \binom{y}{n-x-a+1} \).

(Note that if \( y = n-x-1 \) then \( |\mathcal{A}| = g(x+1) \).) Using the Kruskal–Katona theorem, we have

\[ |\mathcal{A}| \leq h(x) - \binom{y}{n-x-a+1}. \]

Now define a real valued function \( f(y) := -\binom{y}{n-x-a+1} + \binom{y}{n-x-a+1} \) for \( n-x-a+1 \leq y \leq n-x-1 \). Our goal is to show \( f(y) \leq \max\{f(n-x-a+1), f(n-x-1)\} \).

First we settle the case \( x = a \). In this case, we have \( f(y) = -\binom{y}{n-2a+1} \). Since \( n > 2a \),

\( f(y) \) is an increasing function. Thus, \( f(y) \leq f(n-a-1) \) holds.

From now on, we assume \( x < a \). Set \( s := a-x+1, t := x+a-1, \) and \( v := n-2x \).

Using Lemma 4(i), one can check that \( f'(y) < 0 \) holds for \( y \leq n-2x \). Thus, we have

\( f'(n-x-a+1) < 0 \). Therefore, \( f(y) \leq \max\{f(n-x-a+1), f(n-x-1)\} \) follows from Lemma 4(ii). \( \square \)

By Lemmas 10, 11, 12, we have

\[ P(|\mathcal{A}|) \leq \max\{P(g(2)), P(g(a+1))\} < P(2). \]
7. Proof of Theorem 3

Recall that \( P(|\mathcal{A}|) = \max \{|\mathcal{A}| + |\mathcal{B}|\} \) (see Section 5). If \( b \geq a + 2 \), then the theorem follows from Theorem 1.

**Case 1.** \( b = a + 1 \).

If \( |\mathcal{A}| \leq \binom{n-1}{a-1} + \binom{n-2}{a-1} \) then the desired inequality \( P(2) > P(|\mathcal{A}|) \) for \( 2 < |\mathcal{A}| \leq \binom{n-1}{a-1} + \binom{n-2}{a-1} \) follows from Theorem 2. So we may assume \( |\mathcal{A}| \leq \binom{n-1}{a-1} + \binom{n-2}{a-1} \). Then, by the Kruskal–Katona theorem, we have

\[
|\mathcal{B}| \leq \binom{n}{b} - |\Delta_6(\mathcal{A})| \leq (\binom{n-1}{a-1} + \binom{n-2}{a-1}) - (\binom{n-2}{a-1}) = (\binom{n-1}{a-1} + \binom{n-2}{a-1}) - (\binom{n-3}{a-2} + \binom{n-2}{a-1}).
\]

Define

\[
Q(t) := \max \{|\mathcal{A}| + |\mathcal{B}| : |\mathcal{B}| = t, \mathcal{A} \text{ and } \mathcal{B} \text{ are cross-intersecting and } \mathcal{B} \text{ is nontrivial}\}.
\]

Let \( |\mathcal{B}| = \binom{n-y}{a-y-1} + 1 \) for \( n-a-1 \leq y < n-2 \). Then we have \( Q(|\mathcal{B}|) \leq f(y) + \text{(constant)} \), where \( f(y) := -\binom{n-y}{a-y-1} \). Using Lemma 5, one can check that \( f(y) < f(n-a-1) \) for \( n-a-1 < y < n-2 \), that is, \( Q(2) > Q(|\mathcal{B}|) \) for \( 2 < |\mathcal{B}| \leq \binom{n-2}{a-2} + \binom{n-2}{a-1} \). Using Lemma 3, we have \( P(2) > Q(2) \). This completes the proof of this case.

**Case 2.** \( b = a \).

Without loss of generality, we may assume that \( |\mathcal{A}| \leq |\mathcal{B}| \). If \( |\mathcal{A}| \geq \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1 \) then \( |\mathcal{B}| \leq \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1 \). Then the result follows from Theorem 2. □

References