THE PATH TO RECENT PROGRESS ON SMALL GAPS BETWEEN PRIMES

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1. Introduction

In the articles *Primes in Tuples I & II* ([13], [14]) we have presented the proofs of some assertions about the existence of small gaps between prime numbers which go beyond the hitherto established results. Our method depends on tuple approximations. However, the approximations and the way of applying the approximations has changed over time, and some comments in this paper may provide insight as to the development of our work.

First, here is a short narration of our results. Let

$$\theta(n) := \begin{cases} \log n & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and

$$\Theta(N; q, a) := \sum_{\substack{n \le N \\ n \equiv a \ (\mathrm{mod}\ q)}} \theta(n). \tag{2}$$

In this paper N will always be a large integer, p will denote a prime number, and p_n will denote the n-th prime. The prime number theorem says that

$$\lim_{x \to \infty} \frac{|\{p : \ p \le x\}|}{\frac{x}{\log x}} = 1, \tag{3}$$

and this can also be expressed as

$$\sum_{n \le x} \theta(n) \sim x \qquad \text{as } x \to \infty. \tag{4}$$

It follows trivially from the prime number theorem that

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} \le 1. \tag{5}$$

By combining former methods with a construction of certain (rather sparsely distributed) intervals which contain more primes than the expected number by a factor of e^{γ}, Maier [25] had reached the best known result in this direction that

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} \le 0.24846... \ . \tag{6}$$

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It is natural to expect that modulo q the primes would be almost equally distributed in the reduced residue classes. The deepest knowledge on primes which plays a role in our method concerns a measure of the distribution of primes in reduced residue classes referred to as the level of distribution of primes in arithmetic progressions. We say that the primes have *level of distribution* α if

$$\sum_{q\le Q}\max_{\substack{a\\(a,q)=1}}\left|\Theta(N;q,a)-\frac{N}{\phi(q)}\right| \ll \frac{N}{(\log N)^A} \tag{7}$$

holds for any $A>0$ and any arbitrarily small fixed $\epsilon>0$ with

$$Q=N^{\alpha-\epsilon}. \tag{8}$$

The *Bombieri-Vinogradov theorem* provides the level $\frac{1}{2}$, while the *Elliott-Halberstam conjecture* asserts that the primes have level of distribution 1.

The Bombieri-Vinogradov theorem allows taking $Q=N^{\frac{1}{2}}(\log N)^{-B(A)}$ in (7), by virtue of which we have proved unconditionally in [13] that for any fixed $r\ge 1$,

$$\liminf_{n\to\infty}\frac{p_{n+r}-p_n}{\log p_n}\le(\sqrt{r}-1)^2\ ; \tag{9}$$

in particular,

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}=0. \tag{10}$$

In fact, assuming that the level of distribution of primes is α, we obtain more generally than (9) that, for $r\ge 2$,

$$\liminf_{n\to\infty}\frac{p_{n+r}-p_n}{\log p_n}\le(\sqrt{r}-\sqrt{2\alpha})^2. \tag{11}$$

Furthermore, assuming that $\alpha>\frac{1}{2}$, there exists an explicitly calculable constant $C(\alpha)$ such that for $k\ge C(\alpha)$ any sequence of k-tuples

$$\{(n+h_1,n+h_2,\dots,n+h_k)\}_{n=1}^{\infty}, \tag{12}$$

with the set of distinct integers $\mathcal{H}=\{h_1,h_2,\dots,h_k\}$ *admissible* in the sense that $\prod_{i=1}^{k}(n+h_i)$ has no fixed prime factor for every n, contains at least two primes infinitely often. For instance if $\alpha\ge 0.971$, then this holds for $k\ge 6$, giving

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\le 16, \tag{13}$$

in view of the shortest admissible 6-tuple $(n,n+4,n+6,n+10,n+12,n+16)$.

We note that the gaps obeying Eq.s (9)-(11) constitute a positive proportion of all gaps of the corresponding kind. By incorporating Maier's method into ours we improved (9) to

$$\liminf_{n\to\infty}\frac{p_{n+r}-p_n}{\log p_n}\le e^{-\gamma}(\sqrt{r}-1)^2, \tag{14}$$

but for these gaps we don't have a proof of there being a positive proportion of all gaps of this kind. (These results will appear in forthcoming articles).

In [14] the result (10) was considerably improved to

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^{\frac{1}{2}}(\log\log p_n)^2}<\infty. \tag{15}$$

In fact, the methods of [14] lead to a much more general result: When $\mathcal{A} \subseteq \mathbb{N}$ is a sequence satisfying $\mathcal{A}(N) := |\{n; n \le N, n \in \mathcal{A}\}| > C(\log N)^{1/2}(\log\log N)^2$ for all sufficiently large N, infinitely many of the differences of two elements of $\mathcal{A}$ can be expressed as the difference of two primes.

2. Former approximations by truncated divisor sums

The von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & \text{if} \quad n = p^m,\ m \in \mathbb{Z}^+, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

can be expressed as

$$\Lambda(n) = \sum_{d|n} \mu(d)\log(\frac{R}{d}) \qquad \text{for } n > 1. \tag{17}$$

Since the proper prime powers contribute negligibly, the prime number theorem (4) can be rewritten as

$$\psi(x) := \sum_{n \le x} \Lambda(n) \sim x \qquad \text{as } x \to \infty. \tag{18}$$

It is natural to expect that the truncated sum

$$\Lambda_R(n) := \sum_{\substack{d|n \\ d \le R}} \mu(d)\log(\frac{R}{d}) \qquad \text{for } n \ge 1. \tag{19}$$

mimics the behaviour of $\Lambda(n)$ on some averages.

The beginning of our line of research is Goldston's [6] alternative rendering of the proof of Bombieri and Davenport's theorem on small gaps between primes. Goldston replaced the application of the circle method in the original proof by the use of the truncated divisor sum (19). The use of functions like $\Lambda_R(n)$ goes back to Selberg's work [27] on the zeros of the Riemann zeta-function $\zeta(s)$. The most beneficial feature of the truncated divisor sums is that they can be used in place of $\Lambda(n)$ on some occasions when it is not known how to work with $\Lambda(n)$ itself. The principal such situation arises in counting the primes in tuples. Let

$$\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \ \text{ with } \ 1 \le h_1, \ldots, h_k \le h \ \text{ distinct integers} \tag{20}$$

(the restriction of h_i to positive integers is inessential; the whole set $\mathcal{H}$ can be shifted by a fixed integer with no effect on our procedure), and for a prime p denote by $\nu_p(\mathcal{H})$ the number of distinct residue classes modulo p occupied by the elements of $\mathcal{H}$. The singular series associated with the k-tuple $\mathcal{H}$ is defined as

$$\mathfrak{S}(\mathcal{H}) := \prod_p (1 - \frac{1}{p})^{-k}(1 - \frac{\nu_p(\mathcal{H})}{p}). \tag{21}$$

Since $\nu_p(\mathcal{H}) = k$ for $p > h$, the product is convergent. The admissibility of $\mathcal{H}$ is equivalent to $\mathfrak{S}(\mathcal{H}) \ne 0$, and to $\nu_p(\mathcal{H}) \ne p$ for all primes. Hardy and Littlewood [23] conjectured that

$$\sum_{n \le N} \Lambda(n; \mathcal{H}) := \sum_{n \le N} \Lambda(n+h_1)\cdots\Lambda(n+h_k) = N(\mathfrak{S}(\mathcal{H}) + o(1)), \quad \text{as } N \to \infty. \tag{22}$$

The prime number theorem is the $k = 1$ case, and for $k \geq 2$ the conjecture remains unproved. (This conjecture is trivially true if $\mathcal{H}$ is inadmissible).

A simplified version of Goldston's argument in [6] was given in [17] as follows. To obtain information on small gaps between primes, let

(23)
$$\psi(n,h) := \psi(n+h) - \psi(n,h) = \sum_{n<m\leq n+h} \Lambda(m), \quad \psi_R(n,h) := \sum_{n<m\leq n+h} \Lambda_R(m),$$

and consider the inequality

$$\sum_{N<n\leq 2N} (\psi(n,h) - \psi_R(n,h))^2 \geq 0. \tag{24}$$

The strength of this inequality depends on how well $\Lambda_R(n)$ approximates $\Lambda(n)$. On multiplying out the terms and using from [6] the formulas

(25)
$$\sum_{n\leq N} \Lambda_R(n)\Lambda_R(n+k) \sim \mathfrak{S}(\{0,k\})N, \quad \sum_{n\leq N} \Lambda(n)\Lambda_R(n+k) \sim \mathfrak{S}(\{0,k\})N \quad (k \neq 0)$$

(26)
$$\sum_{n\leq N} \Lambda_R(n)^2 \sim N\log R, \quad \sum_{n\leq N} \Lambda(n)\Lambda_R(n) \sim N\log R,$$

valid for $|k| \leq R \leq N^{\frac{1}{2}}(\log N)^{-A}$, gives, taking $h = \lambda \log N$ with $\lambda \ll 1$,

(27)
$$\sum_{N<n\leq 2N} (\psi(n+h) - \psi(n))^2 \geq (hN\log R + Nh^2)(1 - o(1)) \geq (\frac{\lambda}{2} + \lambda^2 - \epsilon)N(\log N)^2$$

(in obtaining this one needs the two-tuple case of Gallagher's singular series average given in (46) below, which can be traced back to Hardy and Littlewood's and Bombieri and Davenport's work). If the interval $(n, n+h]$ never contains more than one prime, then the left-hand side of (27) is at most

$$\log N \sum_{N<n\leq 2N} (\psi(n+h) - \psi(n)) \sim \lambda N(\log N)^2, \tag{28}$$

which contradicts (27) if $\lambda > \frac{1}{2}$, and thus one obtains

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \leq \frac{1}{2}. \tag{29}$$

Later on Goldston et al. in [3], [4], [7], [15], [16], [18] applied this lower-bound method to various problems concerning the distribution of primes and in [8] to the pair correlation of zeros of the Riemann zeta-function. In most of these works the more delicate divisor sum

$$\lambda_R(n) := \sum_{r\leq R} \frac{\mu^2(r)}{\phi(r)} \sum_{d|(r,n)} d\mu(d) \tag{30}$$

was employed especially because it led to better conditional results which depend on the Generalized Riemann Hypothesis.

The left-hand side of (27) is the second moment for primes in short intervals. Gallagher [5] showed that the Hardy-Littlewood conjecture (22) implies that the moments for primes in intervals of length $h \sim \lambda \log N$ are the moments of a Poisson distribution with mean λ. In particular, it is expected that

$$\sum_{n \le N} (\psi(n+h) - \psi(n))^2 \sim (\lambda + \lambda^2) N (\log N)^2 \tag{31}$$

which in view of (28) implies (10) but is probably very hard to prove. It is known from the work of Goldston and Montgomery [12] that assuming the Riemann Hypothesis, an extension of (31) for $1 \le h \le N^{1-\epsilon}$ is equivalent to a form of the pair correlation conjecture for the zeros of the Riemann zeta-function. We thus see that the factor $\frac{1}{2}$ in (27) is what is lost from the truncation level R, and an obvious strategy is to try to improve on the range of R where (25)-(26) are valid. In fact, the asymptotics in (26) are known to hold for $R \le N$ (the first relation in (26) is a special case of a result of Graham [21]). It is easy to see that the second relation in (25) will hold with $R = N^{\alpha-\epsilon}$, where α is the level of distribution of primes in arithmetic progressions. For the first relation in (25) however, one can prove the the formula is valid for $R = N^{1/2+\eta}$ for a small $\eta > 0$, but unless one also assumes a somewhat unnatural level of distribution conjecture for Λ_R, one can go no further. Thus increasing the range of R in (25) is not currently possible.

However, there is another possible approach motivated by Gallagher's work [5]. In 1999 the first and third authors discovered how to calculate some of the higher moments of the short divisor sums (19) and (30). At first this was achieved through straightforward summation and only the triple correlations of $\Lambda_R(n)$ were worked out in [17]. In applying these formulas, the idea of finding approximate moments with some expressions corresponding to (24) was eventually replaced with

$$\sum_{N < n \le 2N} (\psi(n,h) - \rho \log N)(\psi_R(n,h) - C)^2 \tag{32}$$

which if positive for some $\rho > 1$ implies that for some n we have $\psi(n,h) \ge 2 \log N$. Here C is available to optimize the argument. Thus the problem was switched from trying to find a good fit for $\psi(n,h)$ with a short divisor sum approximation to the easier problem of trying to maximize a given quadratic form, or more generally a mollification problem. With just third correlations this resulted in (29), thus giving no improvement over Bombieri and Davenport's result. Nevertheless the new method was not totally fruitless since it gave

$$\liminf_{n \to \infty} \frac{p_{n+r} - p_n}{\log p_n} \le r - \frac{\sqrt{r}}{2}, \tag{33}$$

whereas the argument leading to (29) gives $r - \frac{1}{2}$. Independently of us, Sivak [29] incorporated Maier's method into [17] and improved upon (33) by the factor $e^{-\gamma}$ (cf. (6) and (14)).

Following [17], with considerable help from other mathematicians, in [20] the k-level correlations of $\Lambda_R(n)$ were calculated. This leap was achieved through replacing straightforward summation with complex integration upon the use of Perron type formulae. Thus it became feasible to approximate $\Lambda(n, \mathcal{H})$ which was defined in (22) by

$$\Lambda_R(n; \mathcal{H}) := \Lambda_R(n+h_1)\Lambda_R(n+h_2)\cdots\Lambda_R(n+h_k). \tag{34}$$

Writing

$$\Lambda_R(n;\mathbf{H}) := (\log R)^{k-|\mathcal{H}|}\Lambda_R(n;\mathcal{H}), \quad \psi_R^{(k)}(n,h) := \sum_{1\le h_1,\dots,h_k\le h}\Lambda_R(n;\mathbf{H}), \tag{35}$$

where the distinct components of the k-dimensional vector $\mathbf{H}$ are the elements of the set $\mathcal{H}$, $\psi_R^{(j)}(n,h)$ provided the approximation to $\psi(n,h)^j$, and the expression

$$\sum_{N<n\le 2N}(\psi(n,h)-\rho\log N)(\sum_{j=0}^{k}a_j\psi_R^{(j)}(n,h)(\log R)^{k-j})^2 \tag{36}$$

could be evaluated. Here the a_j are constants available to optimize the argument. The optimization turned out to be a rather complicated problem which will not be discussed here, but the solution was recently completed in [19] with the result that for any fixed $\lambda > (\sqrt{r}-\sqrt{\frac{\alpha}{2}})^2$ and N sufficiently large,

$$\sum_{\substack{n\le N\\ p_{n+r}-p_n\le\lambda\log p_n}} 1 \gg_r \sum_{\substack{p\le N\\ p:\,\text{prime}}} 1. \tag{37}$$

In particular, unconditionally, for any fixed $\eta > 0$ and for all sufficiently large $N > N_0(\eta)$, a positive proportion of gaps $p_{n+1}-p_n$ with $p_n \le N$ are smaller than $(\frac{1}{4}+\eta)\log N$. This is numerically a little short of Maier's result (6), but (6) was shown to hold for a sparse sequence of gaps. The work [19] also turned out to be instrumental in Green and Tao's [22] proof that the primes contain arbitrarily long arithmetic progressions.

The efforts made in 2003 using divisor sums which are more complicated than $\Lambda_R(n)$ and $\lambda_R(n)$ gave rise to more difficult calculations and didn't meet with success. During this work Granville and Soundararajan provided us with the idea that the method should be applied directly to individual tuples rather than sums over tuples which constitute approximations of moments. They replaced the earlier expressions with

$$\sum_{N<n\le 2N}(\sum_{h_i\in\mathcal{H}}\Lambda(n+h_i)-r\log 3N)(\tilde{\Lambda}_R(n;\mathcal{H}))^2, \tag{38}$$

where $\tilde{\Lambda}_R(n;\mathcal{H})$ is a short divisor sum which should be large when $\mathcal{H}$ is a prime tuple. This is the type of expression which is used in the proof of the result described in connection with (12)–(13) above. However, for obtaining the results (9)–(11). we need arguments based on using (32) and (36).

3. Detecting prime tuples

We call the tuple (12) a *prime tuple* when all of its components are prime numbers. Obviously this is equivalent to requiring that

$$P_{\mathcal{H}}(n) := (n+h_1)(n+h_2)\cdots(n+h_k) \tag{39}$$

is a product of k primes. As the generalized von Mangoldt function

$$\Lambda_k(n) := \sum_{d|n}\mu(d)(\log\frac{n}{d})^k \tag{40}$$

vanishes when n has more than k distinct prime factors, we may use

$$\frac{1}{k!}\sum_{\substack{d|P_{\mathcal{H}}(n)\\ d\le R}}\mu(d)(\log\frac{R}{d})^{k} \tag{41}$$

for approximating prime tuples. (Here $1/k!$ is just a normalization factor. That (41) will be also counting some tuples by including proper prime power factors doesn't pose a threat since in our applications their contribution is negligible). But this idea by itself brings restricted progress: now the right-hand side of (6) can be replaced with $1-\frac{\sqrt{3}}{2}$.

The efficiency of the argument is greatly increased if instead of trying to include tuples composed only of primes, one looks for tuples with primes in many components. So in [13] we employ

$$\Lambda_R(n;\mathcal{H},\ell):=\frac{1}{(k+\ell)!}\sum_{\substack{d|P_{\mathcal{H}}(n)\\ d\le R}}\mu(d)(\log\frac{R}{d})^{k+\ell}, \tag{42}$$

where $|\mathcal{H}|=k$ and $0\le\ell\le k$, and consider those $P_{\mathcal{H}}(n)$ which have at most $k+\ell$ distinct prime factors. In our applications the optimal order of magnitude of the integer ℓ turns out to be about $\sqrt{k}$. To implement this new approximation in the skeleton of the argument, the quantities

$$\sum_{n\le N}\Lambda_R(n;\mathcal{H}_1,\ell_1)\Lambda_R(n;\mathcal{H}_2,\ell_2), \tag{43}$$

and

$$\sum_{n\le N}\Lambda_R(n;\mathcal{H}_1,\ell_1)\Lambda_R(n;\mathcal{H}_2,\ell_2)\theta(n+h_0), \tag{44}$$

are calculated as $R,\ N\to\infty$. The latter has three cases according as $h_0\notin\mathcal{H}_1\cup\mathcal{H}_2$, or $h_0\in\mathcal{H}_1\setminus\mathcal{H}_2$, or $h_0\in\mathcal{H}_1\cap\mathcal{H}_2$. Here $M=|\mathcal{H}_1|+|\mathcal{H}_2|+\ell_1+\ell_2$ is taken as a fixed integer which may be arbitrarily large. The calculation of (43) is valid with R as large as $N^{\frac{1}{2}-\epsilon}$ and $h\le R^C$ for any constant $C>0$. The calculation of (44) can be carried out for R as large as $N^{\frac{\alpha}{2}-\epsilon}$ and $h\le R$. It should be noted that in [19] in the same context the usage of (34) which has k truncations, restricted the range of the divisors greatly, for then $R\le N^{\frac{1}{4k}-\epsilon}$ was needed. Moreover the calculations were more complicated compared to the present situation of dealing with only one truncation.

Requiring the positivity of the quantity

$$\sum_{n=N+1}^{2N}\Big(\sum_{1\le h_0\le h}\theta(n+h_0)-r\log 3N\Big)\Big(\sum_{\substack{\mathcal{H}\subset\{1,2,\ldots,h\}\\ |\mathcal{H}|=k}}\Lambda_R(n;\mathcal{H},\ell)\Big)^2,\qquad (h=\lambda\log 3N), \tag{45}$$

which can be calculated easily from asymptotic formulas for (43) and (44), and Gallagher's [5] result that with the notation of (20) for fixed k

$$\sum_{\mathcal{H}}\mathfrak{S}(\mathcal{H})\sim h^k\qquad \text{as}\quad h\to\infty, \tag{46}$$

yields the results (9)–(11). For the proof of the result mentioned in connection with (12), the positivity of (38) with $r=1$ and $\Lambda_R(n;\mathcal{H},\ell)$ for an $\mathcal{H}$ satisfying (20) in place of $\tilde{\Lambda}_R(n;\mathcal{H})$ is used. For (13), the positivity of an optimal linear combination of the quantities for (12) is pursued.

The proof of (15) in [14] also depends on the positivity of (45) for $r = 1$ and $h = \frac{C\log N}{k}$ modified with the extra restriction

$$(P_{\mathcal{H}}(n), \prod_{p\le\sqrt{\log N}} p) = 1 \tag{47}$$

on the tuples to be summed over, but involves some essential differences from the procedure described above. Now the size of k is taken as large as $c\frac{\sqrt{\log N}}{(\log\log N)^2}$ (where c is a sufficiently small explicitly calculable absolute constant). This necessitates a much more refined treatment of the error terms arising in the argument, and in due course the restriction (47) is brought in to avoid the complications arising from the possibly irregular behaviour of $\nu_p(\mathcal{H})$ for small p. In the new argument a modified version of the Bombieri-Vinogradov theorem is needed. Roughly speaking, in the version developed for this purpose, compared to (7) the range of the moduli q is curtailed a little bit in return for a little stronger upper-bound. Moreover, instead of Gallagher's result (46) which was for fixed k (though the result may hold for k growing as some function of h, we do not know exactly how large this function can be in addition to dealing with the problem of non-uniformity in k), the weaker property that $\sum_{\mathcal{H}}\mathfrak{S}(\mathcal{H})/h^k$ is non-decreasing (apart from a factor of $1+o(1)$) as a function of k is proved and employed. The whole argument is designed to give the more general result which was mentioned after (15).

4. Small gaps between almost primes

In the context of our work by *almost prime* we mean an E_2*-number*, i.e. a natural number which is a product of two distinct primes. We have been able to apply our methods to finding small gaps between almost primes in collaboration with S. W. Graham. For this purpose a Bombieri-Vinogradov type theorem for $\Lambda * \Lambda$ is needed, and the work of Motohashi [26] on obtaining such a result for the Dirichlet convolution of two sequences is readily applicable (see also [1]). In [9] alternative proofs of some results of [13] such as (10) and (13) are given couched in the formalism of the Selberg sieve. Denoting by q_n the n-th E_2-number, in [9] and [10] it is shown that there is a constant C such that for any positive integer r,

$$\liminf_{n\to\infty}(q_{n+r}-q_n) \le Cre^r; \tag{48}$$

in particular

$$\liminf_{n\to\infty}(q_{n+1}-q_n) \le 6. \tag{49}$$

Furthermore in [11] proofs of a strong form of the Erdös–Mirsky conjecture and related assertions have been obtained.

5. Further remarks on the origin of our method

In 1950 Selberg was working on applications of his sieve method to the twin prime and Goldbach problems and invented a weighted sieve method that gave results which were later superseded by other methods and thereafter largely neglected. Much later in 1991 Selberg published the details of this work in Volume II of his Collected Works [28], describing it as "by now of historical interest only". In 1997

Heath-Brown [24] generalized Selberg's argument from the twin prime problem to the problem of almost prime tuples. Heath-Brown let

$$\Pi = \prod_{i=1}^{k}(a_i n + b_i) \tag{50}$$

with certain natural conditions on the integers a_i and b_i. Then the argument of Selberg (for the case $k = 2$) and Heath-Brown for the general case is to choose $\rho > 0$ and the numbers λ_d of the Selberg sieve so that, with τ the divisor function,

$$Q = \sum_{n\le x}\{1 - \rho\sum_{i=1}^{k}\tau(a_i n + b_i)\}(\sum_{d|\Pi}\lambda_d)^2 > 0. \tag{51}$$

From this it follows that there is at least one value of n for which

$$\sum_{i=1}^{k}\tau(a_i n + b_i) < \frac{1}{\rho}. \tag{52}$$

Selberg found in the case $k = 2$ that $\rho = \frac{1}{14}$ is acceptable, which shows that one of n and $n + 2$ has at most two, while the other has at most three prime factors for infinitely many n. Remarkably, this is exactly the same type of tuple argument of Granville and Soundararajan which we have used, and the similarity doesn't end here. Multiplying out, we have $Q = Q_1 - \rho Q_2$ where

$$Q_1 = \sum_{n\le x}(\sum_{d|\Pi}\lambda_d)^2 > 0, \qquad Q_2 = \sum_{i=1}^{k}\sum_{n\le x}\tau(a_i n + b_i)\}(\sum_{d|\Pi}\lambda_d)^2 > 0. \tag{53}$$

The goal is now to pick λ_d optimally. As usual, the λ_d are first made 0 for $d > R$. At this point it appears difficult to find the exact solution to this problem. Further discussion of this may be found in [28] and [24]. Heath-Brown, desiring to keep Q_2 small, made the choice

$$\lambda_d = \mu(d)(\frac{\log(R/d)}{\log R})^{k+1}, \tag{54}$$

and with this choice we see

$$Q_1 = \frac{((k+1)!)^2}{(\log R)^{2k+2}}\sum_{n\le x}(\Lambda_R(n;\mathcal{H},1))^2. \tag{55}$$

Hence Heath-Brown used the approximation for a k-tuple with at most $k + 1$ distinct prime factors. This observation was the starting point for our work with the approximation $\Lambda_R(n;\mathcal{H},\ell)$. The evaluation of Q_2 with its τ weights is much harder to evaluate than Q_1 and requires Kloosterman sum estimates. The weight Λ in Q_2 in place of τ requires essentially the same analysis as Q_1 if we use the Bombieri-Vinogradov theorem. Apparently these arguments were never viewed as directly applicable to primes themselves, and this connection was missed until now.

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