Previous Test Questions

On elementary properties of graphs, trees, Eulerian and Hamiltonian Cycles, elementary probability theory

Graph theory

1. Prove that there does not exist a graph on 11 vertices that is isomorphic with its complement.

2. For a natural number n let V_n be the set of $(0, 1)$-sequences of length n and let $\mathcal{G}_n$ be the graph on V_n such that there is an edge between $s, t \in V_n$ iff the sequences s and t differ from each other in exactly one position (for example, for $n = 3$, there is an edge between 011 and 010 but there is no edge between 000 and 011). Determine all values of n for which $\mathcal{G}_n$ has an Eulerian cycle.

3. Show that if a graph has n vertices and at least $\frac{n^2-2n+2}{2}$ edges, then the graph has a Hamiltonian cycle (hint: check the Dirac condition).

4. The set of vertices V of the graph $\mathcal{G}$ is the 3-element subsets of $\{1, 2, 3, 4, 5\}$ and for $a \neq b \in V$ there is an edge between a and b iff $a \cap b \neq \emptyset$. Prove that $\mathcal{G}$ and K_{10} are isomorphic (K_{10} is the complete graph on 10 vertices).

5. Suppose that the graph $\mathcal{G}$ has at least 3 vertices and deleting any of its vertices the remaining graph contains an Eulerian cycle. Prove that $\mathcal{G}$ is a complete graph (that is, it is isomorphic with K_n for some natural number n).

7. The set of vertices V of the graph $\mathcal{G}$ is the set of 3-element subsets of $\{1, 2, 3, ..., 2016\}$ and for $a \neq b \in V$ there is an edge between a and b iff the sum of elements of a and that of b are both odd or both even. Show that $\mathcal{G}$ is not connected.

8. Give an example for a graph that contains a Hamiltonian cycle but does not

satisfy the Ore condition.

9. Give an example for a graph that contains a Hamiltonian cycle but does not contain an Eulerian cycle.

10. Prove that if $\mathcal{G} = \langle V, E \rangle$ is a tree on n vertices then

$$\sum_{x \in V} d(x) = 2n - 2.$$

11. Suppose $\mathcal{G}$ is a 2017-regular graph whose complement is 2016-regular. Show that $\mathcal{G}$ has a Hamiltonian cycle.

12. The set of vertices V of the graph $\mathcal{G}$ is the 2-element subsets of $A = \{1, 2, ..., 10\}$ and there is an edge between $a \neq b \in V$ iff $|x \cap y| = 1$. Does $\mathcal{G}$ contain an Eulerian cycle? (Prove your statement, a simple yes/no answer is not enough.)

13. Is there exist a graph $\mathcal{G}$ on 100 vertices such that both $\mathcal{G}$ and its complement contain an Eulerian cycle? (Prove your statement, a simple yes/no answer is not enough.)

14. Show that if a graph $\mathcal{G}$ has 10 vertices and 38 edges then it contains a Hamiltionian cycle.

15. Give an example for two 3-regular graphs having a same number of vertices such that the two graphs are not isomorphic.

16. Assume that the graph $\mathcal{G}$ has n vertices and the degree of each vertex of $\mathcal{G}$ is at least $\frac{2}{3}n$. Show that $\mathcal{G}$ contains at least $\frac{n}{12}$ many pairwise different Hamiltonian cycles.

17. Does there exist a tree on 5 vertices that is isomorphic with its complement? (Prove your statement, a simple yes/no answer is not enough.)

18. Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be graphs. Assume that $\mathcal{G}_1$ is isomorphic with $\mathcal{G}_2$ and $\mathcal{G}_2$ is isomorphic with $\mathcal{G}_3$. Prove that $\mathcal{G}_1$ is isomorphic with $\mathcal{G}_3$.

19. Suppose $\mathcal{G}$ is a 3-regular graph having at least 6 vertices. Prove that the complement of $\mathcal{G}$ is connected (hint: check the Dirac condition to the complement of $\mathcal{G}$).

Probability Theory

20. We are rolling a red and a blue dice. Let A be the event that the result in the blue dice is even and let B be the event that the sum of the results in the blue and red dice is at least 10. Decide if A and B are independent or not.

21. In public transportation a bus goes through in its route 16 times in each day. Assume, in each route, $p = 0.1$ is the probability that it arrives to its last station later than scheduled. Give the probability that in a day it is late for its last station 3 times.

22. Let ξ and η be uniformly distributed random variables in the unit interval $[0,1]$. Let A be the event that $\xi + \eta \leq \frac{1}{2}$ and let B the event that $\xi \leq \frac{1}{4}$. Decide if A and B are independent or not.

23. We are choosing randomly 5 elements $\{a_1, a_2, a_3, a_4, a_5\}$ from the set $\{1, 2, ..., 90\}$ (each element may be chosen at most once). Let

$$\xi = |\{i \leq 5 : 3 \text{ divides } a_i\}|.$$

Give the probability distribution of ξ.

24. Consider a permanently operating bus in public transportation. Assume p

is the probability that a controller gets on in the next station. Let ξ be the number of the first station where a controller gets on. Provide the probability distribution of ξ.

25. We are rolling a red and a blue dice. Let ξ be the maximum of the results on the red and blue dice. Give the probability distribution of ξ.

27. Let ξ and η be uniformly distributed random variables in the unit interval $[0, 1]$. Compute the probability that the matrix

$$\begin{pmatrix} \xi & 1 \\ \eta & \xi \end{pmatrix}$$

has non-negative determinant.

29. Let ξ and η be uniformly distributed random variables in the unit interval $[0, 1]$. Compute the probability that $min\{\xi, \eta\} \leq \xi^3$.

30. Let ξ and η be uniformly distributed random variables in the interval $[-1, 1]$ Compute the probability that the length of the vector $\underline{v} = [1, \xi, \eta]$ is at most $\sqrt{2}$

32. Let ξ and η be uniformly distributed random variables in the interval $[-1, 1]$ and let q be the polynomial $q(x) = x^2 + \xi \cdot x + \eta$. Compute the probability that $q(1) \geq 1$.

33. Let ξ and η be uniformly distributed random variables in the unit interval $[0, 1]$. The coordinates of the points A, B, C in the plane are $A = (1, 0)$, $B = (1+\xi, 0)$ and $C = (0, \eta)$. Compute the probability that the area of the triangle ABC is at most $\frac{1}{4}$.