Previous Test Questions – Hints for solutions

On elementary properties of graphs, trees, Eulerian and Hamiltonian Cycles, elementary probability theory

Graph theory

2. For a natural number n let V_n be the set of $(0,1)$-sequences of length n and let $\mathcal{G}_n$ be the graph on V_n such that there is an edge between $s,t \in V_n$ iff the sequences s and t differ from each other in exactly one position (for example, for $n=3$, there is an edge between 011 and 010 but there is no edge between 000 and 011). Determine all values of n for which $\mathcal{G}_n$ has an Eulerian cycle.

Hint. Fix n. Then a vertex of $\mathcal{G}_n$ is an n-termed esquence of 0's and 1's, hence the degree of each vertex of $\mathcal{G}_n$ is n (because there are n many possibilities do modify one coordiante of a vertex).

It is easy to see that $\mathcal{G}_n$ is connected, hence it contains an Eulerian cycle if and only if all of its vertices have even degree. But, by the previous paragraph, each vertex has degree n, hence $\mathcal{G}_n$ contains an Eulerian cycle if and only if n is even.

10. Prove that if $\mathcal{G} = \langle V, E \rangle$ is a tree on n vertices then

$$\sum_{x \in V} d(x) = 2n - 2.$$

Hint. As we learned, for any graph $\sum_{x \in V} d(x) = 2 \cdot |E|$ and (as we learned) for a tree $|E| = n - 1$.

11. Suppose $\mathcal{G}$ is a 2017-regular graph whose complement is 2016-regular. Show that $\mathcal{G}$ has a Hamiltonian cycle.

Hint. Let a be a vertex of $\mathcal{G}$. Since $\mathcal{G}$ is 2017-regular, a has 2017 neighbors. Since the complement of $\mathcal{G}$ is 2016-regular, there are 2016 vertices which are not neighbors of a. Thus the number of vertices of $\mathcal{G}$ is $1 + 2017 + 2016 = 4034$ (1 stands for a). Since $\mathcal{G}$ is 2017-regular, it satisfies the Dirac condition, hence it has a Hamiltonian cycle.

13. Is there exist a graph $\mathcal{G}$ on 100 vertices such that both $\mathcal{G}$ and its complement contain an Eulerian cycle? (Prove your statement, a simple yes/no answer is not enough.)

Hint. There is no such a graph: let $\mathcal{G}$ be any graph on 100 vertices and denote the complement of $\mathcal{G}$ by $\overline{\mathcal{G}}$. If $\mathcal{G}$ or $\overline{\mathcal{G}}$ is not connected then it does not contain an Eulerian cycle. Assume both are connected and let a be any vertex of $\mathcal{G}$. Then $d^{\mathcal{G}}(a) + d^{\overline{\mathcal{G}}}(a) = 99$ hence either $d^{\mathcal{G}}(a)$ or $d^{\overline{\mathcal{G}}}(a)$ is odd, hence either $\mathcal{G}$ or $\overline{\mathcal{G}}$ does not contain an Eulerian cycle.

16. Assume that the graph $\mathcal{G}$ has n vertices and the degree of each vertex of $\mathcal{G}$ is at least $\frac{2}{3}n$. Show that $\mathcal{G}$ contains at least $\frac{n}{12}$ many pairwise different Hamiltonian cycles.

Hint. Clearly, $\mathcal{G}$ satisfies the Dirac condition, hence it has a Hamiltonian cycle. Let $\mathcal{G}_1$ be the graph obtained from $\mathcal{G}$ by removing the edges of a Hamiltonian cycle from $\mathcal{G}$. Then, for all $a \in V(\mathcal{G})$ we have $d^{\mathcal{G}_1}(a) = d^{\mathcal{G}}(a) - 2$. If $\mathcal{G}_1$ satisfies the Dirac condition, then it contains a Hamiltonian cycle; removing its edges from $\mathcal{G}_1$ we obtain $\mathcal{G}_2$ and so on.

Clearly, $d^{\mathcal{G}_k}(a) = d^{\mathcal{G}}(a) - 2k$, so we can keep going and repeat the process described in the previous paragrph until $\frac{2}{3}n - 2k \geq \frac{n}{2}$, that is, we can proceed while $k \leq \frac{n}{12}$.

18. Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be graphs. Assume that $\mathcal{G}_1$ is isomorphic with $\mathcal{G}_2$ and $\mathcal{G}_2$ is isomorphic with $\mathcal{G}_3$. Prove that $\mathcal{G}_1$ is isomorphic with $\mathcal{G}_3$.

Hint. Suppose f is an isomorphism from $\mathcal{G}_1$ onto $\mathcal{G}_2$ and g is an isomorphism from $\mathcal{G}_2$ onto $\mathcal{G}_3$. Then their composition $g \circ f$ is an isomorphism from $\mathcal{G}_1$ onto $\mathcal{G}_3$.

Probability Theory

20. We are rolling a red and a blue dice. Let A be the event that the result in

the blue dice is even and let B be the event that the sum of the results in the blue and red dice is at least 10. Decide if A and B are independent or not.

Hint. $P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{6}{36} = \frac{1}{6}$ and $P(A \cap B) = \frac{4}{36} = \frac{1}{9}$. Since $P(A \cap B) \neq P(A) \cdot P(B)$, A and B are NOT independent.

23. We are choosing randomly 5 elements $\{a_1, a_2, a_3, a_4, a_5\}$ from the set $\{1, 2, ..., 90\}$ (each element may be chosen at most once). Let

$$\xi = |\{i \leq 5 : 3 \text{ divides } a_i\}|.$$

Give the probability distribution of ξ.

Hint. Let $A = \{1, 2, 3, ..., 90\}$ and let $B = \{x \in A : 3 \text{ divides } x\} = \{3, 6, 9, ..., 90\}$. The possible values of ξ are $0, 1, 2, ..., 5$. If $0 \leq k \leq 5$ then

$$P(\xi = k) = \frac{\binom{30}{k} \cdot \binom{60}{5-k}}{\binom{90}{5}}$$

because if $\xi = k$ then you have chosen k elements from B and $5 - k$ elements from $A \setminus B$. (Remember $\binom{n}{k}$ is the number of k-element subsets of an n-element set).

32. Let ξ and η be uniformly distributed random variables in the interval $[-1, 1]$ and let q be the polynomial $q(x) = x^2 + \xi \cdot x + \eta$. Compute the probability that $q(1) \geq 1$.

Hint. $q(1) = 1^2 + 1 \cdot \xi + \eta = \xi + \eta$ so the question is $P(\xi + \eta \geq 1)$. Computing the areas of the appropriate square and triangle we get

$$P(\xi + \eta \geq 1) = \frac{\frac{1}{2}}{4} = \frac{1}{8}.$$