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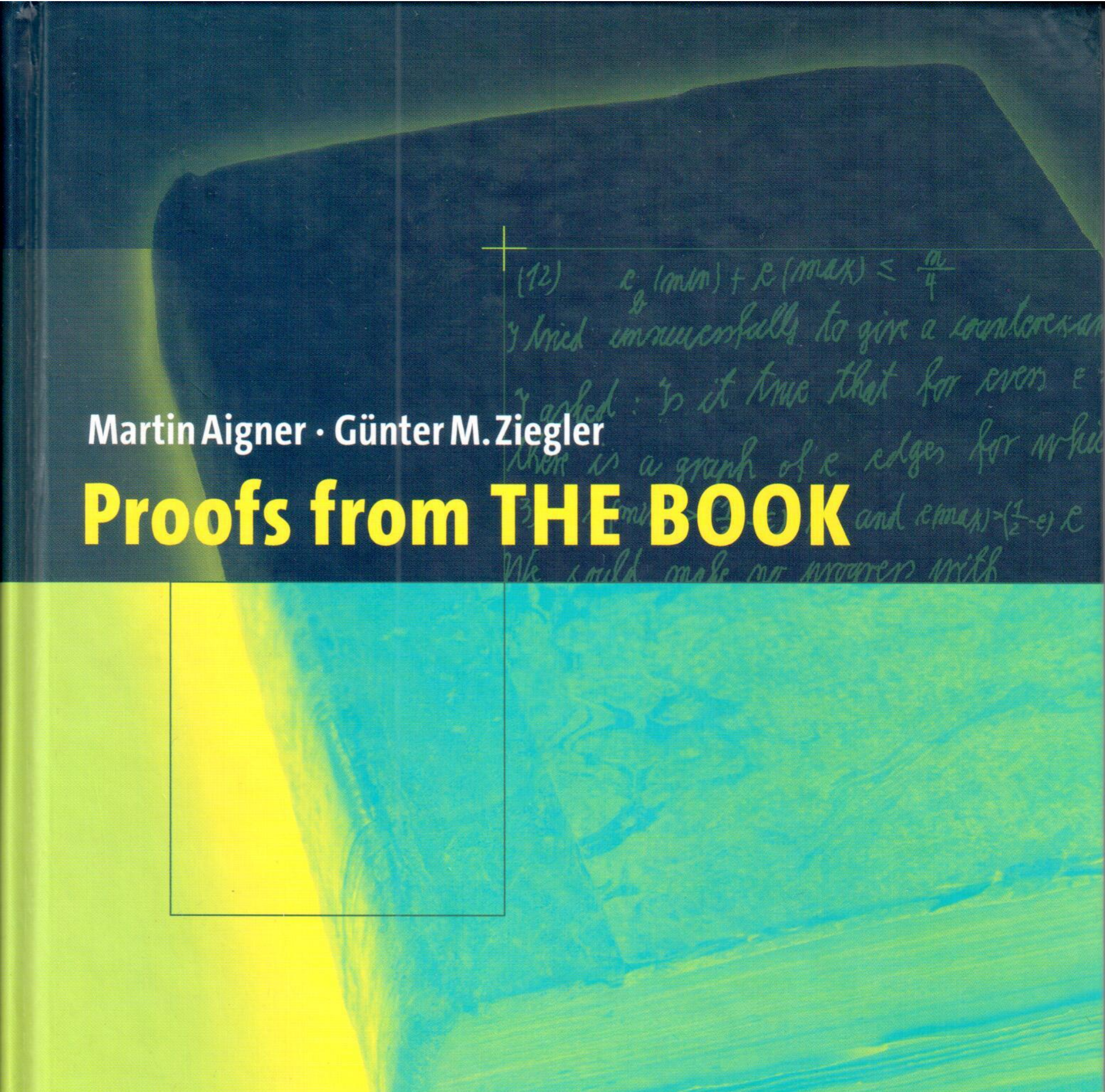
Proofs from THE BOOK



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and x' are contained in precisely one set A_i, hence

$$BB^T = \begin{pmatrix} r_{x_1}-1 & 0 & \dots & 0 \\ 0 & r_{x_2}-1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & r_{x_n}-1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

where r_x is defined as above. Since the first matrix is positive definite (it has only positive eigenvalues) and the second matrix is positive semi-definite (it has the eigenvalues n and 0), we deduce that BB^T is positive definite and thus, in particular, invertible, implying rank$(BB^T) = n$. It follows that the rank of the $(n \times m)$-matrix B is at least n, and we conclude that indeed $n \leq m$, since the rank cannot exceed the number of columns.

Let us go a little beyond and turn to graph theory. (We refer to the review of basic graph concepts in the appendix to this chapter.) A moment's thought shows that the following statement is really the same as Theorem 3:

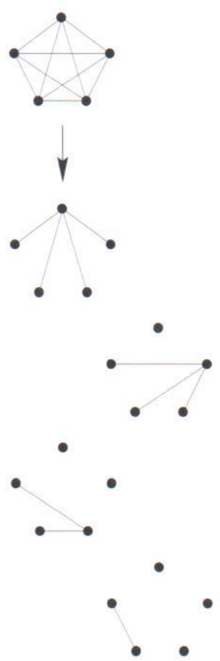
> *If we decompose a complete graph K_n into m cliques different from K_n, such that every edge is in a unique clique, then $m \geq n$.*

Indeed, let X correspond to the vertex set of K_n and the sets A_i to the vertex sets of the cliques, then the statements are identical.

Our next task is to decompose K_n into complete bipartite graphs such that again every edge is in exactly one of these graphs. There is an easy way to do this. Number the vertices $\{1, 2, \dots, n\}$. First take the complete bipartite graph joining 1 to all other vertices. Thus we obtain the graph $K_{1,n-1}$ which is called a *star*. Next join 2 to $3, \dots, n$, resulting in a star $K_{1,n-2}$. Going on like this, we decompose K_n into stars $K_{1,n-1}, K_{1,n-2}, \dots, K_{1,1}$. This decomposition uses $n-1$ complete bipartite graphs. Can we do better, that is, use fewer graphs? No, as the following result of Ron Graham and Henry O. Pollak says:

Theorem 4. *If K_n, $n \geq 2$, is decomposed into complete bipartite subgraphs $H_1, \dots, H_m$, then $m \geq n-1$.*

The interesting thing is that, in contrast to the Erdős-de Bruijn theorem, no combinatorial proof for this result is known! All of them use linear algebra in one way or another. Of the various more or less equivalent ideas let us look at the proof due to Tverberg which may be the most transparent.



A decomposition of K_5 into 4 complete bipartite subgraphs

■ **Proof.** Let the vertex set of K_n be $\{1, \dots, n\}$, and let L_j, R_j be the defining vertex sets of the complete bipartite graph H_j, $j = 1, \dots, m$. To every vertex i we associate a variable x_i. Since $H_1, \dots, H_m$ decompose K_n, we find

$$\sum_{i<j} x_i x_j = \sum_{k=1}^{m} \Big(\sum_{a \in L_k} x_a \cdot \sum_{b \in R_k} x_b\Big). \qquad (1)$$

Now suppose the theorem is false, $m < n-1$. Then the system of linear equations

$$\begin{aligned} x_1 + \ldots + x_n &= 0, \\ \sum_{a \in L_k} x_a &= 0 \qquad (k = 1, \ldots, m) \end{aligned}$$

has fewer equations than variables, hence there exists a non-trivial solution $c_1, \ldots, c_n$. From (1) we infer

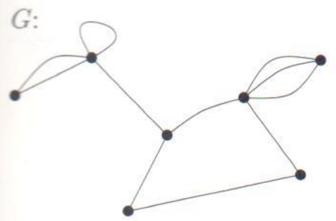
$$\sum_{i<j} c_i c_j \;=\; 0.$$

But this implies

$$0 \;=\; (c_1 + \ldots + c_n)^2 \;=\; \sum_{i=1}^{n} c_i^2 + 2\sum_{i<j} c_i c_j \;=\; \sum_{i=1}^{n} c_i^2 \;>\; 0,$$

a contradiction, and the proof is complete. □

Appendix: Basic graph concepts

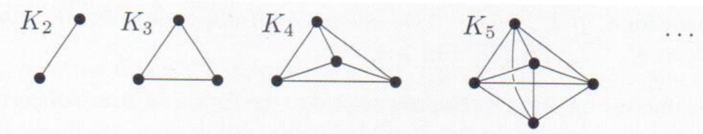


A graph G with 7 vertices and 11 edges. It has one loop, one double edge and one triple edge.

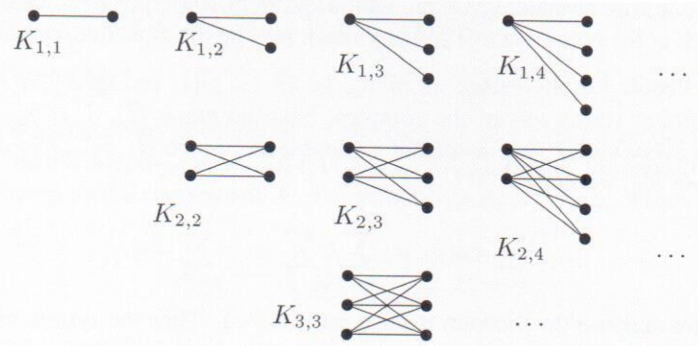
Graphs are among the most basic of all mathematical structures. Correspondingly, they have many different versions, representations, and incarnations. Abstractly, a *graph* is a pair $G = (V, E)$, where V is the set of *vertices*, E is the set of *edges*, and each edge $e \in E$ "connects" two vertices $v, w \in V$. We consider only finite graphs, where V and E are finite.

Usually, we deal with *simple graphs*: Then we do not admit *loops*, i. e., edges for which both ends coincide, and no *multiple edges* that have the same set of endvertices. Vertices of a graph are called *adjacent* or *neighbors* if they are the endvertices of an edge. A vertex and an edge are called *incident* if the edge has the vertex as an endvertex.

Here is a little picture gallery of important (simple) graphs:



The *complete graphs* K_n on n vertices and $\binom{n}{2}$ edges



The *complete bipartite graphs* K_{m+n} with $m + n$ vertices and mn edges