3.3 Kneser's Conjecture

One of the earliest and most spectacular applications of topological methods in combinatorics is Lovász's 1978 proof [Lov78] of a conjecture of Kneser. Kneser posed the following problem in 1955:

> **Aufgabe 360:** k und n seien zwei natürliche Zahlen, $k \leqq n$; N sei eine Menge mit n Elementen, N_k die Menge derjenigen Teilmengen von N, die genau k Elemente enthalten; f sei eine Abbildung von N_k auf eine Menge M, mit der Eigenschaft, daß $f(K_1) \neq f(K_2)$ ist falls der Durchschnitt $K_1 \cap K_2$ leer ist; $m(k, n, f)$ sei die Anzahl der Elemente von M und $m(k, n) = \operatorname{Min}_f m(k, n, f)$. Man beweise: Bei festem k gibt es Zahlen $m_0 = m_0(k)$ und $n_0 = n_0(k)$ derart, daß $m(k, n) = n - m_0$ ist für $n \geqq n_0$; dabei ist $m_0(k) \geqq 2k - 2$ und $n_0(k) \geqq 2k - 1$; in beiden Ungleichungen ist vermutlich das Gleichheitszeichen richtig.
>
> Heidelberg. MARTIN KNESER.

> Let k and n be two natural numbers, $k \leq n$; let N be a set with n elements, N_k the set of all subsets of N with exactly k elements; let f be a map from N_k to a set M with the property that $f(K_1) \neq f(K_2)$ if the intersection $K_1 \cap K_2$ is empty; let $m(k, n, f)$ be the number of elements of M, and $m(k, n) = \min_f m(k, n, f)$. Prove that for fixed k there are numbers $m_0 = m_0(k)$ and $n_0 = n_0(k)$ such that $m(k, n) = n - m_0$ for $n \geq n_0$; here $m_0(k) \geq 2k-2$ and $n_0(k) \geq 2k-1$; both inequalities probably hold with equality.

We will use a slightly different notation, and recast this in a graph-theoretic language. We take $N = [n]$, we write $\binom{[n]}{k}$ instead of N_k for the collection of all k-subsets of $[n]$, we take $\binom{[n]}{k}$ as the vertex set of a graph, and we connect two vertices by an edge if the corresponding k-sets are disjoint. Then the mapping f becomes a *coloring* of the graph, where M is the set of colors, and Kneser asks for the *chromatic number* of the graph!

We recall that a *(proper) k-coloring* of a graph $G = (V, E)$ is a mapping $c: V \to [k]$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$ is an edge. The *chromatic number of* G, denoted by $\chi(G)$, is the smallest k such that G has a k-coloring.

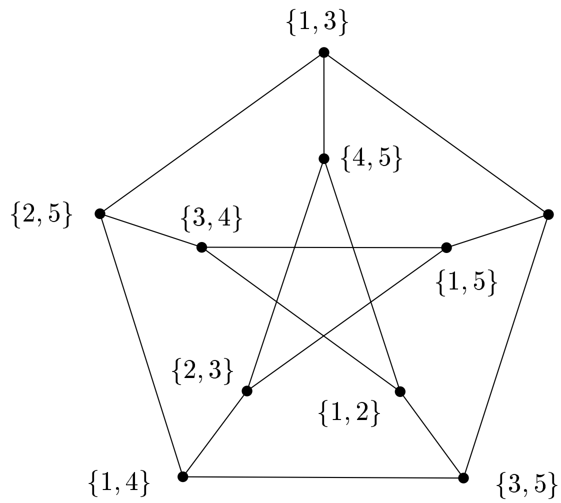
Let X be a finite ground set and let $\mathcal{F} \subseteq 2^X$ be a set system. The *Kneser graph of* $\mathcal{F}$, denoted by $\mathrm{KG}(\mathcal{F})$, has $\mathcal{F}$ as the vertex set, and two sets $F_1, F_2 \in \mathcal{F}$ are adjacent iff $F_1 \cap F_2 = \emptyset$. In symbols,

$$\mathrm{KG}(\mathcal{F}) = \Big(\mathcal{F}, \{\{F_1, F_2\} : F_1, F_2 \in \mathcal{F},\, F_1 \cap F_2 = \emptyset\}\Big).$$

Let $\mathrm{KG}_{n,k}$ denote the Kneser graph of the system $\mathcal{F} = \binom{[n]}{k}$ (all k-element subsets of $[n]$). Then Kneser's conjecture is $\chi(\mathrm{KG}_{n,k}) = n-2k+2$ for $n \geq 2k-1$.

3.3.1 Examples.

- $\mathrm{KG}_{n,1}$ is the complete graph K_n with $\chi(K_n) = n$.
- $\mathrm{KG}_{2k-1,k}$ is a graph with no edges, and so $\chi(\mathrm{KG}_{2k-1,k}) = 1$.
- $\mathrm{KG}_{2k,k}$ is a matching (every set is adjacent only to its complement), and $\chi(\mathrm{KG}_{2k,k}) = 2$ for all $k \geq 1$.
- The first interesting example is $\mathrm{KG}_{5,2}$, which turns out to be the ubiquitous *Petersen graph*:



This graph serves as a "(counter)example for almost everything" in graph theory (see, e.g., [CHW92], [HS93]). Check that 3 colors suffice and are necessary!

As we have already mentioned, Kneser's conjecture was first proved by Lovász.

3.3.2 Theorem (Lovász–Kneser theorem [Lov78]). *For all* $k > 0$ *and* $n \geq 2k-1$*, the chromatic number of the Kneser graph* $\mathrm{KG}_{n,k}$ *is* $\chi(\mathrm{KG}_{n,k}) = n-2k+2$.

The Kneser graphs $\mathrm{KG}_{n,k}$ are very interesting examples of graphs with high chromatic number. For example, note that for $n = 3k-1$, they have no triangles, and yet the chromatic number is $k+1$. One of the main reasons for their importance, and also probably a reason why the proof of Kneser's conjecture is difficult, is that there is a large gap between the chromatic number and the *fractional chromatic number*. (There are *very* few examples of such graphs known.)

The fractional chromatic number $\chi_f(G)$ of a graph G is defined as the infimum (actually minimum) of the fractions $\frac{a}{b}$ such that $V(G)$ can be covered by a independent sets in such a way that every vertex is covered at least

b times. We always have $\chi_f(G) \le \chi(G)$, and many methods for bounding $\chi(G)$ from below actually estimate $\chi_f(G)$. This means that they do not give good results for graphs that have high chromatic number $\chi(G)$, but low fractional chromatic number $\chi_f(G)$, as in the case of the Kneser graphs.

For example, the well-known lower bound in terms of the maximal size of independent sets, $\chi(G) \ge |V(G)|/\alpha(G)$, is just a part of the chain

$$\frac{|V|}{\alpha(G)} \;\le\; \chi_f(G) \;\le\; \chi(G),$$

where $\alpha(G)$, the *independence number* of G, is the maximum size of an independent set in G. However, for the Kneser graph, we have $\chi_f(\mathrm{KG}_{n,k}) = \frac{n}{k}$ (Exercise 1). So, for example, $\chi_f(\mathrm{KG}_{3k-1,k}) < 3$.

Upper bound for the chromatic number. It is simple to show that the chromatic number of $\mathrm{KG}_{n,k}$ cannot be larger than $n-2k+2$. We color the vertices of the Kneser graph by

$$\chi(F) := \min\{\min(F), n-2k+2\}.$$

This assigns a color $\chi(F) \in \{1, 2, \ldots, n-2k+2\}$ to each subset $F \in \binom{[n]}{k}$. If two sets F, F' get the same color $\chi(F) = \chi(F') = i < n-2k+2$, then they cannot be disjoint, since they both contain the element i. If the two k-sets both get color $n-2k+2$, then they are both contained in the set $\{n-2k+2, \ldots, n\}$, which has only $2k-1$ elements, and hence they cannot be disjoint either.

All known proofs of the tight lower bound for $\chi(\mathrm{KG}_{n,k})$ are topological or at least imitate the topological proofs. We begin with the simplest known proof, recently discovered by Greene.

First proof of the Lovász–Kneser theorem. Let us consider the Kneser graph $\mathrm{KG}_{n,k}$ and set $d := n-2k+1$. Let $X \subset S^d$ be an n-point set such that no hyperplane passing through the center of S^d contains more than d points of X. This condition is easily met by a set in a suitably general position, since we deal with points in $\mathbb{R}^{d+1}$ and require that no $d+1$ of them lie on a common hyperplane passing through the origin.

Let us suppose that the vertex set of $\mathrm{KG}_{n,k}$ is $\binom{X}{k}$, rather than the usual $\binom{[n]}{k}$ (in other words, we identify elements of $[n]$ with points of X).

We proceed by contradiction. Suppose that there is a proper coloring of $\mathrm{KG}_{n,k}$ by at most $n-2k+1 = d$ colors. We fix one such proper coloring and we define sets $A_1, \ldots, A_d \subseteq S^d$: For a point $\boldsymbol{x} \in S^d$, we have $\boldsymbol{x} \in A_i$ if there is at least one k-tuple $F \in \binom{X}{k}$ of color i contained in the open hemisphere $H(\boldsymbol{x})$ centered at $\boldsymbol{x}$ (formally, $H(\boldsymbol{x}) = \{\boldsymbol{y} \in S^d : \langle \boldsymbol{x}, \boldsymbol{y} \rangle > 0\}$). Finally, we put $A_{d+1} = S^d \setminus (A_1 \cup \cdots \cup A_d)$.

Clearly, A_1 through A_d are open sets, while A_{d+1} is closed. By the version of the Lyusternik–Shnirel'man theorem mentioned in Exercise 2.1.6, there exist $i \in [d+1]$ and $\boldsymbol{x} \in S^d$ such that $\boldsymbol{x}, -\boldsymbol{x} \in A_i$.

If $i \leq d$, we get two disjoint k-tuples colored by color i, one in the open hemisphere $H(\boldsymbol{x})$ and one in the opposite open hemisphere $H(-\boldsymbol{x})$. This means that the considered coloring is not a proper coloring of the Kneser graph.

If $i = d{+}1$, then $H(\boldsymbol{x})$ contains at most $k{-}1$ points of X, and so does $H(-\boldsymbol{x})$. Therefore, the complement $S^d \setminus (H(\boldsymbol{x}) \cup H(-\boldsymbol{x}))$, which is an "equator" (the intersection of S^d with a hyperplane through the origin), contains at least $n{-}2k{+}2 = d{+}1$ points of X, and this contradicts the choice of X.

Notes. Kneser's conjecture was formulated in [Kne55]. Garey and Johnson [GJ76] established the case $k = 3$ by elementary means; also see Stahl [Sta76]. As was already mentioned, the conjecture was proved by Lovász [Lov78]; a variation on his proof will be shown in Section 5.9. The short proof explained in this section by Greene [Gre02] was inspired by a proof by Bárány [Bár78], which we will present in Section 3.5. Still other proofs were found by Dol'nikov [Dol'81] (see Section 3.4) and by Sarkaria [Sar90] (see Section 5.8). In [Mat03], Kneser's conjecture was derived from Tucker's lemma by a direct combinatorial argument, without using a continuous result of Borsuk–Ulam type. Since the required instance of Tucker's lemma also has a combinatorial proof, the resulting proof of the Lovász–Kneser theorem is purely combinatorial, although the topological inspiration remains notable.

Generalizations of the Kneser conjecture to hypergraphs and related results will be discussed in Section 6.7.

Exercises

1. (a) Show that the fractional chromatic number of the Kneser graphs satisfies
$$\chi_f(\mathrm{KG}_{n,k}) \leq \frac{n}{k} \quad (n \geq 2k > 0).$$
(b) Show that the inequality in (a) is actually an equality. Hint: (Look up and) use the Erdős–Ko–Rado theorem.
2. Show that $\mathrm{KG}_{n,k}$ has no odd cycles of length shorter than $1 + 2i\left\lceil \frac{k}{n-2k} \right\rceil$. What about even cycles?
3. What is the maximum number of vertices in a complete bipartite subgraph of $\mathrm{KG}_{n,k}$?

3.4 More General Kneser Graphs: Dol'nikov's theorem

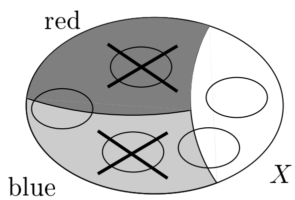
The proof of the Lovász–Kneser theorem shown in the previous section provides a more general result for free: a lower bound for the chromatic number of the Kneser graph $\mathrm{KG}(\mathcal{F})$ for an arbitrary finite set system $\mathcal{F}$.

First we recall the important notion of the *chromatic number of a hypergraph* (or of a set system). If $\mathcal{F}$ is a system of subsets of a set X, a coloring $c: X \to [m]$ is a (proper) m-coloring of $(X, \mathcal{F})$ if no edge is monochromatic under c ($|c(F)| > 1$ for all $F \in \mathcal{F}$). The chromatic number $\chi(\mathcal{F})$ is the smallest m such that $(X, \mathcal{F})$ is m-colorable. In this section we are interested only in 2-colorability.

Next, we define a less standard parameter of the set system $\mathcal{F}$. Let the *m-colorability defect*, denoted by $\mathrm{cd}_m(\mathcal{F})$, be the minimum size of a subset $Y \subseteq X$ such that the system of the sets of $\mathcal{F}$ that contain no points of Y is m-colorable. In symbols,

$$\mathrm{cd}_m(\mathcal{F}) = \min\Big\{|Y| : \big(X \setminus Y, \{F \in \mathcal{F} : F \cap Y = \varnothing\}\big) \text{ is } m\text{-colorable}\Big\}.$$

For example, for $m = 2$, we want to color each point of X red, blue, or white in such a way that no set of $\mathcal{F}$ is completely red or completely blue (but it may be completely white), and $\mathrm{cd}_2(\mathcal{F})$ is the minimum required number of white points for such a coloring.



3.4.1 Theorem (Dol'nikov's theorem [Dol'81]). *For any finite set system $(X, \mathcal{F})$, we have*

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F}).$$

It is fair to remark that this bound for $\chi(\mathrm{KG}(\mathcal{F}))$ need not be tight, and that $\mathrm{cd}_2(\mathcal{F})$ is not easy to determine in general.

If $\mathcal{F}$ consists of all the k-point subsets of $[n]$, $n \geq 2k$, then after deleting any $n-2k+1$ points we are left with the system of all k-element subsets of a $(2k-1)$-element set. In any red–blue coloring of that set, one of the colors has at least k points and contains a monochromatic k-element set. Thus $\mathrm{cd}_2(\mathcal{F}) \geq n-2k+2$, and we see that Theorem 3.4.1 generalizes the Lovász–Kneser theorem.

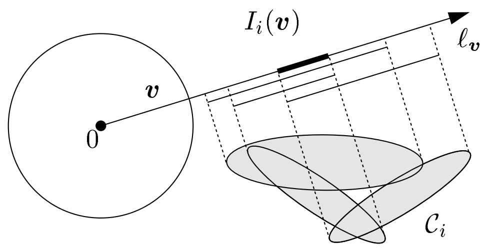
Proof of Dol'nikov's theorem. Let $d := \chi(\mathrm{KG}(\mathcal{F}))$. As in the above proof of the Lovász–Kneser theorem, we identify the ground set of $\mathcal{F}$ with a point set $X \subset S^d$ in general position (no $d+1$ points on an "equator"). For $\boldsymbol{x} \in S^d$, we define $\boldsymbol{x} \in A_i$ if the open hemisphere $H(\boldsymbol{x})$ contains a set $F \in \mathcal{F}$ colored by color i, $i \in [d]$. As before, we set $A_{d+1} = S^d \setminus (A_1 \cup \cdots \cup A_d)$. The appropriate version of Lyusternik–Shnirel'man yields an $\boldsymbol{x}$ with $\boldsymbol{x}, -\boldsymbol{x} \in A_i$ for some i.

We cannot have $i \leq d$, for otherwise, we would have two sets of $\mathcal{F}$ of color i lying in opposite open hemispheres. So $i = d{+}1$. We color the points of X in $H(\boldsymbol{x})$ red, those in $H(-\boldsymbol{x})$ blue, and the remaining ones (on the “equator” separating the two hemispheres) white. There are at most d white points by the general position of X, and so $\mathrm{cd}_2(\mathcal{F}) \leq d$.

Another proof of Dol’nikov’s theorem. Let us explain Dol’nikov’s original proof, somewhat more complicated but elegant. It is based on a geometric statement slightly resembling the ham sandwich theorem.

3.4.2 Proposition. *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_d$ be families of nonempty compact convex sets in $\mathbb{R}^d$, and suppose that for each $i = 1, 2, \ldots, d$, the system $\mathcal{C}_i$ is intersecting; that is, $C \cap C' \neq \emptyset$ for $C, C' \in \mathcal{C}_i$. Then there is a hyperplane (transversal) intersecting all sets of $\bigcup_{i=1}^{d} \mathcal{C}_i$.*

Proof. For a direction vector $\boldsymbol{v} \in S^{d-1}$, let $\ell_{\boldsymbol{v}}$ denote the line containing $\boldsymbol{v}$ and passing through the origin, oriented from the origin toward $\boldsymbol{v}$. Consider the system of the orthogonal projections of the sets of $\mathcal{C}_i$ on the line $\ell_{\boldsymbol{v}}$:



Each of these projections is a closed and bounded interval, and any two of them intersect. It is easy to see (directly, or by the one-dimensional Helly theorem) that the intersection of all these intervals is a nonempty interval, which we denote by $I_i(\boldsymbol{v})$. Let $m_i(\boldsymbol{v})$ denote the midpoint of $I_i(\boldsymbol{v})$.

We define an antipodal mapping $f: S^{d-1} \to \mathbb{R}^d$, by letting $f(\boldsymbol{v})_i = \langle m_i(\boldsymbol{v}), \boldsymbol{v} \rangle$ be the oriented distance of $m_i(\boldsymbol{v})$ from the origin. This is a continuous antipodal map, and we claim that for any such map, there is a point $\boldsymbol{v} \in S^{d-1}$ with $f_1(\boldsymbol{v}) = f_2(\boldsymbol{v}) = \cdots = f_d(\boldsymbol{v})$. To see this, we define a new antipodal map g, this time into $\mathbb{R}^{d-1}$, by letting $g_i = f_i - f_d$, $i = 1, 2, \ldots, d{-}1$. This g has a zero by the Borsuk–Ulam theorem, and if $g(\boldsymbol{v}) = \mathbf{0}$, then $f_1(\boldsymbol{v}) = f_2(\boldsymbol{v}) = \cdots = f_d(\boldsymbol{v})$ as required. For a $\boldsymbol{v}$ with this property, all the d midpoints $m_i(\boldsymbol{v})$ coincide, and so the hyperplane passing through them and perpendicular to $\ell_{\boldsymbol{v}}$ is the desired transversal of all sets of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_d$.

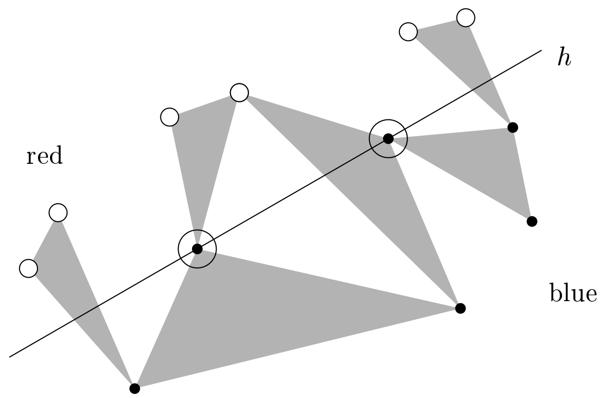
Second proof of Theorem 3.4.1. Suppose that there is a d-coloring of the Kneser graph $\mathrm{KG}(\mathcal{F})$. This means that $\mathcal{F}$ can be partitioned into set systems $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_d$ such that each two sets in $\mathcal{F}_i$ have a common point, $i = 1, 2, \ldots, d$.

We place the points of the ground set X into $\mathbb{R}^d$ (note that in the first proof the points were placed in $\mathbb{R}^{d+1}$!). We require general position: X is such that no $d+1$ points lie on a common hyperplane. We define the d families of convex sets in $\mathbb{R}^d$ by

$$\mathcal{C}_i = \{\mathrm{conv}(F) : F \in \mathcal{F}_i\}.$$

These $\mathcal{C}_i$ satisfy the assumptions of Proposition 3.4.2 above, and so there is a hyperplane h intersecting the convex hulls of all $F \in \mathcal{F}$.

We color the points of X in one of the open half-spaces bounded by h red, those in the opposite open half-space blue, and those lying on h white.



There are at most d white points, and this coloring shows that $\mathrm{cd}_2(\mathcal{F}) \leq d$. Theorem 3.4.1 is proved.

Notes. Theorem 3.4.1 is a special case of results of Dol'nikov [Dol'81] (also see [Dol'92], [Dol'94]). It was also independently found by Kříž [Kri92], in a more general form for hypergraphs (see Section 6.7).

The first proof in the text is a straightforward generalization of Greene's proof. For yet another proof of Dol'nikov's theorem see Section 5.8.

Exercises

1. For set systems $\mathcal{F}$ with $\chi(\mathrm{KG}(\mathcal{F})) \leq 2$, prove Dol'nikov's theorem by a direct combinatorial argument.
2. Find 2-colorable set systems $\mathcal{F}$ with $\chi(\mathrm{KG}(\mathcal{F}))$ arbitrarily large.
3. (a) Show that every graph is a Kneser graph. That is, given a (finite) graph G, construct a set system $\mathcal{F}$ such that $\mathrm{KG}(\mathcal{F})$ is isomorphic to G.
(b) Generalize the definition of $\mathrm{KG}(\mathcal{F})$, in the obvious way, to the case where $\mathcal{F}$ is a multiset of sets (some sets may occur several times in $\mathcal{F}$). For example, the complete graph K_n is isomorphic to the Kneser graph of the collection $\mathcal{F}$ consisting of n copies of $\emptyset$. Given a graph G, we want to find a multiset $\mathcal{F}$ of sets with $\mathrm{KG}(\mathcal{F})$ isomorphic to G and with $|\bigcup \mathcal{F}|$

as small as possible. Rephrase this problem in graph-theoretic notions speaking about G. (Hint: It is a minimum-cover problem.)

3.5 Gale's Lemma and Schrijver's Theorem

Here we present another geometric proof of the Lovász–Kneser theorem. An extension of this approach leads to a result that the methods considered in the previous two sections seem unable to provide.

This proof was found by Bárány [Bár78] soon after the announcement of Lovász's breakthrough. It is similar to Greene's proof shown in Section 3.3, or rather, Greene's proof is similar to Bárány's, which came much earlier. But the points are placed on a sphere of one dimension lower, using the following lemma.

3.5.1 Lemma (Gale's lemma [Gal56]). *For every $d \geq 0$ and every $k \geq 1$, there exists a set $X \subset S^d$ of $2k+d$ points such that every open hemisphere of S^d contains at least k points of X.*

First let us see how this implies the Lovász–Kneser theorem.

Another proof of the Lovász–Kneser theorem. We consider the Kneser graph $\mathrm{KG}_{n,k}$ and we set $d := n-2k$ (this dimension is one lower than in Greene's proof). Let $X \subset S^d$ be the set as in Gale's lemma. We identify $[n]$ with X, so that the vertices of $\mathrm{KG}_{n,k}$ are k-point subsets of X.

For contradiction, let us suppose that a proper $(d+1)$-coloring of $\mathrm{KG}_{n,k}$ has been chosen. We define sets $A_1, \ldots, A_{d+1} \subseteq S^d$ by letting $\boldsymbol{x} \in A_i$ if there is at least one k-tuple $F \in \binom{X}{k}$ of color i contained in the open hemisphere $H(\boldsymbol{x})$ centered at $\boldsymbol{x}$.

This time $A_1, \ldots, A_{d+1}$ form an open cover of S^d, since each $H(\boldsymbol{x})$ contains at least one k-tuple by Gale's lemma. By (LS-o) (Lyusternik–Shnirel'man for open covers), there are $i \in [d+1]$ and $\boldsymbol{x} \in S^d$ with $\boldsymbol{x}, -\boldsymbol{x} \in A_i$. This leads to a contradiction as before: We have two disjoint k-tuples of color i, one in $H(\boldsymbol{x})$ and one in $H(-\boldsymbol{x})$.

Proof of Gale's lemma. We prove the following version (equivalent to the above formulation using the central projection to S^d): There exist points $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{2k+d}$ in $\mathbb{R}^{d+1}$ such that every open half-space whose boundary hyperplane passes through $\mathbf{0}$ contains at least k of them.

The construction uses the moment curve (Definition 1.6.3), but we lift it one dimension higher, into the hyperplane $x_1 = 1$. That is, let

$$\bar{\gamma} := \{(1, t, t^2, \ldots, t^d) \in \mathbb{R}^{d+1} : t \in \mathbb{R}\}.$$

We take any $2k+d$ distinct points on $\bar{\gamma}$ and label them $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{2k+d}$ in the order in which they occur along the curve. For example, we can take

$\boldsymbol{w}_i := \bar{\gamma}(i)$ for $1 \leq i \leq 2k+d$. We call the points $\boldsymbol{w}_2, \boldsymbol{w}_4, \ldots$ *even* and the points $\boldsymbol{w}_1, \boldsymbol{w}_3, \ldots$ *odd*. Further we define $\boldsymbol{v}_i := (-1)^i \boldsymbol{w}_i$.

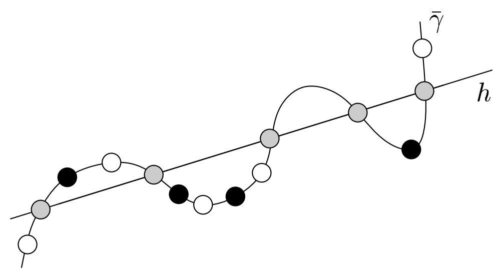
Let h be a hyperplane passing through $\mathbf{0}$, and let $h^{\oplus}$ and $h^{\ominus}$ be the two open half-spaces determined by it. We want to argue that both $h^{\oplus}$ and $h^{\ominus}$ contain at least k points among the $\boldsymbol{v}_i$; we formulate the argument for $h^{\oplus}$. Since $\boldsymbol{v}_i = \boldsymbol{w}_i$ for i even and $\boldsymbol{v}_i = -\boldsymbol{w}_i$ for i odd, we need to prove that *the number of even points $\boldsymbol{w}_i$ in $h^{\oplus}$ plus the number of odd points $\boldsymbol{w}_i$ in $h^{\ominus}$ is at least k.*

Using Lemma 1.6.4, we see that every hyperplane h through the origin intersects $\bar{\gamma}$ in no more than d points. Moreover, if there are d intersections, then $\bar{\gamma}$ crosses h at each of the intersections.

Given an arbitrary hyperplane h through the origin, we move it continuously to a position where it contains the origin and exactly d points of $W := \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{d+2k}\}$, while no point of W crosses from one side to the other during the motion. This is possible: Having already some $j < d$ points of W on h, we rotate h around some $(d-2)$-flat containing these points and $\mathbf{0}$, until we hit another point of W.

We thus suppose that h intersects $\bar{\gamma}$ in exactly d points, which all lie in W. Let $W_{\rm on}$ be the subset of the d points of W lying on h, and let $W_{\rm off} := W \setminus W_{\rm on}$ be the remaining $2k$ points. At every point of $W_{\rm on}$, $\bar{\gamma}$ crosses from one side of h to the other.

We color a $\boldsymbol{w}_i \in W_{\rm off}$ black if either it is even and lies in $h^{\oplus}$ or it is odd and lies in $h^{\ominus}$. Otherwise, we color $\boldsymbol{w}_i$ white. It is easy to see that as we follow $\bar{\gamma}$, black and while points of $W_{\rm off}$ alternate:



Indeed, let $\boldsymbol{w}$ and $\boldsymbol{w}'$ be two consecutive points of $W_{\rm off}$ along $\bar{\gamma}$ with j points of $W_{\rm on}$ between them. For j even, both $\boldsymbol{w}$ and $\boldsymbol{w}'$ are in the same half-space, and one of them is odd and the other is even, so one is black and one white. If j is odd, then $\boldsymbol{w}$ and $\boldsymbol{w}'$ are in different half-spaces, but they are both even or both odd, and so again one is black and one white. So the number of black points is at least $\lfloor \frac{1}{2} |W_{\rm off}| \rfloor \geq k$. This proves Gale's lemma.

A strengthening. Almost the same proof establishes a stronger theorem, found by Schrijver [Sch78] soon after Kneser's conjecture was proved.

3.5.2 Definition (Schrijver graph). *Let us call a subset $S \in \binom{[n]}{k}$* **stable** *if it does not contain any two adjacent elements modulo n (if $i \in S$, then*

$i{+}1 \notin S$, and if $n \in S$, then $1 \notin S$). In other words, S corresponds to an independent set in the cycle C_n. We denote by $\binom{[n]}{k}_{\text{stab}}$ the family of stable k-subsets of $[n]$. The **Schrijver graph** *is*

$$\mathrm{SG}_{n,k} := \mathrm{KG}\left(\binom{[n]}{k}_{\text{stab}}\right).$$

It is an induced subgraph of the Kneser graph $\mathrm{KG}_{n,k}$, and as it turns out, it has the same chromatic number. For example, for $\mathrm{KG}_{5,2}$, the Petersen graph, $\mathrm{SG}_{5,2}$ is a 5-cycle.

3.5.3 Theorem (Schrijver's theorem [Sch78]). *For all $n \geq 2k \geq 0$, we have $\chi(\mathrm{SG}_{n,k}) = \chi(\mathrm{KG}_{n,k}) = n{-}2k{+}2$.*

In fact, Schrijver showed that $\mathrm{SG}_{n,k}$ is a *vertex-critical* subgraph of $\mathrm{KG}_{n,k}$; that is, the chromatic number decreases if any single vertex (stable k-set) from $\mathrm{SG}_{n,k}$ is deeted (Exercise 1).

Proof of Schrijver's theorem. We proceed exactly as above for the Lovász–Kneser theorem, with the following strengthening of Gale's lemma:

> *There exists a $(2k{+}d)$-point set $X \subset S^d$ such that under a suitable identification of X with $[n]$, every open hemisphere contains a stable k-tuple.*

And this is precisely what the above proof of Gale's lemma provides: The black points form a stable set if the points of X are numbered along $\bar{\gamma}$.

Notes. Gale's proof of Lemma 3.5.1 is different from the one shown; it goes by induction on d and k. On the other hand, our argument is also based on Gale's work, namely, on the investigation of cyclic polytopes, which are convex hulls of finite point sets on the moment curve. The possibility of proving both Gale's lemma and the stronger version needed for Schrijver's graphs by the above simple construction was observed by Ziegler.

As was shown in [MZ02], Bárány's method of proof (together with the Gale transform, well-known in the theory of convex polytopes) yields the following "generalized Bárány bound" for the chromatic number of Kneser graphs: Given a set system $\mathcal{F}$ on a finite set X, we define the abstract simplicial complex $\mathsf{K} := \{S \subseteq X : F \not\subseteq S \text{ for all } F \in \mathcal{F}\}$. If K is isomorphic to a subcomplex of the boundary complex of a d-dimensional simplicial convex polytope P, then $\chi(\mathrm{KG}(\mathcal{F})) \geq |X| - d$. In particular, if we choose P as the cyclic polytope, we obtain Schrijver's theorem.

Exercises

1.* (a) Show that the graph $\mathrm{SG}_{n,k}$ is vertex-critical (for chromatic number); that is, for every k-tuple $A \in V(\mathrm{SG}_{n,k})$, there is a proper coloring of the vertex set of $\mathrm{SG}_{n,k}$ by $n-2k+2$ colors that uses the color $n-2k+2$ only at A. (This is not easy; a solution can be found in Schrijver's paper.)
   (b) Show that not all $\mathrm{SG}_{n,k}$ are edge-critical (an edge may be removed without decreasing the chromatic number).
2. Show that the Schrijver graph $\mathrm{SG}_{n,k}$ is not regular in general; that is, its vertices need not all have the same degree. What can you say about the symmetries of the Schrijver graphs?
3.* (Due to Anders Björner) Let $\mu(n,k)$ be the minimal number of monochromatic edges in a coloring of $\mathrm{KG}_{n,k}$ by $n-2k+1$ colors. Show that:
   (a) $\mu(n,k) \le \binom{2k-1}{k}$.
   (b) Equality holds for the cases $k=2$ and $n=2k+1$. (Hint: Use Schijver's theorem.)