Information Theory

Exercises to calculate channel capacity

1. Erasure channel with four input letters.

   Let the input alphabet be $\mathcal{X} = \{0, 1, 2, 3\}$ and the output alphabet be $\mathcal{Y} = \{0, 1, 2, 3, *\}$. The transition probabilities are given by $W(i|i) = 1 - p$ and $W(*|i) = p$ for every $i \in \mathcal{X}$, while $W(j|i) = 0$ for $j \notin \{i, *\}$. Give the capacity of this channel.

   Solution. We claim that $C = 2(1 - p)$.

   First we show $C \geq 2(1 - p)$. This can be done by assuming the input distribution to be uniform and calculating $I(X, Y)$ as follows.

   $$I(X, Y) = H(Y) - H(Y|X) =$$

   $$H((1 - p)/4, (1 - p)/4, (1 - p)/4, (1 - p)/4, p) - h(p) =$$

   $$h(p) + (1 - p) \log 4 - h(p) = 2(1 - p).$$

   To prove this is also an upper bound, we introduce the indicator variable for erasure: Let $E = 1$ if $Y = *$ and $E = 0$ otherwise. Then we can write

   $$H(Y) = H(Y, E) = H(E) + H(Y|E) =$$

   $$h(p) + (1 - p)H(Y|E = 0) \leq h(p) + (1 - p) \log 4,$$

   where the first equality is by E being determined by Y and the last inequality is implied by the fact that Y can take four values that are different of *, that is, four values when $E = 0$. We also used that $H(Y|E = 1) = 0$, since if $E = 1$ then Y is determined to be $*$. Thus

   $$I(X, Y) = H(Y) - H(Y|X) \leq h(p) + (1 - p) \log 4 - h(p) = 2(1 - p).$$

   So $2(1-p)$ bounds C both from above and from below, thus it is the exact value of the capacity C.

2. Let the input alphabet be $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let the output alphabet $\mathcal{Y}$ be the same. The transition probabilities are given by $W(i|i) = 1-p-q$, $W(i+1|i) = p$, and $W(i+2|i) = q$ for every $i \in \mathcal{X}$, where addition is performed modulo 8. All other transition probabilities are 0. Give the capacity of this channel.

   Solution:

   $$I(X, Y) = H(Y) - H(Y|X) =$$

   $$H(Y) - H(p, q, 1 - p - q) \leq \log 8 - H(p, q, 1 - p - q).$$

   The right had side can be attained, since by the symmetry of the system uniform input distribution results in uniform output distribution, and then $H(Y) = \log 8 = 3$ holds. Thus $C = 3 - H(p.q, 1 - p - q)$.