Fractional chromatic number

Definition: Denoting the set of independent sets of a graph G by $S(G)$ a *fractional coloring* is a function (a weighting) $w : S(G) \to R_{+,0}$, such that

$$\forall v \in V(G) : \sum_{A \ni v, A \in S(G)} w(A) \geq 1.$$

The *fractional chromatic number* $\chi_f(G)$ is the value

$$\inf \sum_{A \in S(G)} w(A)$$

taken under the above conditions. Instead of inf one can write min as the infimum is always attained.

The above can be formulated as a linear program as follows: Let A be a matrix with $n := |V(G)|$ rows and $s := |S(G)|$ columns in which the columns are the characteristic vectors of the independent sets. This means that $A[i,j] = 1$ if vertex $v_i \in A_j$, where A_j denotes the j^{th} independent set, and $A[i,j] = 0$ otherwise. Then $\chi_f(G) = \min(\mathbf{c} \cdot \mathbf{x})$, where $\mathbf{c} = (1, \ldots, 1)$ is the s-dimensional all-1 vector and the minimization is under the constraints

$$A\mathbf{x} \geq \mathbf{b}$$

for the n-dimensional all-1 vector $\mathbf{b} = (1, \ldots, 1)^T$ and

$$\mathbf{x} \geq \mathbf{0}.$$

All the inequalities are meant coordinatewise.

A linear program as the above has a dual:

$$\mathbf{y}A \leq \mathbf{c},\ \mathbf{y} \geq \mathbf{0},$$

and we seek

$$\max(\mathbf{y} \cdot \mathbf{b})$$

under these constraint. With the above matrix a vector $\mathbf{y}$ satisfying the constraints is a non-negative weighting of the vertices such that the total weight in any independent set is at most 1. Such a weighting is called a *fractional clique* and the maximum possible total weight (that is the maximum of $\mathbf{y} \cdot \mathbf{b}$) is the *fractional clique number* $\omega_f(G)$.

By the Duality Theorem of linear programming

$$\chi_f(G) = \omega_f(G).$$

This is a minimax theorem as any *feasible solution* for the first linear program gives an upper bound for the value given by any feasible solution of the dual program:

$$\mathbf{y} \cdot \mathbf{b} \leq \mathbf{y}(A\mathbf{x}) = (\mathbf{y}A)\mathbf{x} \leq \mathbf{c} \cdot \mathbf{x}.$$

Therefore if we present a fractional coloring and a fractional clique for a graph giving the same value, then they are necessarily optimal.

An alternative definition for $\chi_f(G)$ can be given by b-fold colorings.

Definition: For a positive integer b a b-fold coloring of a graph G is an attachment of b distinct colors to each vertex such that adjacent vertices get disjoint sets of colors. The minimum nuber of colors needed for this is the *b-fold chromatic number* $\chi_b(G)$.

A b-fold coloring is easy to turn to a fractional coloring: just attach weight $\frac{1}{b}$ to every independent set that is a color class in your b-fold coloring. (Note that you may use two different colors on exactly the same vertices. Then the two color classes coincide and the corresponding independent set gets the weight $\frac{1}{b}$ twice. Or several times if there are other color classes that are the same.)

It is also easy to see that a fractional clique with all weights rational can also be turned into a b-fold coloring for some appropriate b. Since irrationals can arbitrarily well approximated by rationals, this leads to the fact that

$$\chi_f(G) = \inf_b \frac{\chi_b(G)}{b},$$

where again we can write min in place of inf.

A graph homomorphism from graph F to graph G is a mapping $\varphi : V(F) \to V(G)$ that preserves edges that is for which

$$uv \in E(F) \Rightarrow \varphi(u)\varphi(v) \in E(G).$$

The existence of a homomorphism from F to G is denoted by $F \to G$. It is worth noting that a proper coloring of a graph G is equivalent to a homomorphism of G to the complete graph K_n. Generalizing this, one can observe that a b-fold coloring of a graph G using a colors is equivalent to a homomorphism to the Kneser graph $\mathrm{KG}(a, b)$. Thus we get that

$$\chi_f(G) = \inf\left\{\frac{a}{b} : G \to \mathrm{KG}(a, b)\right\},$$

where again, we can write min in place of inf.

Definition: A graph G is called *vertex-transitive* if for all pairs of vertices u and v one can give an automorphism (meaning an isomorphism to itself) of G that maps u to v. Intuitively, this means that all vertices "look the same", that is the graph is highly symmetric.

Theorem: If G is vertex-transitive, then

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)}.$$

Proof: First we show that $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$ for any graph G. Indeed, giving weight $\frac{1}{\alpha(G)}$ to every vertex no independent set gets more weight than 1, so this is a fractional clique with total weight $\frac{|V(G)|}{\alpha(G)}$. Thus

$$\chi_f(G) = \omega_f(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

Now we prove that for vertex-transitive graphs the reverse inequality also holds. If G is vertex-transitive, then all vertices are contained in the same number of maximum independent sets. Call this number t and give every maximum independent set (that is those of size $\alpha(G)$) weight $\frac{1}{t}$. By the definition of t this is a fractional colouring: all vertices get total weight $t\frac{1}{t} = 1$. If the number of maximum independent sets is ℓ then we distributed altogether $\ell\frac{1}{t} = \frac{\ell}{t}$ total weight, thus this is an upper bound on $\chi_f(G)$. Now we show that this upper bound is equal to $\frac{|V(G)|}{\alpha(G)}$.

To this end we calculate the number of pairs (v, A) where A is an independent set of size $\alpha(G)$ and $v \in A$. We have ℓ such A each containing $\alpha(G)$ vertices, so the number of such pairs is $\ell\alpha(G)$. On the other hand, we have $|V(G)|$ vertices and each is contained in t independent sets of size $\alpha(G)$, so the number of such pairs is $|V(G)|t$. Thus

$$\ell\alpha(G) = |V(G)|t,$$

that is we obtained

$$\chi_f(G) \leq \frac{\ell}{t} = \frac{|V(G)|}{\alpha(G)}.$$

By the two inequalities the statement $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ follows. QED