Homework 10

Due April 23, 2025

1. Let $R(3;n)$ denote the Ramsey number $R(3,3,\dots,3)$, where the number of 3's in the argument is n. Prove that
$$R(3;n) \leq 1 + n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right).$$

2. Let $R(3;n)$ denote the same Ramsey number as in the previous problem. Prove that for every positive integer k we have
$$R(3;2k) > 5^k.$$

3. A famous (and still open) conjecture due to Gyárfás and Sumner states that given any fixed finite tree T there exists a function $f_T : \mathbf{N} \to \mathbf{N}$ (here $\mathbf{N}$ is the set of natural numbers) satisfying that if G is any graph that does not contain T as an induced subgraph, then
$$\chi(G) \leq f_T(\omega(G)).$$
Prove that the conjecture is true when T is a star, that is $T = K_{1,r}$ for some fixed number r.

4. Let n, k be two positive integers such that $n \geq 2k \geq 2$. Prove that
$$\mathrm{KG}(n,k) \to \mathrm{KG}(n-2,k-1),$$
that is, that a homomorphism exists from the Kneser graph $\mathrm{KG}(n,k)$ to the other Kneser graph $\mathrm{KG}(n-2,k-1)$.

   (Reminder: A homomorphism from graph G to a graph H is a function $f : V(G) \to V(H)$ satisfying $\{a,b\} \in E(G) \Rightarrow \{f(a), f(b)\} \in E(H)$ for every $a, b \in V(G)$.)

5. Let $[m]$ denote the set $\{1,2,\dots,m\}$ and let the graph $U(m,r)$ be defined as follows.
$$V(U(m,r)) = \{(x,A) : A \subseteq [m], |A| = r-1, x \in [m], x \notin A\},$$
$$E(U(m,r)) = \{\{(x,A),(y,B)\} : x \in B \text{ and } y \in A\}.$$
Determine the value of $\chi_f(U(m,r))$.