Homework 11

Due December 5, 2024

1. Let A_n be the graph obtained from K_n by replacing each of its edges by a path of length three. (Thus A_n has $n + 2\binom{n}{2} = n^2$ vertices and $3\binom{n}{2}$ edges.) Prove that for any fixed b there is an $n(b)$ threshold such that if $n > n(b)$ then we have $\chi_b(A_n) = 3b$ for the b-fold chromatic number $\chi_b(A_n)$ of the graph A_n.

2. Determine the girth (the length of a shortest cycle) $g(\mathrm{KG}(n, k))$ of the Kneser graph $\mathrm{KG}(n, k)$ for every pair of positive integers n, k satisfying $n \geq 2k + 1$.

3. Let n, k be two positive integers such that $n \geq 2k \geq 2$. Prove that

$$\mathrm{KG}(n + 2, k + 1) \to \mathrm{KG}(n, k),$$

   that is, that a homomorphism exists from the Kneser graph $\mathrm{KG}(n+2, k+1)$ to the other Kneser graph $\mathrm{KG}(n, k)$.

4. Let $\mu(n, k)$ (for positive integers n, k satisfying $n \geq 2k$) denote the minimum number of edges having both endvertices colored with the same color in any coloring of the Kneser graph $\mathrm{KG}(n, k)$ with $n-2k+1$ colors. (Thus we minimize the number of "badly" colored edges over all $(n - 2k + 1)$-colorings of $\mathrm{KG}(n, k)$ and denote this minimum by $\mu(n, k)$.) Prove that

$$\mu(n, k) \leq \binom{2k - 1}{k}$$

   holds for every n and k satisfying $n \geq 2k$.

5. Prove that for $n = 2k + 1$ equality holds in the inequality stated in the previous problem.