Homework 8

Due April 9, 2025

1. A graph G is called an interval graph if it can be represented as the intersection graph of $|V(G)|$ closed intervals on a line. That is, one can put $|V(G)|$ closed intervals on a line and attach each to a vertex of G so that two vertices of G are adjacent if and only if the corresponding two intervals intersect.

   Prove without using the Strong Perfect Graph Theorem that interval graphs are perfect.

2. For every positive integer n we define the following graph G_n. Let

   $$V(G) = [n] = \{1, 2, \dots, n\}$$

   and

   $$E(G) = \{\{i, j\} : i, j \in [n],\ \gcd(i, j) = 1\},$$

   where $\gcd(i, j)$ denotes the greatest common divisor of the positive integers i and j.

   (Thus two numbers are adjacent in G_n iff they are coprime.)

   a) Prove that $\chi(G_n) = \omega(G_n)$ holds for every positive integer n.

   b) Is the graph G_n perfect for every positive integer n?

3. Prove without using the Strong Perfect Graph Theorem that there exists no minimal imperfect graph on (exactly) 200 vertices.

   (Reminder: A graph is called minimal imperfect if it is imperfect but deleting any of its vertices the remaining graph is perfect.)

4. Call a graph G normal if it has a family $\mathcal{A}$ of independent sets and a family $\mathcal{B}$ of cliques satisfying the following two properties.

   1. $\cup_{A\in\mathcal{A}} A = V(G) = \cup_{B\in\mathcal{B}} B$;

   2. $\forall A \in \mathcal{A},\ \forall B \in \mathcal{B} :\ A \cap B \neq \emptyset$.

   Prove that all perfect graphs are normal.

5. Prove that a minimal imperfect graph can be normal and a minimal imperfect graph can also be non-normal (where normal graphs are meant to be those defined in the previous problem).