Information Theory
Second midterm with sketches of solutions

1) State the theorem called *Chain rule*.

2. Give the definition of information stability of a stationary source.

3) Two fair coins are tossed simultaneously and this is done twice. Let X be the number of times when both the results are heads and Y be the number of times when both results are tails. (So the possible values of X are $0, 1$, and 2 and the same holds for Y.) Determine the value of the mutual information $I(X, Y)$.

4) Let us have a source with alphabet $\mathcal{X} = \{a, b, c\}$. Encode the source sequence

$$abcabcbabcabc$$

with the Lempel-Ziv-Welch algorithm. (The dictionary originally contains the codewords (i.e., the indices) for the one character sequences: 1 for a, 2 for b, 3 for c.)

5) Let X be a source whose outcome is given by the state of a binary stationary Markov chain. The transition probabilities of the Markov chain are given by

$$P(X_2 = 0|X_1 = 0) = p,\ P(X_2 = 1|X_1 = 0) = 1 - p,$$

$$P(X_2 = 0|X_1 = 1) = \frac{1-p}{2},\ P(X_2 = 1|X_1 = 1) = \frac{1+p}{2}.$$

What is $Prob(X_i = 0)$? Give the entropy of this source.
(Note that since the Markov chain is stationary, it is also homogenous and we have that $Prob(X_i = 0)$ does not depend on i.)

Solution of 3)

The joint distribution of X and Y is as follows:

$$P(X = 0, Y = 0) = 1/4,\quad P(X = 0, Y = 1) = 1/4,\quad P(X = 0, Y = 2) = 1/16,$$

$$P(X = 1, Y = 0) = 1/4,\quad P(X = 1, Y = 1) = 1/8,\quad P(X = 1, Y = 2) = 0,$$

$$P(X = 2, Y = 0) = 1/16,\quad P(X = 2, Y = 1) = 0,\quad P(X = 2, Y = 2) = 0.$$

These values are obtained by investigating the number of elementary events leading to the corresponding values of X and Y. From the above we have

$$Prob(X = 0) = Prob(Y = 0) = 9/16,\quad Prob(X = 1) = Prob(Y = 1) = 3/8,$$

$$Prob(X = 2) = Prob(Y = 2) = 1/16.$$

From here we can calculate

$$I(X, Y) = H(X) + H(Y) - H(X, Y).$$

We have

$$H(X) = H(Y) = H\left(\frac{9}{16}, \frac{3}{8}, \frac{1}{16}\right)$$

thus

$$I(X, Y) = 2H\left(\frac{9}{16}, \frac{3}{8}, \frac{1}{16}\right) - H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\right).$$

After some calculation this simplifies to $\frac{39}{8} - 3\log 3 (\approx 0.12)$.

Solution of 4)

Using the algorithm we learnt one obtains the code 1,2,3,4,3,2,7,7. The dictionary becomes: 1: a; 2: b; 3: c; 4: ab; 5: bc; 6: ca; 7: abc; 8: cb; 9: ba; 10: abca.

Solution of 5)

Let $Prob(X_i = 0) = q, Prob(X_i = 1) = 1 - q$. Then we have

$$q = qp + (1 - q)\frac{1 - p}{2}.$$

From here we get

$$(1 - p)q = \frac{1}{2}(1 - p)(1 - q).$$

Here $(1 - p)$ cancels out and we obtain $q = \frac{1}{3}$. Thus

$$Prob(X_i = 0) = \frac{1}{3}.$$

Now we can express the source entropy:

$$H(X) = H(X_{i+1}|X_i) = \frac{1}{3}h(p) + \frac{2}{3}h\left(\frac{1 - p}{2}\right).$$