Information Theory
Third Midterm with sketches of solutions

1) Give the definition of quadratic distortion of a quantizer Q.

2. State the converse part of the Channel Coding Theorem

3) Let the random variable X have density function $f(x)$ given as follows.

$$f(x) = x + 1 \text{ if } x \in [-1, 0], \quad f(x) = -x + 1 \text{ if } x \in [0, 1],$$

and $f(x)$ is 0 outside the interval $[-1, 1]$. Perform the first iteration of the Lloyd-Max algorithm for a two-level quantizer of the variable X starting with initial quantization values $-0.5, 0.5$.

4) U, Z, W are independent binary ($0-1$ valued) random variables on the figure and $\oplus$ means modulo 2 sum. The scheme shown by the figure gives a channel with binary input U and binary output V. What is the capacity of this channel if $P(Z = 1) = p$ and $P(W = 1) = q$?

5) Let the input alphabet of a discrete memoryless channel be $\mathcal{X} = \{0, 1, 3, 4\}$ and the output alphabet be $\mathcal{Y} = \{0, 1, 2, 3, 4, 5\}$. The transition probabilities are given by $W(i|i) = 1 - p$ for all $i \in \{0, 1, 3, 4\}$, $W(2|0) = W(2|1) = p$ and $W(5|3) = W(5|4) = p$. All other transition probabilities are equal to 0. Give the capacity of this channel.

Solution of 3): The arithmetic mean of the two quantization levels is 0, so we have $B_1 = [-1, 0), B_2 = [0, 1]$.

$$\frac{\int_{-1}^{0} xf(x)dx}{\int_{-1}^{0} f(x)dx} = \frac{\int_{-1}^{0} x^2 + xdx}{1/2} = 2\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} = -\frac{1}{3}.$$

Similarly,

$$\frac{\int_{0}^{1} xf(x)dx}{\int_{0}^{1} f(x)dx} = \frac{\int_{0}^{1} -x^2 + 1dx}{1/2} = 2\left[-\frac{x^3}{3} + \frac{x^2}{2}\right]_{0}^{1} = \frac{1}{3}.$$

Remark: The B_i's do not change, so this is a Lloyd-Max quantizer.

Solution of 4): We have $V = U$ if and only if $W = Z$, that has probability $pq + (1 - p)(1 - q) = 1 - p - q + 2pq$. Thus we have a BSC with probability of changing a letter to the opposite equal to $p + q - 2pq$. Putting this into the capacity formula we have for a BSC, we obtain that the capacity is

$$C = 1 - h(p + q - 2pq).$$

Solution of 5): If the input distribution is uniform, then

$$H(Y) = H((1 - p)/4, (1 - p)/4, (1 - p)/4, (1 - p)/4, p/2, p/2) =$$

$$-(1 - p)\log(1 - p) + (1 - p)\log 4 - p\log p + p\log 2 = h(p) + 2 - p.$$

Thus

$$C = \max I(X, Y) = \max(H(Y) - H(Y|X)) \geq h(p) + 2 - p - h(p) = 2 - p.$$

We show this lower bound is also an upper bound on C. Let E be a random variable taking value 0 if $Y = X$, that is, if $Y = 0, 1, 3$ or 4 and $E = 1$ otherwise, that is, when $Y = 3$ or 5. As E is determined by Y we have $H(Y) = H(E, Y)$. We can continue:

$$H(E, Y) = H(E) + H(Y|E) =$$

$$h(p) + pH(Y|E = 1) + (1 - p)H(Y|E = 0) \leq$$

$$h(p) + p \log 2 + (1 - p) \log 4 = h(p) + 2 - p.$$

Thus

$$I(X, Y) \leq h(p) + 2 - p - h(p) = 2 - p.$$

So

$$C = 2 - p.$$