Graphs, Capacities, Entropies, Exam items, Fall 2022

1. Shannon capacity of graphs (motivation and formal definition, $C_{\mathrm{OR}}(G)$ and $C_{\mathrm{AND}}(G)$, their relation); estimates of $C_{\mathrm{OR}}(G)$ by the clique number and by the chromatic number. Improving the lower bound in case of C_5.

2. Perfect graphs. Four basic graph classes that are perfect, perfectness of comparability graphs, the perfect graph conjectures.

3. Vertex packing polytope and fractional vertex packing polytope, their relation, Substitution Lemma and proof of the Perfect Graph Theorem.

4. Fractional chromatic number and fractional clique number, connection to linear programming, their value for vertex-transitive graphs. Connection between the fractional chromatic number and the zero-error capacity of a noisy channel with feedback.

5. Lovász's stronger (than the PGT) theorem characterizing perfect graphs (with Gasparian's proof).

6. Lovász theta number, some of its basic properties and its relation to Shannon capacity. (Here the relation to Shannon capacity is needed with proof, the properties not needed for that and not proven in class naturally need not be proven in the exam either.)

7. Fractional chromatic number and Kneser graphs, the largest possible gap between the chromatic number and the fractional chromatic number, Lovász's theorem giving an upper bound on the chromatic number in terms of the fractional chromatic number and the independence number. McEliece-Posner theorem.

8. Witsenhausen rate, motivation and properties, its value for C_5.

9. Shannon capacity and Ramsey numbers, theorem of Erdős, McEliece and Taylor.

10. Types and typical sequences, the number of sequences of a given type and the entropy function, graph parameters within a distribution, relation of within-a-distribution capacity and within-a-distribution-Witsenhausen-rate.

11. Graph entropy, motivation and defining formulas. Basic properties: monotonicity, sub-additivity, entropy of the complete graph.

12. Application of graph entropy to estimate the monotone formula size of the function Th_2^n.

13. Entropy of convex corners, antiblockers and generating pairs. Characterizing additivity of graph entropy for complementary graphs.

14. Applying corner entropy to a job scheduling problem (theorem of Denardo-Hoffman-MacKenzie-Pulleyblank).

15. Kahn and Kim's application of graph entropy to sorting and to enforce many comparisons.

16. Sperner capacity of directed graphs. Basic lower and upper bounds, Alon's theorem, the Sperner capacity of cyclically oriented cycles.

17. Local chromatic number, a generalization of Alon's theorem and its consequences to the Sperner capacity of oriented cycles.

18. Sperner capacity of the alternating oriented C_5 and the maximum Sperner capacity of orientations of vertex-transitive self-complementary graphs.

19. Shannon and Sperner capacity of graph families. Their meaning in information theory and combinatorics. Upper bound in terms of within-a-distribution capacity values.

20. The Gargano-Körner-Vaccaro theorem. Calculation rules for the formula it gives and sketch of proof.