ON SOME NEW THEOREMS IN THE THEORY OF DIOPHANTINE APPROXIMATIONS

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1. In a recent book[1] one of us based a series of applications on the following three theorems.

I. If

$$|z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|;$$

m is a non-negative integer, the b_j's are arbitrary complex numbers, there is then an integer ν_1 such that

$$m+1 \leqq \nu_1 \leqq m+n$$

and

$$|b_1 z_1^{\nu_1} + b_2 z_2^{\nu_1} + \cdots + b_n z_n^{\nu_1}| \geqq |z_n|^{\nu_1} \left(\frac{n}{2e(m+n)}\right)^n |b_1 + \cdots + b_n|.$$

II. With the above notations there is an integer ν_2 such that

$$m+1 \leqq \nu_2 \leqq m+n$$

and

$$|b_1 z_1^{\nu_2} + b_2 z_2^{\nu_2} + \cdots + b_n z_n^{\nu_2}| \geqq |z_1|^{\nu_2} \left(\frac{n}{24e^2(m+2n)}\right)^n \min_{j=1,\ldots,n} |b_1 + \cdots + b_j|.$$

III. With the above notations and

$$\omega(z) = \prod_{\nu=1}^{n} (z - z_\nu)$$

there is an integer ν_3 such that

$$m+1 \leqq \nu_3 \leqq m+n$$

and

$$|b_1 z_1^{\nu_3} + b_2 z_2^{\nu_3} + \cdots + b_n z_n^{\nu_3}| \geqq$$

$$\geqq \left(\sum_{j=1}^{n} |b_j|\,|z_j|^{\nu_3}\right) \cdot \prod_{l=1}^{n} \frac{1}{1+|z_l|} \cdot \frac{1}{\max\limits_{k=0,1,\ldots,(n-1)} \sum\limits_{l=1}^{n} \frac{|z_l|^k}{|\omega'(z_l)|(1+|z_l|)}}.$$

In mentioned book[1] **III** was discussed mainly as a matter of orientation and had only one application in the investigation of integral functions of type

$$\sum_{\nu=1}^{n} a_\nu F(c_\nu z)$$

[1] P. TURÁN, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953), Akadémiai Kiadó.

where $F(z)$ is an integral function. Since that time the second of us found some more applications of it. To show the essence of these three Theorems **I**, **II**, **III** we call the quantities

$$\sum_{j=1}^{n} |b_j|\,|z_j|^{\nu}, \qquad |z_n|^{\nu}, \qquad |z_1|^{\nu} \tag{1.1}$$

$N_l(\nu, f)$ norms of $f(\nu) = \sum_{j=1}^{n} b_j z_j^{\nu}$ $(l = 1, 2, 3)$. Then the Theorems **I**, **II** and **III** can be expressed by saying that, for a suitable integer ν, $\left|\sum_{j=1}^{n} b_j z_j^{\nu}\right|$ is estimated from below by the $N_l(\nu, f)$ norms (1.1) so, that the lower estimation of their quotient should be independent

(1.2) a) of the z_j-values or b) of the b_j coefficients.

Theorems **I** and **II** are of a)-type, Theorem **III** is of b)-type. This formulation of the theory is more symmetrical than that given in [1], where only problems of a)-type were *systematically* treated. In connection with an application[2] the necessity of dual theorems emerged where *non-trivial upper* estimations of

$$\min_{\substack{m+1 \leqq \nu \leqq m+n \\ \nu \text{ integer}}} \frac{|f(\nu)|}{N_l(\nu, f)}$$

are needed even at the rate of simple geometrical restrictions on the z_j's. In [1] one can find detailed motivation, on which way Theorems **I**, **II** and **III** can be considered as generalisations of KRONECKER's and DIRICHLET's classical theorems in the theory of diophantine approximations.

2. In [1] the emphasis was laid upon the applicability of these theorems and no care was taken to *best-possible* inequalities, though these have a significance for some applications, too. One can show this e. g. on the estimation[3] of $N(\alpha, T)$, the number of zeros of $\zeta(s)$ $(s = \sigma + it)$ in the parallelogram $\sigma \geqq \alpha$, $0 < t \leqq T$

$$N(\alpha, T) = O(T^{2(1-\alpha)+(1-\alpha)^{1,14}}), \tag{2.1}$$

which is uniformly valid for

$$1 - \delta \leqq \alpha \leqq 1 \tag{2.2}$$

with a (small) numerical positive δ. This constitutes the best-known estima-

[2] P. TURÁN, On Lindelöf's conjecture, *Acta Math. Acad. Sci. Hung.*, 5 (1954), pp. 145—163.

[3] This is an unpublished sharpening of Theorem XXXVIII of [1].

tion in this range today. The proof of this is largely based on the case

$$b_1 = b_2 = \cdots = b_n = 1 \tag{2.3}$$

of Theorem **II**; thereby the decrease of the numerical factor $24e^2$, since n is "large" in this case, would result an increase of δ in (2. 2) which in turn would result a decrease of the smallest known Θ with the property

$$p_{n+1} - p_n = O(p_n^{\Theta})$$

where p_n denotes the n^{th} prime. The main aim of this paper is to review the a)-type results in the first part of [1] from this point of view, in particular in the case (2. 3), and to study certain $(z_1, z_2, \ldots, z_n)$-systems, which will play a role in these questions. We suppose without loss of generality

$$z_n = 1, \qquad |z_j| \geqq 1 \qquad (j = 1, \ldots, n) \tag{2.4}$$

in **I** and

$$z_1 = 1, \qquad |z_j| \leqq 1 \qquad (j = 1, \ldots, n) \tag{2.5}$$

in **II**. In the case (2. 5) we ask for the "smallest" numerical positive value A_1, for which

$$\max_{\substack{m+1 \leqq \nu \leqq m+n \\ \nu \text{ integer}}} |z_1^\nu + z_2^\nu + \cdots + z_n^\nu| \geqq \left(\frac{n}{A_1(m+n)}\right)^n \tag{2.6}$$

holds for an arbitrary non-negative integer m and positive integer n. We shall show in **6—9** the following

THEOREM. *We have for the A_1 defined in* (2. 6) *the inequality*

$$1{,}321 < A_1 < 2e^{1+\frac{4}{e}} (< 24).$$

The gap is still large but the upper bound is much better than the previous $24e^2 \sim 177$. That $A_1 > 1{,}1$, can already be shown taking

$$m = 0, \quad n = 2, \quad z_1 = 1, \quad z_2 = \frac{1}{2} e^{\frac{2\pi i}{3}};$$

in this case we have

$$|z_1 + z_2| = \frac{\sqrt{3}}{2}, \qquad |z_1^2 + z_2^2| = \frac{\sqrt{13}}{4},$$

i. e.

$$\frac{\sqrt{13}}{4} \geqq \frac{1}{A_1^2}, \quad A_1 \geqq \frac{2}{\frac{4}{\sqrt{13}}} > 1{,}1.$$

An interesting feature of the proof of the upper bound is the avoiding of the use of H. CARTAN's theorem and replacing it by a lemma of CHEBYSEV-type. The new proof furnishes mutatis mutandis the following improvement of **II**.

Under the conditions of **II** we have

$$(2.7)\quad |b_1 z_1^{\nu_2}+\cdots+b_n z_n^{\nu_2}| \geqq |z_1|^{\nu_2}\left(\frac{n}{2e^{1+\frac{4}{e}}(m+n)}\right)^n \cdot \min_{j=1,\ldots,n} |b_1+\cdots+b_j|.$$

It will be sufficient to indicate only the changes, the proof of (2.7) needs compared to that of the theorem.

3. A further refinement of the upper estimation in the Theorem would be given if the constant $2e$ in **I** could be diminished, even only in the special case $b_1=\cdots=b_n=1$. Asking for the "smallest" numerical positive value A_2, for which in the case (2.4)

$$(3.1)\quad \max_{\substack{m+1\leqq\nu\leqq m+n\\ \nu \text{ integer}}} |z_1^\nu+z_2^\nu+\cdots+z_n^\nu| \geqq \left(\frac{n}{A_2(m+n)}\right)^n$$

holds for any positive integer n and non-negative integer m, nothing better than

$$(3.2)\quad 1\leqq A_2\leqq 2e$$

can be asserted at the present. Something better can be said on the "smallest" numerical positive value A_3 for which in the case (2.4)

$$(3.3)\quad \max_{\substack{m+1\leqq\nu\leqq m+n\\ \nu \text{ integer}}} |b_1 z_1^\nu+\cdots+b_n z_n^\nu| \geqq \left(\frac{n}{A_3(m+n)}\right)^n |b_1+\cdots+b_n|$$

holds for any complex b_ν's, positive integer n and non-negative integer m. We shall show in **10** that

$$(3.4)\quad 1,27\sim\frac{4}{\pi}\leqq A_3\leqq 2e\sim 5,44.$$

4. The lower limitation of A_1 in the Theorem will be proved in **9** by refining an idea of P. ERDŐS, i.e. considering $(z_1, z_2, \ldots, z_n)$-systems with the property

$$(4.1)\quad s_2=s_3=\cdots=s_{n-1}=0,\quad z_1=1$$

where s_ν stands for $z_1^\nu+z_2^\nu+\cdots+z_n^\nu$. This suggests for the sake of counter examples the usefulness of the study of all $(z_1, \ldots, z_n)$-systems with the property

$$s_2=s_3=\cdots=s_{n-1}=s_n=0,$$

or more generally with a prescribed non-negative integer m the determination of all those with

$$(4.2)\quad s_{m+1}=s_{m+2}=\cdots=s_{m+n-1}=0.$$

We mention another reason why (4.2) is interesting. The whole theory emerged from the necessity to diminish the interval for ν_j in **I**, **II** and **III** as much as possible; the question arises now, whether or not for *some* integer values m there is a non-trivial lower limitation for

$$\max_{\substack{m+1 \leqq \nu \leqq m+n-1 \\ \nu \text{ integer}}} \frac{|s_\nu|}{|z_1|^\nu}$$

or for

$$\max_{\substack{m+1 \leqq \nu \leqq m+n-1 \\ \nu \text{ integer}}} \frac{|s_\nu|}{|z_n|^\nu}.$$

That such a reduction of the interval for ν is *generally* impossible, is trivial, since in the case $m \equiv 0 \bmod n$ we have for the n^{th} roots of unity

$$s_{m+1} = s_{m+2} = \cdots = s_{m+n-1} = 0.$$

Probably the same holds for *all* non-negative integer values m. As the Newton—Girard formulae show at once, for $m = 0$ *all the* $(z_1, \ldots, z_n)$-systems with

$$s_1 = s_2 = \cdots = s_{n-1} = 0 \tag{4.3}$$

are given by the zeros of an equation

$$z^n + a = 0 \qquad (a \text{ arbitrary complex}). \tag{4.4}$$

We can determine all systems satisfying (4.2) with $m = 1$ and $m = 2$. For $m = 1$ we assert that *all* $(z_1, z_2, \ldots, z_n)$*-systems with the property*

$$s_2 = s_3 = \cdots = s_n = 0 \tag{4.5}$$

are formed by the zeros of an equation

$$\varphi_n(z, a) = z^n + \frac{a}{1!} z^{n-1} + \cdots + \frac{a^n}{n!} = 0 \qquad (a \text{ arbitrary complex}). \tag{4.6}$$

An asymptotical determination of these systems for fixed a and $n \to \infty$ follows at once from SZEGŐ's[4] results. For $m = 2$ we shall see in **11** that *all the* $(z_1, z_2, \ldots, z_n)$*-systems with*

$$s_3 = s_4 = \cdots = s_n = s_{n+1} = 0 \tag{4.7}$$

are formed by the zeros of an equation

$$f_n(z, a, \lambda) = z^n + \frac{H_1(\lambda)}{1!} a z^{n-1} + \cdots + \frac{H_n(\lambda)}{n!} a^n = 0, \tag{4.8}$$

where $H_\nu(y)$ *stands for the* ν^{th} *Hermite polynomial defined by*

$$H_\nu(y) = (-1)^\nu e^{y^2} \frac{d^\nu}{dy^\nu} e^{-y^2},$$

λ *denotes any zero of* $H_{n+1}(y) = 0$ *and* a *is an arbitrary complex number.* An asymptotical determination of the zeros of $f_n(z, a, \lambda)$ is not known. The

[4] G. SZEGŐ, Über eine Eigenschaft der Exponentialreihe, *Sitzungsber. der Berl. Math. Ges.*, **23** (1924), pp. 50—63.

result (4. 5)—(4. 6) can be easily[5] proved by Newton—Girard formulae so we omit the details. It is interesting to remark a characteristic difference between the cases $m=1$ and $m=2$. All solutions of (4. 5) can be derived from the single equation

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}=0,$$

all solutions of (4. 7), however, can be derived from the $(n+1)$ equations

$$\sum_{\nu=0}^{n}\frac{H_\nu(\lambda)}{\nu!}z^\nu=0 \quad \text{where} \quad H_{n+1}(\lambda)=0.$$

5. Owing to the applicability to the approximative solution of algebraic equations it is of interest to study **II** in the special case

(5. 1) $$m=0, \qquad b_1=b_2=\cdots=b_n=1.$$

In this case we have to determine

(5. 2) $$M_n=\min_{\substack{|z_j|\leqq 1\\ j=1,\ldots,n}}\ \max_{\nu=1,\ldots,n}|1+z_2^\nu+\cdots+z_n^\nu|.$$

In[1] we have shown

(5. 3) $$M_n\geqq\frac{\log 2}{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}};$$

according to a written communication of DE BRUIJN this can be replaced by

(5. 4) $$M_n>c\,\frac{\log\log n}{\log n} \qquad (c \text{ numerical positive constant}),$$

what is for large n somewhat better. This makes still more probable the conjecture that $M_n\geqq d$, independently of n. As to the upper estimation of M_n MR. HYLTÉN-CAVALLIUS showed $M_2=\frac{\sqrt{5}-1}{\sqrt{2}}\sim 0{,}86$ and found by considering $z_2=0{,}1295+i\,0{,}7063$, $z_3=-0{,}5128+i\,0{,}1508$ the estimation

$$M_3<0{,}831.$$

6. Before turning to the proof of the Theorem we need the following simple

LEMMA. *Let be* $0\leqq\delta\leqq 1$ *and*

(6. 1) $$f(z)=z^n+\cdots=\prod_{\nu=1}^{n}(z-z_\nu).$$

[5] Another very elegant verification is due to E. EGERVÁRY. Using Euler's formulae $\sum_{j=1}^{n}\frac{z_j^\nu}{\omega'(z_j)}=0$ $(\nu=0,1,\ldots,n-2)$ valid for any polynomial $\omega(z)$ with simple z_j-zeros of degree n and using $\varphi_n'(z_j,a)=(n!\,z_j^2)^{-1}a^{n+1}$ the result follows at once.

Then, there is a circle $|z|=r_0$ *with* $\delta \leqq r_0 \leqq 1$ *such that on the whole periphery*

$$|f(z)| \geqq 2\left(\frac{1-\delta}{4}\right)^n.$$

For the proof we introduce

$$f^*(z)=\prod_{\nu=1}^{n}(z-|z_\nu|). \tag{6.2}$$

Owing to the classical theorem of ČEBYŠEV there is a ξ with $\delta \leqq \xi \leqq 1$ and

$$|f^*(\xi)| \geqq 2\left(\frac{1-\delta}{4}\right)^n. \tag{6.3}$$

But on the circle $|z|=\xi$ we have

$$|z-z_j| \geqq ||z|-|z_j||=|\xi-|z_j|| \qquad (j=1,2,\ldots,n),$$

i. e. multiplying, further using (6. 2) and (6. 3) we get

$$|f(z)| \geqq |f^*(\xi)| \geqq 2\left(\frac{1-\delta}{4}\right)^n.$$

Hence the above ξ can be chosen as r_0. Q. e. d.

7. Next we turn to the proof of our Theorem. We may suppose that $n \geqq 2$. Let be, with our numbers z_j,

$$\omega(z)=\prod_{j=1}^{n}(z-z_j) \tag{7.1}$$

and let δ be a positive number $0 \leqq \delta \leqq 1$ to be determined later. To this $\omega(z)$ and δ there is, according to the Lemma, an r_0 with $\delta \leqq r_0 \leqq 1$ and such that for $|z|=r_0$

$$|\omega(z)| \geqq 2\left(\frac{1-\delta}{4}\right)^n. \tag{7.2}$$

Since each factor $|z-z_j|$ is at most 2, we have from (7. 2) for $|z|=r_0$ and for any choice of the indices $(1 \leqq)\ i_1<i_2<\cdots<i_k \leqq n$ $(1 \leqq k \leqq n)$

$$\prod_{\nu=1}^{k}|z-z_{i_\nu}| \geqq 2\left(\frac{1-\delta}{4}\right)^n \frac{1}{2^{n-k}}=2\left(\frac{1-\delta}{8}\right)^n 2^k. \tag{7.3}$$

We investigate two cases.

Case a) All z_j's are absolutely $\geqq r_0$. Then, owing to **I**, there is an integer ν_1 with $m+1 \leqq \nu_1 \leqq m+n$ and

$$|z_1^{\nu_1}+z_2^{\nu_1}+\cdots+z_n^{\nu_1}| \geqq r_0^{\nu_1}\left(\frac{n}{2e(m+n)}\right)^n \geqq \delta^{m+n}\left(\frac{n}{2e(m+n)}\right)^n. \tag{7.4}$$

Case b) There is an index l with $1 \leqq l < n$ and

$$1=|z_1| \geqq |z_2| \geqq \cdots \geqq |z_l| > r_0 > |z_{l+1}| \geqq \cdots \geqq |z_n|. \tag{7.5}$$

In the treatment of this case we shall suppose first that the z_j's are all different.

8. Let first be

$$f_1(z) = \prod_{j=l+1}^{n} (z-z_j) = \sum_{j=0}^{n-l} c_j^{(1)} z^{n-l-j}, \qquad c_0^{(1)} = 1. \tag{8. 1}$$

We have obviously

$$|c_j^{(1)}| = \left| \sum_{l+1 \leqq i_1 < \cdots < i_j \leqq n} z_{i_1} z_{i_2} \cdots z_{i_j} \right| \leqq \binom{n-l}{j}. \tag{8. 2}$$

Next let $f_2(z)$ be that polynomial of degree $\leqq l-1$, which assumes for $z = z_1, z_2, \ldots, z_l$ the values

$$\frac{1}{z_1^{m+1} f_1(z_1)}, \quad \frac{1}{z_2^{m+1} f_1(z_2)}, \ldots, \frac{1}{z_l^{m+1} f_1(z_l)},$$

respectively. If $l = 1$, then

$$f_2(z) \equiv \frac{1}{z_1^{m+1} f_1(z_1)} \equiv c_0^{(2)};$$

if $1 < l < n$, then we can represent $f_2(z)$ as Newton-interpolatorical polynomial

$$f_2(z) = c_0^{(2)} + c_1^{(2)}(z-z_1) + c_2^{(2)}(z-z_1)(z-z_2) + \cdots + c_{l-1}^{(2)}(z-z_1)(z-z_2)\cdots(z-z_{l-1}).$$

Since the function $\dfrac{1}{z^{m+1} f_1(z)}$ is regular for $|z| > r_0$ and vanishes for $z = \infty$, we have according to NÖRLUND's representation

$$c_j^{(2)} = \frac{1}{2\pi i} \int_{|w|=r_0} \frac{dw}{w^{m+1} f_1(w)(w-z_1)(w-z_2)\cdots(w-z_{j+1})} \tag{8. 3}$$

$$(j = 0, 1, \ldots, (l-1)).$$

But $f_1(w)(w-z_1)\cdots(w-z_{j+1})$ is of type (7. 3) with $k = n-l+j+1$ and thus owing to (7. 3) we have for $|w| = r_0$

$$|f_1(w)(w-z_1)(w-z_2)\cdots(w-z_{j+1})| \geqq 2\left(\frac{1-\delta}{8}\right)^n 2^{n-l+j+1} = 2\left(\frac{1-\delta}{4}\right)^n \frac{1}{2^{l-j-1}},$$

i. e. from (8. 3)

$$|c_j^{(2)}| \leqq \frac{1}{2r_0^m}\left(\frac{4}{1-\delta}\right)^n 2^{l-j-1} \leqq \frac{1}{2\delta^m}\left(\frac{4}{1-\delta}\right)^n 2^{l-j-1}. \tag{8. 4}$$

But we need also $f_2(z)$ in the form

$$f_2(z) = \sum_{j=0}^{l-1} c_j^{(3)} z^j$$

and we have to estimate the coefficients $c_j^{(3)}$. We have

$$c_{l-1}^{(3)} = c_{l-1}^{(2)}$$

and if $l > 1$, for $j = 0, 1, \ldots, (l-2)$

$$c_j^{(3)} = c_j^{(2)} - c_{j+1}^{(2)} \sum_{1 \leqq r_1 \leqq j+1} z_{r_1} + c_{j+2}^{(2)} \sum_{1 \leqq r_1 < r_2 \leqq j+2} z_{r_1} z_{r_2} - \cdots +$$
$$+ (-1)^{l-j-1} c_{l-1}^{(2)} \sum_{1 \leqq x_1 < r_2 < \ldots < r_{l-j-1} \leqq l-1} z_{r_1} z_{r_2} \cdots z_{r_{l-j-1}},$$

i. e.

$$|c_j^{(3)}| \leqq |c_j^{(2)}| + |c_{j+1}^{(2)}| \binom{j+1}{1} + |c_{j+2}^{(2)}| \binom{j+2}{2} + \cdots + |c_{l-1}^{(2)}| \binom{l-1}{l-j-1}.$$

Thus from (8. 4)

$$|c_j^{(3)}| \leqq \frac{2^{l-j-1}}{2\delta^m} \left(\frac{4}{1-\delta}\right)^n \left\{ 1 + \binom{j+1}{1} \frac{1}{2} + \binom{j+2}{2} \frac{1}{2^2} + \cdots + \binom{l-1}{l-1-j} \frac{1}{2^{l-j-1}} \right\} <$$

$$(8.5) \qquad < \frac{2^{l-j-1}}{2\delta^m} \left(\frac{4}{1-\delta}\right)^n \sum_{d=0}^{\infty} \binom{j+d}{d} \frac{1}{2^d} =$$

$$= \frac{2^{l-j-1}}{2\delta^m} \left(\frac{4}{1-\delta}\right)^n \sum_{d=0}^{\infty} (-1)^d \binom{-j-1}{d} \frac{1}{2^d} = \frac{2^{l-1}}{\delta^m} \left(\frac{4}{1-\delta}\right)^n.$$

Let finally be

$$(8.6) \qquad f_3(z) = z^{m+1} f_1(z) f_2(z) = \sum_{j=m+1}^{m+n} c_j^{(4)} z^j.$$

It follows from the definition of $f_1(z)$ and $f_2(z)$ that

$$(8.7) \qquad f_3(z_1) = f_3(z_2) = \cdots = f_3(z_l) = 1, \; f_3(z_{l+1}) = \cdots = f_3(z_n) = 0.$$

Replacing z in (8. 6) by z_j and[6] summing for $j = 1, 2, \ldots, n$ we get, using (8. 7) and writing

$$(8.8) \qquad s_\nu = z_1^\nu + z_2^\nu + \cdots + z_n^\nu,$$

the identity

$$\left| \sum_{\nu=m+1}^{m+n} c_\nu^{(4)} s_\nu \right| = \sum_{\nu=m+1}^{m+n} c_\nu^{(4)} s_\nu = l,$$

i. e.

$$(8.9) \qquad l \leqq \max_{\substack{m+1 \leqq \nu \leqq m+n \\ \nu \text{ integer}}} |s_\nu| \left(\sum_{j=m+1}^{m+n} |c_j^{(4)}| \right).$$

But from (8. 6) we have

$$c_j^{(4)} = \sum_{n-l-j_1+j_2=j-m-1} c_{j_1}^{(1)} c_{j_2}^{(3)},$$

i. e.

$$\sum_{j=m+1}^{m+n} |c_j^{(4)}| \leqq \left(\sum_{j_1=0}^{n-l} |c_{j_1}^{(1)}| \right) \left(\sum_{j_2=0}^{l-1} |c_{j_2}^{(3)}| \right);$$

[6] If we want to prove (2. 7) instead of the upper estimation of the Theorem, we have also to multiply by b_j and then to sum for j.

thus, using (8. 2) and (8. 5), we get

$$\sum_{j=m+1}^{m+n} |c_j^{(4)}| \leqq 2^{n-l}\, l\, \frac{2^{l-1}}{\delta^m}\left(\frac{4}{1-\delta}\right)^n = \frac{l}{2\delta^m}\left(\frac{8}{1-\delta}\right)^n.$$

Then we have from (8. 9)

$$\max_{\substack{m+1\leqq \nu \leqq m+n \\ \nu \text{ integer}}} |s_\nu| \geqq 2\delta^m \left(\frac{1-\delta}{8}\right)^n. \tag{8. 10}$$

Now we choose δ so, that

$$\delta^{m+n}\left(\frac{n}{2e(m+n)}\right)^n = \delta^m \left(\frac{1-\delta}{8}\right)^n,$$

i. e.

$$\delta = \frac{1}{1+\frac{4}{e}\frac{n}{m+n}} \equiv \delta_0;$$

then from (7. 4) and (8. 10) we got

$$\max_{\nu=m+1,\, m+2, \ldots ,\, m+n} |z_1^\nu + z_2^\nu + \cdots + z_n^\nu| \geqq 2\delta_0^{m+n}\left(\frac{n}{2e(m+n)}\right)^n.$$

But

$$\left(1+\frac{4n}{e(m+n)}\right)^{m+n} \leqq e^{\frac{4n}{e}}, \quad \text{i. e.} \quad \delta_0^{m+n} \geqq e^{-\frac{4n}{e}}$$

whence

$$\max_{\nu=m+1,\, m+2, \ldots ,\, m+n} |z_1^\nu + \cdots + z_n^\nu| \geqq 2\left(\frac{n}{2e^{1+\frac{4}{e}}(m+n)}\right)^n,$$

i. e. our Theorem is for different z_j's proved. We can get rid of the restriction $z_\mu \neq z_\nu$ $(\mu \neq \nu)$ exactly on the same way as in the previous proof in[1] and we do not detail it.

9. Now we turn to the lower estimation in the Theorem. Let ϑ be a positive constant, less than 1 and to be determined later, let $z_1 = 1$ and $z_2, z_3, \ldots, z_n$ be determined by the conditions (with the notation (8. 8))

$$s_1 = \vartheta n,\ s_2 = s_3 = \cdots = s_{n-1} = 0. \tag{9. 1}$$

It is well known that these conditions determine uniquely the numbers z_j $(j=2, 3, \ldots, n)$; let us denote them by ζ_ν $(\zeta_1 = 1,\ \nu = 2, 3, \ldots, n)$. Let further be

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

the polynomial with the zeros ζ_n $(\nu = 1, \ldots, n)$. Then the Newton—Girard formulae give successively

$$a_1 = -s_1,\ a_2 = \frac{s_1^2}{2!},\ a_3 = -\frac{s_1^3}{3!}, \ldots, a_{n-1} = (-1)^{n-1}\frac{s_1^{n-1}}{(n-1)!}. \tag{9. 2}$$

Since $\zeta_1 = 1$ is a zero of $f(z)$, we have

$$a_n = -(1 + a_1 + \cdots + a_{n-1}) = -\left(1 - \frac{s_1}{1!} + \frac{s_1^2}{2!} - \cdots + (-1)^{n-1} \frac{s_1^{n-1}}{(n-1)!}\right) =$$

$$= -e^{-s_1} + \sum_{j=n}^{\infty} (-1)^j \frac{1}{j!} s_1^j. \tag{9.3}$$

Further, we have from Newton—Girard formulae

$$s_n = -a_{n-1}s_1 - n a_n = (-1)^n \frac{s_1^n}{(n-1)!} + n e^{-s_1} - \sum_{j=n}^{\infty} n(-1)^j \frac{s_1^j}{j!} =$$

$$= n\left(e^{-s_1} - \sum_{j=n+1}^{\infty} (-1)^j \frac{s_1^j}{j!}\right) = n \sum_{j=0}^{n} (-1)^j \frac{s_1^j}{j!} \tag{9.4}$$

and from (9. 3) and (9. 4)

$$s_{n+1} = -a_1 s_n - a_n s_1 = s_1(s_n - a_n) =$$

$$= s_1 \left\{(n+1)e^{-s_1} + (-1)^{n+1} \frac{s_1^n}{n!} - (n+1) \sum_{j=n+1}^{\infty} (-1)^j \frac{s_1^j}{j!}\right\}. \tag{9.5}$$

Since $0 < \vartheta < 1$, the terms $\frac{s_1^j}{j!}$ decrease monotonically if $j \geqq n$, it follows

$$\left|\sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{s_1^j}{j!}\right| < \frac{s_1^{n+1}}{(n+1)!},$$

i. e. from (9. 4) and (9. 5)

$$|s_n| \leqq n\left(e^{-\vartheta n} + \frac{(\vartheta n)^{n+1}}{(n+1)!}\right), \qquad |s_{n+1}| \leqq \vartheta n \left\{(n+1)e^{-\vartheta n} + \frac{(\vartheta n)^n}{n!} + \frac{(\vartheta n)^{n+1}}{n!}\right\}.$$

Since $k! > \left(\frac{k}{e}\right)^k e$, we get for $n > 10$

$$|s_n| \leqq n\left\{e^{-\vartheta n} + \frac{1}{e}(e\vartheta)^{n+1}\right\}, \qquad |s_{n+1}| \leqq (n+1)^2(e^{-\vartheta n} + 4(e\vartheta)^n)$$

whence with $m = 1$

$$\max_{m+1 \leqq \nu \leqq m+n} |s_\nu| < 4(n+1)^2\{e^{-\vartheta n} + (e\vartheta)^n\}.$$

If ϑ is the (only) positive zero α_0 of the transcendental equation

$$x = e^{-x-1}, \tag{9.6}$$

then we have $0{,}2784 \leqq \vartheta \leqq 0{,}2785$ and for our ζ_ν's

$$\max_{\substack{2 \leqq \nu \leqq n+1 \\ \nu \text{ integer}}} |s_\nu| \leqq 8(n+1)^2 e^{-\alpha_0 n}. \tag{9.7}$$

It may occur that (2. 5) is not fulfilled by the ζ_ν-system, i. e. some of them are absolutely $\geqq 1$. If so, we can construct simply by contraction a ζ_ν^*-system satisfying (2. 5) and for which (9. 7) holds a fortiori. Thus for an arbitrary small $\varepsilon > 0$ and for $n > n_0(\varepsilon)$ there is a ζ_ν-system satisfying (2. 5) such that

$$\max_{\substack{2 \leqq \nu \leqq n+1 \\ \nu \text{ integer}}} |s_\nu| \leqq e^{-(\alpha_0-\varepsilon)n}.$$

Hence, if for each integer $n \geqq 1$ and $m \geqq 0$ the estimation (2. 6) is valid, then

$$\left(\frac{n}{A_1(1+n)}\right)^n \leqq e^{(-\alpha_0+\varepsilon)n}, \quad \text{i. e.} \quad A_1 \geqq e^{\alpha_0} \sim 1{,}321,$$

indeed.

10. We are going to prove (3. 4). Obviously we need only to show

$$A_3 \geqq \frac{4}{\pi}.$$

We choose e. g. n big odd, $m = n^3$ and with a slight modification of idea of P. STEIN[7]

(10. 1) $$z_j = e^{\frac{\pi i}{2(m+n)}(2j-n-1)}, \quad b_j = \frac{1}{2^{n-1}}\binom{n-1}{j-1} e^{\frac{\pi i n}{4(m+n)}(2j-n-1)} \qquad (j = 1, 2, \ldots, n).$$

Then we get

$$f_0(y) = \sum_{j=1}^{n} b_j z_j^y = \frac{1}{2^{n-1}} \sum_{j=1}^{n} \binom{n-1}{j-1} e^{\frac{\pi i}{2(m+n)}(2j-n-1)\left(y+\frac{n}{2}\right)} =$$

$$= \left(\frac{e^{\frac{\pi i}{2(m+n)}\left(y+\frac{n}{2}\right)} + e^{-\frac{\pi i}{2(m+1)}\left(y+\frac{n}{2}\right)}}{2}\right)^{n-1} = \cos^{n-1}\left\{\frac{\pi}{2(m+n)}\left(y+\frac{n}{2}\right)\right\}.$$

Let us observe that for all sufficiently large n's

$$\left|\sum_{j=1}^{n} b_j\right| = \cos^{n-1}\frac{\pi n}{4(m+n)} = \cos^{n-1}\frac{\pi}{4(n^2+1)} > \cos^{n-1}\frac{\pi}{4n^2} > \frac{1}{1+\frac{1}{n}}.$$

Hence for $m \leqq y \leqq m+n$ we obtain

$$|f_0(y)| \leqq \left(1+\frac{1}{n}\right)\left|\sum_{j=1}^{n} b_j\right| \sin^{n-1}\frac{\pi n}{4(m+n)} < \left(1+\frac{1}{n}\right)\left(\frac{\pi}{4}\cdot\frac{n}{m+n}\right)^{n-1}\left|\sum_{j=1}^{n} b_j\right| =$$

$$= \left(1+\frac{1}{n}\right)\frac{4}{\pi}(n^2+1)\left(\frac{n}{\frac{4}{\pi}(m+n)}\right)^n \left|\sum_{j=1}^{n} b_j\right|.$$

Since for an arbitrary small $\varepsilon > 0$ we have for $n > n_0(\varepsilon)$

$$\left(1+\frac{1}{n}\right)\frac{4}{\pi}(n^2+1) < \frac{1}{(1-\varepsilon)^n},$$

we got

$$A_3 \geqq \frac{4}{\pi}(1-\varepsilon). \qquad \text{Q. e. d.}$$

[7] See J. E. LITTLEWOOD, Math. Notes (12), An inequality for a sum of cosines, *Journ. of London Math. Soc.*, **12** (1937), pp. 217—222.

11. In order to show (4. 7)—(4. 8) we take

(11. 1) $$s_1 = -2u, \quad s_2 = 2v^2 \qquad (v \neq 0).$$

Then Newton—Girard's formulae give

$$a_1 = 2u = vH_1\left(\frac{u}{v}\right),$$

$$2a_2 = -s_2 - a_1 s_1 = -2v^2 + 4u^2 = v^2\left\{4\left(\frac{u}{u}\right)^2 - 2\right\} = v^2 H_2\left(\frac{u}{v}\right),$$

$$a_2 = \frac{v^2}{2!} H_2\left(\frac{u}{v}\right),$$

denoting by $H_\nu(y)$ the ν^{th} Hermite polynomial defined in (4. 9). Suppose we showed already

(11. 2) $$a_m = \frac{v^m}{m!} H_m\left(\frac{u}{v}\right) \qquad (m = 1, 2, \ldots, k)$$

for a $k < n$. Then

$$(k+1)a_{k+1} = -(s_{k+1} + a_1 s_k + \cdots + a_k s_1)$$

and, using (4. 7) and (11. 2),

$$(k+1)a_{k+1} = -(a_{k-1}s_2 + a_k s_1) = -\frac{v^{k+1}}{(k-1)!} H_{k-1}\left(\frac{u}{v}\right)2 + \frac{v^k}{k!} H_k\left(\frac{u}{v}\right)2u =$$

(11. 3) $$= \frac{v^{k+1}}{k!}\left\{-2kH_{k-1}\left(\frac{u}{v}\right) + 2\frac{u}{v} H_k\left(\frac{u}{v}\right)\right\}.$$

But as well known

(11. 4) $$2xH_k(x) - 2kH_{k-1}(x) = H_{k+1}(x),$$

i. e. from (11. 3)

$$a_{k+1} = \frac{v^{k+1}}{(k+1)!} H_{k+1}\left(\frac{u}{v}\right),$$

what shows that (11. 2) is true for $m = 1, 2, \ldots, n$. Conversely these values a_ν assure that $s_3 = s_4 = \cdots = s_n = 0$. For s_{n+1} we have

$$s_{n+1} = -(a_1 s_n + \cdots + a_n s_1) = -(a_{n-1}s_2 + a_n s_1),$$

i. e. from (11. 1) and (11. 2)

$$s_{n+1} = -\frac{v^{n+1}}{(n-1)!} H_{n-1}\left(\frac{u}{v}\right)2 + \frac{v^n}{n!} H_n\left(\frac{u}{v}\right)2u =$$

$$= \frac{v^{n+1}}{n!}\left\{2\frac{u}{v} H_n\left(\frac{u}{v}\right) - 2nH_{n-1}\left(\frac{u}{v}\right)\right\},$$

or from (11. 4)

$$s_{n+1} = \frac{v^{n+1}}{n!} H_{n+1}\left(\frac{u}{v}\right).$$

Since $v \neq 0$, $s_{n+1} = 0$ implies $u = \lambda v$ where

$$H_{n+1}(\lambda) = 0.$$

This proves already (4.7)—(4.8).

For the value s_{n+2} we have in our case

$$s_{n+2} = -(a_1 s_{n+1} + a_2 s_n + \cdots + a_n s_2) = -a_n s_2 = -2\frac{v^{n+2}}{n!} H_n(\lambda),$$

i. e.

$$\max_{\nu=3,4,\ldots,(n+2)} |s_\nu| = \frac{2}{n!} |v|^{n+2} |H_n(\lambda)|. \tag{11.5}$$

If the minimal absolute value of the zeros of

$$\sum_{\nu=0}^{n} \frac{H_\nu(\lambda)}{\nu!} z^\nu = 0 \tag{11.6}$$

is denoted by $A_n(\lambda)$, then the maximal one of

$$\sum_{\nu=0}^{n} \frac{H_\nu(\lambda)}{\nu!} v^\nu z^{n-\nu}$$

is $\frac{|v|}{A_n(\lambda)}$. Hence, if we choose $v = A_n(\lambda)$, we obtained a $(z_1^*, \ldots, z_n^*)$-system with

$$\max_{j=1,\ldots,n} |z_j^*| = 1$$

and

$$\max_{\nu=3,4,\ldots,(n+2)} |z_1^{*\nu} + \cdots + z_n^{*\nu}| = \frac{2}{n!} |H_n(\lambda)| A_n(\lambda)^{n+2}. \tag{11.7}$$

An asymptotical determination of $A_n(\lambda)$ (or even a good upper estimation of it) and a suitable choice of λ would probably result a better lower bound for A_1 in the Theorem.

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О НЕКОТОРЫХ НОВЫХ ТЕОРЕМАХ ТЕОРИИ ДИОФАНТОВЫХ ПРИБЛИЖЕНИЙ

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Второй из авторов настоящей работы во своей недавно вышедшей книге дал целый ряд применений диофантовых неравенств I, II, III, относящихся к различным ветвьям анализа и аналитической теории чисел. Улучшение этих неравенств важно и с точки зрения применений. В настоящей работе авторы значительно улучшают теорему II. В одном из самых важных для применений случаев это улучшение состоит в следующем: Пусть $\max_{j=1,\ldots,n} |z_j| = 1$, и пусть A означает наименьшую числовую постоянную,

для которой

$$\max_{m+1\leqq \nu\leqq m+n} |s_\nu| \equiv \max_{m+1\leqq \nu\leqq m+n} |z_1^\nu + \cdots + z_n^\nu| \geqq \left(\frac{n}{A(m+n)}\right)^n$$

(ν принимает целые значения), при всех целых неотрицательных m и целых n. Тогда

$$1{,}321 < A < 2e^{1+\frac{4}{e}}.$$

В связи с улучшением оценки снизу возникает вопрос о всех системах $(z_1, \ldots, z_n)$, для которых

a) $s_2 = s_3 = \cdots = s_n = 0$,

b) $s_3 = s_4 = \cdots = s_{n+1} = 0$.

Легко доказать, что с точностью до растяжения и вращения единственная система $(z_1, \ldots, z_n)$, удовлетворяющая условию a) состоит из корней уравнения $\sum_{\nu=0}^{n} \frac{z^{n-\nu}}{\nu!} = 0$. Несколько труднее доказать, что — опять с точностью до растяжения и вращения — все системы $(z_1, \ldots, z_n)$, удовлетворяющие условию b) состоит из корней $n+1$ уравнений

$$\sum_{\nu=0}^{n} \frac{H_\nu(\lambda)}{\nu!} z^{n-\nu} = 0,$$

где $H_\nu(y)$ есть многочлен Эрмита степени ν, а λ любой корень уравнения $H_{n+1}(y) = 0$.