ON THE DISTRIBUTION MOD 1 OF THE SEQUENCE $n\alpha$

By

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1. Let $0<\alpha<1$. Starting out in positive direction from a point 0 of a circle K with unity periphery we put up the arc with length α n-times; the endpoint of this we shall call the $n\alpha$-point. In connection with the problems of diophantine approximation it is obviously important the investigation of the geometrical structure of the $n\alpha$-points. It is possible to give on that way a geometrical theory and generalisation of continued fractions[2] by which the classical theorems of the diophantine approximations can be rather simply treated as well as new results obtained.[3] In this paper we shall give on this way simple proofs for some theorems conjectured by H. STEINHAUS, some in sharper form, and a simple characterization of the geometrical order of the $n\alpha$-points.

To the above defined $n\alpha$-points refers a very surprisingly sounding conjecture of H. STEINHAUS.[4] Considering the above defined $n\alpha$-points for $n=1,2,\ldots,N$, together with 0 they determine $N+1$ disjoint arcs of the periphery; the conjecture of STEINHAUS asserts that their respective lengths can have at most three different values, for *every* N and α.

Denoting among them the maximal resp. minimal length by H_N resp. h_N, the further conjectures of STEINHAUS refer to the behaviour of H_N and h_N if $N\to\infty$ and assert that *if the digits*[5] *of the regular continued fraction of α are unbounded, then*

$$(1.1)\qquad \begin{array}{ll} \underline{\lim}_{N\to\infty} N\cdot h_N=0 & \underline{\lim}_{N\to\infty} N\cdot H_N=1 \\ \overline{\lim}_{N\to\infty} N\cdot h_N=1 & \overline{\lim}_{N\to\infty} N\cdot H_N=\infty. \end{array}$$

[1] The subject of this paper was part of my dissertation, defended at 21. June 1957.

[2] In the present paper the geometrical treatment of the continued fraction will only be sketched; a more detailed version treated in On the theory of diophantine approximations. I, *Acta Math. Acad. Sci. Hung.*, **8** (1957), 461—471.

[3] A lánctörtek egy geometriai interpretációja és alkalmazásai. *Matematikai Lapok*, **8** (1957), 248—263.

[4] This was proved independently in the mean-time by P. ERDŐS, G. HAJÓS, N. SWIECZKOWSKI, P. SZÜSZ and J. SURÁNYI. (See J. SURÁNYI, Über die Anordnung der Vielfachen einer reellen Zahl mod 1, *Annales Univ. Sci. Budapest*, Sectio Math., **1** (1958), 107—111.)

[5] "partial quotients".

These conjectures in (1·1) were proved using the theory of continued fractions first by S. HARTMAN.[6]

As we shall see, the geometrical order of the $n\alpha$-points is determined by the $s\alpha$-points with the property, that one of the two closed arcs on the circle determined by 0 and the $s\alpha$-point contains no $n\alpha$-points with $1 \leqq n < s$. In what follows we shall call these $s\alpha$-points "adjacent to 0" and the corresponding s-multipla "adjacent multipla". For a fixed α let the sequence of the adjacent multipla be denoted by

$$(1.2) \qquad (1 =) s_1 < s_2 < \cdots < s_\nu < \cdots.$$

Obviously for any irrational α the s_ν-s form an infinite sequence. Further we denote the empty arc bordered by 0 and the $s_\nu\alpha$-point by Δ_ν. We call the $s_\nu\alpha$-point also sometimes as the endpoint of Δ_ν. We emphasize, we mean Δ_ν *directed,* i. e. positive resp. negative, when the empty arc goes from 0 in the positive resp. negative direction. The *directed* length of Δ_ν we denote by δ_ν. For the s_ν adjacent multipla we shall see the

LEMMA I. *Let $s_\nu\alpha$ and $s_{\nu-k}\alpha$ (k positive) two adjacent points on the opposite side of* 0 *so that no $n\alpha$-points with $0 < n < s_\nu$ lie on the closed arc, bordered by the $s_\nu\alpha$-point and $s_{\nu-k}\alpha$-point and containing* 0. *Then we have the recursive formulae:*

$$(1.3) \qquad s_{\nu+1} = s_\nu + s_{\nu-k},$$

$$(1.4) \qquad \delta_{\nu+1} = \delta_\nu + \delta_{\nu-k}.$$

This Lemma means obviously that one obtains the endpoint of $\Delta_{\nu+1}$ taking the absolutely greater arc among Δ_ν and $\Delta_{\nu-k}$ and from its endpoint measuring back the absolutely smaller arc.

We can now describe the geometrical order of the $n\alpha$-points ($n = 1, 2, \ldots, N$), i. e. to determine the $(k_1, k_2, \ldots, k_N)$-permutation of $(1, 2, \ldots, N)$ with[7]

$$0 < (k_1\alpha) < (k_2\alpha) < \cdots < (k_N\alpha) < 1,$$

or

$$0 < (k_N\alpha) < (k_{N-1}\alpha) < \cdots < (k_1\alpha) < 1$$

by these s_ν adjacent multipla. Let for the sake of simplicity α be irrational. Then this is given by the

THEOREM I. *For our given N we determine ν by*

$$s_\nu \leqq N < s_{\nu+1}$$

[6] S. HARTMAN, Über die Abstände von Punkten $n\xi$ auf der Kreisperipherie, *Annales de la Société Polonaise de Mathematique,* **25** (1954), 110—115.

[7] (x) denotes, as usual, the fractional part of x:

and $s_{\nu-k}$ *should be defined as in Lemma I. Starting from the point* 0 *in the direction of* $\Delta_{\nu-k}$ *let the consecutive multipla of* α *be* $k_1\alpha = s_{\nu-k}\alpha$, $k_2\alpha, \ldots, k_{N-1}\alpha$, $k_N\alpha = s_\nu\alpha$. *With the numbers* $k_1 = s_{\nu-k}$ *and* $k_N = s_\nu$ *the whole permutation* $(k_1, k_2, \ldots, k_N)$ *is exactly determined, namely:*

A) $$k_{l+1} = k_l + k_1 \quad \text{if} \quad 0 \leqq k_l \leqq N - k_1$$

B) $$k_{l+1} = k_l - (k_N - k_1) \quad \text{if} \quad N - k_1 < k_l < k_N$$

C) $$k_{l+1} = k_l - k_N \quad \text{if} \quad k_N \leqq k_l \leqq N.$$

(This has a sense since from (1.3) $N < k_1 + k_N$.)

Since the directed arc between the $n\alpha$-point and $m\alpha$-point $(m > n)$ equals to that among 0 and the $(m-n)\alpha$-point,[8] the Theorem I obviously gives a proof for STEINHAUS's threelength-conjecture, giving at the same time an explicit determination of the lengths in question. This explicit determination allows e. g. to prove the existence of an infinity of N's for whose the number of different arcs is *only two*.

In **6** we shall give simple proofs for HARTMAN's above mentioned theorems in the frame of the above mentioned considerations.

2. PROOF OF LEMMA I. We consider an $n\alpha$-point in $\Delta_\nu + \Delta_{\nu-k}$; then obviously $n > s_\nu$. Since the length of this arc is $|\delta_\nu| + |\delta_{\nu-k}|$, there are two cases:

a) the distance on the circle of the $n\alpha$-point and $s_\nu\alpha$-point (within $\Delta_\nu + \Delta_{\nu-k}$) is not greater then $|\delta_{\nu-k}|$,

b) the distance of the $n\alpha$-point from the $s_{\nu-k}\alpha$-point (in the above sense) is less than $|\delta_\nu|$.

Consider first the case *a)*. We remark that the directed distance between the $n\alpha$-point and the $s_\nu\alpha$-point is the same as that of the $(n-s_\nu)\alpha$-point and 0; hence the $(n-s_\nu)\alpha$-point lies in $\Delta_{\nu-k}$. Thus from the definition of $s_{\nu-k}$ it follows

(2.1) $$n - s_\nu \geqq s_{\nu-k}, \qquad n \geqq s_\nu + s_{\nu-k}$$

In the case *b)* similarly we get

$$n \geqq s_\nu + s_{\nu-k}.$$

Hence the smallest possible n-value is $s_\nu + s_{\nu-k}$; the remark[8] shows at once, that the $(s_\nu + s_{\nu-k})\alpha$-pont lies indeed in $\Delta_\nu + \Delta_{\nu-k}$, which completes the proof of Lemma I.

3. PROOF OF THEOREM I. We shall treat separately all cases; the common feature of the proofs is that we always show that if an $n\alpha$-point

[8] The content of this remark we shall quote in the sequel as remark [8].

lies "between" the $k_l\,\alpha$-point and the $k_{l+1}\,\alpha$-point,[9] assigned by our theorem, we have always $n > N$.

Case A. We consider separately the n's with

$$n > k_l \tag{3. 1}$$

resp. with

$$n < k_l. \tag{3. 2}$$

In the case (3. 1) using remark[8] to the

$$k_l\,\alpha\text{-},\quad n\alpha\text{-},\quad (k_l+k_1)\,\alpha\text{-}$$

points, the directed distances on circle K are the same as between the

$$0\text{-},\quad (n-k_l)\,\alpha\text{-},\quad k_1\,\alpha\text{-}$$

points, hence owing to the definition of k_1 it follows

$$n-k_l > N,$$

i. e.

$$n > N$$

indeed. In the case (3. 2) applying remark[8] to the

$$(k_l+k_1)\,\alpha\text{-},\quad n\alpha\text{-}$$

points the directed distances are the same as between the

$$(k_l+k_1-n)\,\alpha\text{-},\quad 0\text{-}$$

points, — and thus, owing to the definition of k_1,

$$k_l+k_1-n > N.$$

But then we have a fortiori $k_l+k_1 > N$, which contradicts the restriction of case A).

Case B. Using the remark[8] to the

$$k_l\,\alpha\text{-},\quad n\alpha\text{-},\quad (k_l+k_1-k_N)\,\alpha\text{-}$$

points, it follows that the directed distances on K are the same as between the

$$k_N\,\alpha\text{-},\quad (n+k_N-k_l)\,\alpha\text{-},\quad k_1\,\alpha\text{-} \tag{3. 3}$$

points. Owing to the definition of k_1 and k_N and since now $k_N \geqq k_l$ we have $n+k_N-k_l \geqq 0$, thus (3. 3) can occur either if

$$n+k_N-k_l = 0$$

which implies owing to $k_N \geqq k_l$

$$n = 0$$

or if

$$n+k_N-k_l > N$$

which implies again owing to $k_N \geqq k_l$

$$n > N.$$

[9] I. e. from the $k_l\,\alpha$-point *in the direction of the sign of* $\delta_{\nu-k}$ towards the $k_{l+1}\,\alpha$-point.

Case C. We shall consider separately the n's with

(3. 4) $$n > k_l - k_N$$

and

(3. 5) $$n < k_l - k_N.$$

In the case (3. 4) we use the remark[8] to the

$$k_l\alpha\text{-},\quad n\alpha\text{-},\quad (k_l - k_N)\alpha\text{-}$$

points; according to this the respective directed distances are the same as those among the

$$k_N\alpha\text{-},\quad (n - k_l + k_N)\alpha\text{-},\quad 0\text{-}$$

points. Thus, if the $n\alpha$-point would lie between the $k_l\alpha$ and $(k_l - k_N)\alpha$-point, then the $(n - k_l + k_N)\alpha$-point would lie between the $k_N\alpha$-point and the 0-points, and thus, owing to the definition of k_N

$$n - k_l + k_N > N$$

and owing to $k_l > k_N$ a fortiori $n > N$ indeed. In the case (3. 5) applying the remark[8] to the

(3. 6) $$k_l\alpha\text{-},\quad n\alpha\text{-}$$

points their directed distance on k is the same as that of the

$$(k_l - n)\alpha\text{-},\quad 0\text{-}$$

points. But if the $n\alpha$-point would lie "between" the $k_l\alpha$-point and the $(k_l - k_N)\alpha$-point, then the directed distance on K of the points (3. 6) would be less than the distance on K of the $k_l\alpha$-point and the $(k_l - k_N)\alpha$-point, i. e. less, than the distance of the $k_N\alpha$-point and the 0-point. Hence the $(k_l - n)\alpha$-point would lie "between" the $k_N\alpha$-point and the 0-point, and thus owing to the definition of k_N

$$k_l - n > N$$

in contradiction to $k_l \leqq N$.

4. It follows from Theorem I that the three different lengths of arcs, to which the periphery of K is divided by the $n\alpha$-points ($n = 1, 2, \ldots, N$) are

(4. 1) $$|\delta_\nu|,\quad |\delta_{\nu-k}|,\quad |\delta_\nu| + |\delta_{\nu-k}|.$$

Theorem I gives answer to the question, how often these arc-lengths occur. The arc-length $|\delta_{\nu-k}|$ occurs exactly

$$N - k_1 + 1 = N - s_{\nu-k} + 1$$

times, the arc-length $|\delta_\nu|$ exactly

$$N - k_N + 1 = N - s_\nu + 1$$

times, and finally the arc-length $|\delta_\nu|+|\delta_{\nu-k}|$ exactly

$$k_N+k_1-N-1=s_\nu+s_{\nu-k}-N-1$$

times. Thus, if

$$N=s_\nu+s_{\nu-k}-1=s_{\nu+k}-1, \tag{4.2}$$

then the arc-length $|\delta_\nu|+|\delta_{\nu-k}|$ does not appear, i. e., *there are an infinity of values N for whose there are only two different arc-lengths.*

5. Before turning to the simple proofs of the mentioned theorems of S. HARTMAN, we mention the connection of our above considerations with the theory of regular continued fractions.[10] Our s_ν adjacent multipla are identical with the denominators of the convergents and of the "Nebenbrüche" of the regular continued fraction[11] of α. Those s_ν-adjacent multipla, for whose δ_ν and $\delta_{\nu+1}$ are of opposite sign, are the denominators of the convergents. This subsequence of the s_ν-multipla for whose δ_ν and $\delta_{\nu+1}$ have opposite signs (and thus $|\delta_\nu|<|\delta_{\nu-k}|$), we denote by

$$(1=)\, q_1<q_2<\cdots<q_k<\cdots \tag{5.1}$$

and the corresponding δ_ν quantities we denote by

$$(\alpha=)\, d_1<d_2<\cdots<d_k<\cdots. \tag{5.2}$$

The digits a_k of the regular continued fraction[11] of α are given by

$$a_k=\left[\left|\frac{d_{k-1}}{d_k}\right|\right]. \tag{5.3}$$

Corresponding to the well known recursion-formula[12] of the q_ν-s we have

$$|d_{k+1}|=|d_{k-1}|-a_k|d_k|. \tag{5.4}$$

6. Next we turn to the proofs of the theorems of S. HARTMAN, described in (1. 1). We shall prove them in the following form.

[10] See 1. and 2.

[11] If $\alpha=\cfrac{1}{a_1+\cfrac{1}{a_1+\cfrac{1}{a_2+}\cdots}}$ with positive integer a_k-digits is the regular continued fraction of α, then the finite fractions $\dfrac{p_k}{q_k}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_k}}}$ are the convergents and the fractions $\dfrac{p_{k-1}+\nu p_k}{q_{k-1}+\nu q_k}$ $(\nu=1,2,\ldots,a_k-1)$ the "Nebenbrüche".

[12] Which follows also quite easily from the considerations of [2].

If k is an index, for which a_k is "large", then

I. for $N=N_1\equiv q_k$ the h_N is "small" compared to $\frac{1}{N}$,

II. for $N=N_2\equiv q_{k-1}+(a_k-1)q_k$ the h_N is "nearly 1" compared to $\frac{1}{N}$,

III. for $N=N_3\equiv q_{k-1}+(a_k+1)q_k$ the H_N is "nearly 1" compared to $\frac{1}{N}$,

IV. for $N=N_4\equiv q_{k-1}+\left[\frac{a_k}{2}\right]q_k$ the H_N is "large" compared to $\frac{1}{N}$.

PROOF OF I. First of all we remark that owing to (4. 1), and the definition of q_k in **5**,

$$h_{N_1}=|d_k|.$$

But we assert that $|d_k|$ is the minimal arc-length even for $N=N_2$. Namely in this case the roles $|\delta_\nu|$ and $|\delta_{\nu-k}|$ are played by $|d_k|$ and $|d_{k-1}|-(a_k-1)|d_k|$ according to the remark after Lemma I; but owing to (5. 4) we have

$$|d_{k-1}|-(a_k-1)|d_k|=|d_{k+1}|+|d_k|>|d_k|$$

indeed. But then we have

$$h_{N_1}=h_{N_2}\leqq\frac{1}{N_2}=\frac{1}{q_{k-1}+(a_k-1)q_k}<\frac{1}{a_k-1}\cdot\frac{1}{q_k}=\frac{1}{a_k-1}\cdot\frac{1}{N_1},$$

which proves I.

PROOF OF II. We assert that

$$H_{N_3}=|d_k| \qquad (=h_{N_2}). \tag{6. 1}$$

Namely, for $N=N_3$ the roles of $|\delta_{\nu-k}|$ and $|\delta_\nu|$ in (4. 1) are played by

$$|d_{k-1}|-a_k|d_k|=|d_{k+1}| \quad\text{and}\quad |d_k|-|d_{k+1}|,$$

i. e. the largest arc is $|d_k|$ indeed. Hence

$$h_{N_2}=H_{N_3}\geqq\frac{1}{N_3}=\frac{1}{q_{k-1}+(a_k+1)q_k}=\frac{1}{q_{k-1}+(a_k-1)q_k}\cdot\frac{q_{k-1}+(a_k-1)q_k}{q_{k-1}+(a_k+1)q_k}$$

(6. 2)

$$h_{N_2}=H_{N_3}>\frac{1}{N_2}\cdot\frac{a_k-1}{a_k+1},$$

which proves II.

PROOF OF III.

$$H_{N_3}=h_{N_2}\leqq\frac{1}{N_2}=\frac{1}{q_{k-1}+(a_k-1)q_k}<\frac{a_k+1}{a_k-1}\cdot\frac{1}{N_3}$$

similarly as in II.

PROOF OF IV. For the case $N=N_4$ the role of $|\delta_{\nu-k}|$ and $|\delta_\nu|$ in (4. 1) is played by

$$|d_{k-1}|-\left[\frac{a_k}{2}\right]|d_k| \quad \text{and} \quad |d_k|$$

so we have, owing to (5. 3),

$$H_{N_4}=|d_{k-1}|-\left[\frac{a_k}{2}\right]|d_k|+|d_k|>a_k|d_k|-\left(\left[\frac{a_k}{2}\right]-1\right)|d_k|>$$

$$>\frac{a_k}{2}|d_k|=\frac{a_k}{2}\cdot h_{N_2}>\frac{a_k}{6}\cdot\frac{1}{N_4}$$

using (6. 2), which proves IV.